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Near continuous g-frames for Hilbert C^*-modules

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Abstract. Let $\mathcal{U}$ be a Hilbert $\mathcal{A}$-module and $L(\mathcal{U})$ the set of all adjointable $\mathcal{A}$-linear maps on $\mathcal{U}$. Let $K = \{\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ and $L = \{\Gamma_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ be two continuous g-frames for $\mathcal{U}$, K is said to be similar with L if there exists an invertible operator $J \in L(\mathcal{U})$ such that $\Gamma_x = \Lambda_x J$, for all $x \in \mathscr{X}$. In this paper, we define the concepts of closeness and nearness between two continuous g-frames. In particular, we show that K and L are near, if and only if they are similar.

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1. Introduction

Frames are important subjects in pure and applied mathematics and they have an essential role in digital processing and scientific computations. In 1952, Duffin and Schaeffer [4] introduced the concept of a

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discrete frame in a Hilbert space to study non-harmonic Fourier series. The idea of generalization of frames to a family indexed by locally compact spaces given with a Radon measure was suggested by Kaiser [9]. These frames are known as continuous frames. Gabardo and Han called these frames "frames associated with measurable spaces" [6]. Discrete frames are introduced in Hilbert C^*-modules by Frank and Larson [5]. In the way, Nazari and Rashidi-Kouchi considered continuous g-frame in Hilbert C^*-modules. This theory has a critical role in many parts, specially in wavelet and shearlet frames. For more material about continuous frames, we refer the reader to [1, 2] and references therein. In this paper, we study some relations about continuous g-frames for a Hilbert C^*-module $\mathcal{U}$ and define the notions of closeness bound and pre-distance between them.

2. Preliminaries

In the following, we introduce a few definitions and properties of Hilbert C^*-modules and g-continuous frames in Hilbert C^*-module.

Definition 2.1. *Let $\mathcal{A}$ be a C^*-algebra. An inner product $\mathcal{A}$-module is a right $\mathcal{A}$-module equipped with an inner product $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{U} \to \mathcal{A}$, which satisfies the following properties:*

(i) $\langle h, \alpha f + \beta g \rangle = \alpha \langle h, f \rangle + \beta \langle h, g \rangle$;

(ii) $\langle f, ga \rangle = \langle f, g \rangle a$;

(iii) $\langle f, g \rangle = \langle g, f \rangle^*$;

(iv) $\langle f, f \rangle \geqslant 0$ *for all* $f \in \mathcal{U}$ *and* $\langle f, f \rangle = 0$ *if and only if* $f = 0$;

where $f, g, h \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$. We can define a norm on $\mathcal{U}$ by $\|f\| = \|\langle f, f \rangle\|^{1/2}$. If $\mathcal{U}$ is complete with this norm, it is called a Hilbert C^*-module over $\mathcal{A}$ or a Hilbert $\mathcal{A}$-module.

For more details one can see [10]. An element a of a C^*-algebra $\mathcal{A}$ is positive and denoted by $a \geqslant 0$ if $a^* = a$ and its spectrum is a subset of positive real numbers. Since $\langle f, f \rangle \geqslant 0$ for every $f \in \mathcal{U}$, we can define $|f| = \langle f, f \rangle^{1/2}$.

Frank and Larson [5], defined the standard frames in Hilbert C^*-modules. Let $\mathcal{U}$ be a Hilbert C^*-module and I be a finite or countable set. A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{U}$ is called a frame for $\mathcal{U}$ if there are two constants $A, B > 0$ such that

$$A\langle f, f\rangle \leqslant \sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle \leqslant B\langle f, f\rangle, \tag{1}$$

for every $f \in \mathcal{U}$. The constants A and B are called frame bounds.

Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert $\mathcal{A}$-modules. The operator T from $\mathcal{U}$ into $\mathcal{V}$ is said to be adjointable $\mathcal{A}$-linear map if there exists a map $T^* : \mathcal{V} \to \mathcal{U}$ satisfying $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f \in \mathcal{U}$ and $g \in \mathcal{V}$. The collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ into $\mathcal{V}$ is denoted by $L(\mathcal{U}, \mathcal{V})$ and $L(\mathcal{U})$ for the set of all adjointable $\mathcal{A}$-linear maps on $\mathcal{U}$, when $\mathcal{U} = \mathcal{V}$.

Suppose that $(\mathscr{X}, \mathscr{M}, \mu)$ is a measure space and $\mathscr{B}$ is a Banach space. It is well known a Bochner-measurable function $f : \mathscr{X} \to \mathscr{B}$ is Bochner integrable if and only if $\int_{\mathscr{X}} \|f\|_{\mathscr{B}} d\mu(x) < \infty$. If $T : \mathscr{B} \to \mathscr{B}$ is a continuous linear operator and f is Bochner-integrable, then, Tf is Bochner-integrable and $\displaystyle\int_{\mathscr{X}} Tf d\mu(x) = T\int_{\mathscr{X}} f d\mu(x)$, [14].

Definition 2.2. *Let $(\mathscr{X}, \mathscr{M}, \mu)$ be a measure space, $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert $\mathcal{A}$-modules and $\{\mathcal{V}_x : x \in \mathscr{X}\}$ be a net of closed Hilbert $\mathcal{A}$-submodules of $\mathcal{V}$. A net $\{\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ is said to be a continuous g-frame for Hilbert $\mathcal{A}$-module $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$ if*

(i) for any $f \in \mathcal{U}$ the function $\tilde{f} : \mathscr{X} \to \mathcal{V}_x$ defined by $\tilde{f}(x) = \Lambda_x f$ is measurable,

(ii) there is a pair of constants $A, B > 0$ such that for any $f \in \mathcal{U}$,

$$A\langle f, f\rangle \leqslant \int_{\mathscr{X}} \langle \Lambda_x f, \Lambda_x f\rangle d\mu(x) \leqslant B\langle f, f\rangle. \tag{2}$$

The constants A and B are called continuous g-frame bounds. The continuous g-frame $\{\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ is called a tight continuous g-frame if $A = B$ and said to be a continuous Parseval g-frame if $A = B = 1$.

If only the second inequality in (2) holds, then, $\{\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ is called the continuous g-Bessel with Bessel bound B.

Suppose that $K = \{\Lambda_x : x \in \mathscr{X}\}$ is a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$. Then, the continuous g-frame operator S on $\mathcal{U}$ is defined by

$$Sf = \int_{\mathscr{X}} \Lambda_x^* \Lambda_x f d\mu(x),$$

for each $f \in \mathcal{U}$.

Nazari and Rashidi-Kouchi mentioned in [12] that the frame operator S is bounded, positive, self-adjoint, and invertible. They also showed that $\tilde{K} = \{\tilde{\Lambda}_x : x \in \mathscr{X}\}$ whose elements is defined by $\tilde{\Lambda}_x = \Lambda_x S^{-1}$ is a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$ with continuous g-frame operator S^{-1} with bounds $1/B$ and $1/A$. Then, $\tilde{K}$ is called a continuous canonical dual g-frame of K. If K has only one canonical dual g-frame, then, K is called a Riesz-type frame. Next theorem states the relation between the continuous g-frame and inequalities (2).

Theorem 2.3. *[11, Theorem 2.5] Let $\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x)$ for any $x \in \mathscr{X}$. Then, $K = \{\Lambda_x : x \in \mathscr{X}\}$ is a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$ if and only if there exist constants $A, B > 0$ such that for any $f \in \mathcal{U}$*

$$A\|f\|^2 \leqslant \left\| \int_{\mathscr{X}} \langle \Lambda_x f, \Lambda_x f\rangle d\mu(x) \right\| \leqslant B\|f\|^2. \tag{3}$$

Let

$$\bigoplus_{x\in\mathscr{X}} \mathcal{V}_x = \left\{ g = \{g_x\} : g_x \in \mathcal{V}_x \text{ and} \left\| \int_{\mathscr{X}} |g_x|^2 d\mu(x) \right\| < \infty \right\}.$$

It is well known $\bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$ is a Hilbert $\mathcal{A}$-module, where the $\mathcal{A}$-valued inner product is defined by $\langle f, g\rangle = \int_{\mathscr{X}} \langle f_x, g_x\rangle d\mu(x)$ and the norm is defined by $\|f\| = \|\langle f, f\rangle\|^{\frac{1}{2}}$ for any $f = \{f_x : x \in \mathscr{X}\}$ and $g = \{g_x : x \in \mathscr{X}\}$ in $\bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$.

Let $\{\Lambda_x \in L(\mathcal{U}, \mathcal{V}_x) : x \in \mathscr{X}\}$ be a continuous g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$. Then, the synthesis operator $T : \bigoplus_{x\in\mathscr{X}} \mathcal{V}_x \to \mathcal{U}$, is defined by $T(g)=\int_{\mathscr{X}} \Lambda_x^* g_x d\mu(x)$, for any $g\in \bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$, and the analysis operator $F:\mathcal{U}\to \bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$ is defined by $F(f)=\{\Lambda_x f : x\in\mathscr{X}\}$ for any $f\in\mathcal{U}$. It is obvious to show that $S=TF$ and $F^*=T$, since

$$TF(f)=T(\Lambda_x f)=\int \Lambda_x^*\Lambda_x f=S(f)$$

and also

$$\begin{aligned}\langle F(f),g\rangle=\int_{\mathscr{X}}\langle\Lambda_x f,g_x\rangle d\mu(x) &= \int_{\mathscr{X}}\langle f_x,\Lambda_x^* g_x\rangle d\mu(x)\\ &= \langle f,\int_{\mathscr{X}}\Lambda_x^* g_x d\mu(x)\rangle=\langle f,T(g)\rangle,\end{aligned}$$

for each $f\in\mathcal{U}$, i.e. $F^*=T$.

Lemma 2.4. *The analysis operator $F:\mathcal{U}\to \bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$ has a closed range.*

Proof. Suppose $\{F_n\}_{n\in\mathbb{N}}$ is a sequence in range of F such that converges to ν. Then, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $F_n=F(f_n)=\{\Lambda_x f_n : x\in\mathscr{X}\}$ for each $n\in\mathbb{N}$. We show that $\nu\in Ran\ F$, that is there exists a $f'\in\mathcal{U}$ such that $\nu=F(f')$. The sequence $\{F_n\}_{n\in\mathbb{N}}$ is cauchy so, for given $\varepsilon>0$, there is N_0 such that for each $m,n>N_0, \|F_n-F_m\|<\sqrt{\varepsilon A}$. On the other hand, $\|F_n-F_m\|^2=\|F(f_n)-F(f_m)\|^2=\|\{\Lambda_x(f_n)\}-\{\Lambda_x(f_m)\}\|^2=\|\{\Lambda_x(f_n-f_m)\}\|^2=\|\int_{\mathscr{X}}\langle\Lambda_x(f_n-f_m),\Lambda_x(f_n-f_m)\rangle d\mu(x)\|$. Since $\{\Lambda_x : x\in\mathscr{X}\}$ is a continuous g-frame, by inequality (3) we reach $A\|f_n-f_m\|^2\leqslant \|\int_{\mathscr{X}}\langle\Lambda_x(f_n-f_m),\Lambda_x(f_n-f_m)\rangle d\mu(x)\|=\|F_n-F_m\|^2<\varepsilon$. Hence, $\{f_n\}_{n\in\mathbb{N}}$ is a cauchy sequence in the Hilbert $\mathcal{A}$-module $\mathcal{U}$ and so, is convergent to f_0. Therefore, $F(f_n)\to F(f_0)$. By uniqueness of limit, we obtain $\nu=F(f_0)$, which means that F has closed range. □

3. Geometry of Continuous g-Frames For Hilbert C^*-Module

In this section, we assume that $(\mathscr{X},\mathscr{M},\mu)$ is a measure space, $K=\{\Lambda_x\in L(\mathcal{U},\mathcal{V}_x) : x\in\mathscr{X}\}$ and $L=\{\Gamma_x\in L(\mathcal{U},\mathcal{V}_x) : x\in\mathscr{X}\}$ are two continuous g-frames for Hilbert $\mathcal{A}$-module $\mathcal{U}$ with respect to net $\{\mathcal{V}_x : x \in \mathscr{X}\}$ of closed Hilbert $\mathcal{A}$-submodules of $\mathcal{U}$. We want to investigate some relations between these continuous g-frames.

At first, we recall that

(i) K is partial equivalent with L if there exists an operator $J \in L(\mathcal{U})$ such that $\Gamma_x = \Lambda_x J$, for all $x \in \mathscr{X}$.

(ii) K is similar with L if there exists an invertible operator $J \in L(\mathcal{U})$ such that $\Gamma_x = \Lambda_x J$, for all $x \in \mathscr{X}$.

(iii) K is unitary equivalent with L, if they are similar by a unitary operator $J \in L(\mathcal{U})$ (that is $JJ^* = J^*J = I$, where I stands for the identity operator).

(iv) K is called partial isometric equivalent with L if K is partial equivalent with L via a partial isometry J.

Recall that the synthesis operators are

$$T_K(g) = \int_{\mathscr{X}} \Lambda_x^* g_x d\mu(x), \quad T_L(g) = \int_{\mathscr{X}} \Gamma_x^* g_x d\mu(x),$$

and the analysis operators are

$$F_K(f) = \{\Lambda_x f : x \in \mathscr{X}\}, \quad F_L(f) = \{\Gamma_x f : x \in \mathscr{X}\},$$

where $g = \{g_x : x \in \mathscr{X}\} \in \bigoplus_{x\in\mathscr{X}} \mathcal{V}_x$ and $f \in \mathcal{U}$. Now we want to describe the similarity property by the concept of nearness. For this, we need the following lemma.

Lemma 3.1. *[13, Proposition 19] Let K and L be two continuous g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_x : x \in \mathscr{X}\}$. Then,*

(i) *Ran $F_L \subseteq$ Ran F_K if and only if two continuous g-frames K and L are partial equivalent.*

(ii) *Ran $F_L =$ Ran F_K if and only if two continuous g-frames K and L are similar.*

Now by getting idea from [3], we are going to describe some concepts, such as closeness and nearness between continuous g-frames for Hilbert $\mathcal{A}$-module $\mathcal{U}$ in the following definition.

Definition 3.2. *Let K and L be two continuous g-frames for $\mathcal{U}$. We say that L is close to K if there exists a positive integer $\lambda \geqslant 0$ such that*

$$|(T_L - T_K)g| \leqslant \lambda |T_K g|, \tag{4}$$

for all $g \in \bigoplus_{x\in\mathscr{X}} V_x$. We define

$$C(L,K) = \inf\{\lambda : |(T_L - T_K)g| \leqslant \lambda|T_K g|\}, \tag{5}$$

and is called the closeness bound of continuous g-frame L to the continuous g-frame K. Note that the close bound relation is not an equivalence relation, because is not symmetric, in general. In the following, we show that under special conditions, this relation is symmetric.

Jiang [8] showed that the Hilbert $\mathcal{A}$-module $\mathcal{U}$ has triangle inequality property if and only if $\overline{\langle \mathcal{U},\mathcal{U}\rangle} = \overline{span}\{\langle f,g\rangle : f,g \in \mathcal{U}\}$ is a commutative C^*-subalgebra of $\mathcal{A}$. In this case from (4), we get

$$\begin{aligned} |(T_L - T_K)g| &\leqslant \lambda|T_K g| = \lambda|T_K g - T_L g + T_L g| \\ &\leqslant \lambda|T_L g - T_K g| + \lambda|T_L g|. \end{aligned}$$

Therefore, $|(T_K - T_L)g| \leqslant \dfrac{\lambda}{1-\lambda}|T_L g|$, where $\lambda < 1$, $g \in \bigoplus_{x\in\mathscr{X}} V_x$. Thus, K is close to L.

Definition 3.3. *We say that two continuous g-frames K and L are near, if K is close to L and L is close to K. We define the pre-distance between K and L and denoted by $r^0(K,L)$ as the maximum of the two closeness bounds, that is*

$$r^0(K,L) = \max\{C(K,L), C(L,K)\}.$$

It is easy to check that r^0 is positive and symmetric.

The following example shows that closeness bound and so, the predistance r^0 do not satisfy the triangle inequality.

Example 3.4. Let $K = \{\Lambda_x : x \in \mathscr{X}\}$ be the continuous g-frame for $\mathcal{U}$. It is obvious that $K' = \{2\Lambda_x : x \in \mathscr{X}\}$ and $K'' = \{4\Lambda_x : x \in \mathscr{X}\}$ are two continuous g-frames. Then, (5) of Definition 3.2 implies that

$$C(K', K) = \inf\{\lambda : |(T'_K - T_K)g| \leqslant \lambda |T_K g|\}$$
$$= \inf\{\lambda : |\int_{\mathscr{X}} 2\Lambda_x^* g_x d\mu(x) - \int_{\mathscr{X}} \Lambda_x^* g_x d\mu(x)| \leqslant$$
$$\lambda |\int_{\mathscr{X}} \Lambda_x^* g_x d\mu(x)|\} = 1$$

for all $g \in \bigoplus_{x\in\mathscr{X}} V_x$. Similarly, we reach $C(K'', K') = 1$ and $C(K'', K) = 3$. Hence, $C(K'', K) \nleq C(K'', K') + C(K', K)$.

Theorem 3.5. *Let K and L be two continuous g-frames for Hilbert $\mathcal{A}$-module $\mathcal{U}$. They are near if and only if they are similar via some invertible operator J.*

Proof. Let K and L are near. Since K is close to L, there is $\lambda \geqslant 0$ such that

$$|(T_K - T_L)g| \leqslant \lambda |T_L g|.$$

By Lemma 2.4, the analysis operator F_K has closed range, so, [9, Theorem 3.2] implies that $Ran\ F_K$ is orthogonal complement and by assumption $ker\ T_L \subseteq ker\ T_K$ or $Ran\ F_K = (ker\ T_K)^\perp \subseteq (ker\ T_L)^\perp = Ran\ F_L$. Since L is close to K, in a similar way we reach $Ran\ F_L \subseteq Ran\ F_K$, so $Ran\ F_L = Ran\ F_K$. Now Lemma 3.1, implies that K and L are similar for some invertible operator J. Conversely, if K and L are similar via some invertible operator J, then, $\Gamma_x = \Lambda_x J$ for all $x \in \mathscr{X}$ and

$$T_L(g) = \int_{\mathscr{X}} \Gamma_x^* g_x d\mu(x) = \int_{\mathscr{X}} J^* \Lambda_x^* g_x d\mu(x) = J^* \int_{\mathscr{X}} \Lambda_x^* g_x d\mu(x) = J^* T_K(g).$$

Thus $T_L = J^* T_K$.

By [9, Proposition 1.2], $|(J^* - I)u| \leqslant \|J^* - I\| |u|$ for each $u \in \mathcal{U}$. Let $g \in \bigoplus_{x\in\mathscr{X}} V_x$ be arbitrary. Then,

$$|(T_L - T_K)g| = |(J^* T_K - T_K)g| = |(J^* - I)T_K g| \leqslant \|J^* - I\| |T_K g|,$$

shows that L is close to K. Also

$$|(T_K - T_L)g| = |((J^*)^{-1} T_L - T_L)g| = |((J^*)^{-1} - I)T_L g| \leqslant \|(J^*)^{-1} - I\| |T_L g|$$

shows that K is close to L. Therefore, they are near. $\square$

We borrow the following example, from [7, Example 3.2], and proceed two continuous g-frames for Hilbert $\mathcal{A}$-module $\mathcal{U}$ such that they are close.

Example 3.6. Let l^∞ be the set of all bounded complex-valued sequence. We define

$$ab = \{a_i b_i\}_{i\in\mathbb{N}}\ ,\ a^* = \{\overline{a}_i\}\ _{i\in\mathbb{N}}\ ,\ \|a\| = max|a_i|(\ i \in \mathbb{N}\ ),$$

where $a = \{a_i\}_{i\in\mathbb{N}}$ and $b = \{b_i\}_{i\in\mathbb{N}}$ in l^∞. Therefore, $\mathcal{A} = (l^\infty, \|.\|)$ is a C^*-algebra.

Let $\mathcal{U} = c_0$ be the set of all sequences such that vanishes at infinity. Define

$$\langle f, g\rangle = fg^* = \{f_i \overline{g}_i\}_{i\in\mathbb{N}}$$

for every $f, g \in\ \mathcal{U}$. Then, $\mathcal{U}$ is a Hilbert $\mathcal{A}$-module.

Suppose $\{e_i\}_{i\in\mathbb{N}}$ is the standard orthonormal basis for $\mathcal{U}$, for each $i \in \mathbb{N}$, set $V_i = \overline{span}\{e_i\}$, and we define the adjointable operator $\Lambda_i : \mathcal{U} \to V_i$ by $\Lambda_i f = \langle f, e_i\rangle e_i$. Hence,

$$\textstyle\sum_{i\in\mathbb{N}}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in\mathbb{N}}\langle f, e_i\rangle\langle e_i, e_i\rangle\langle e_i, f\rangle = \{f_i\overline{f_i}\}_{i\in\mathbb{N}} = \langle f, f\rangle,$$

for every $f \in \mathcal{U}$. Thus $K = \{\Lambda_i\}_{i\in\mathbb{N}}$ is a Parseval g-frame for $\mathcal{U}$.

Fix α, $\beta \in \mathbb{R}^+$ and define $J \in L(\mathcal{U})$ by

$$Je_i = \begin{cases} \alpha e_i & i = 2n-1 \\ \beta e_i & i = 2n \end{cases}$$

Thus J is adjointable and $J = J^*$, since

$$\begin{aligned} \langle Jf, g\rangle &= \langle(\alpha f_1, \beta f_2, \alpha\ f_3, \beta f_4, \cdots), (g_1, g_2, g_3, \cdots)\rangle \\ &= \{\alpha f_1 \bar{g}_1, \beta f_2 \bar{g}_2, \cdots\} \\ &= \langle(f_1, f_2, f_3, f_4, \cdots), (\alpha g_1, \beta g_2, \alpha g_3, \beta g_4, \cdots)\rangle \\ &= \langle f, Jg\rangle. \end{aligned}$$

Now, we show that $L = \{\Lambda_i J\}_{i\in\mathbb{N}}$ is a g-frame for $\mathcal{U}$ with the two frame bounds $\min\{\alpha^2,\beta^2\}$ and $\max\{\alpha^2,\beta^2\}$. By straightforward calculation, we reach

$$\begin{aligned}
\sum_{i\in\mathbb{N}}\langle \Lambda_i Jf, \Lambda_i Jf\rangle &= \sum_{i\in\mathbb{N}}\langle Jf, e_i\rangle\langle e_i, e_i\rangle\langle e_i, Jf\rangle \\
&= \sum_{i\in\mathbb{N}}\langle Jf, e_i\rangle\langle e_i, Jf\rangle \\
&= \sum_{i=2n-1}\alpha^2 f_i\bar{f}_i e_i + \sum_{i=2n}\beta^2 f_i\bar{f}_i e_i \\
&\leqslant \max\{\alpha^2,\beta^2\}\sum_{i\in\mathbb{N}} f_i\bar{f}_i e_i = \max\{\alpha^2,\beta^2\}\{f_i\bar{f}_i\} \\
&= \max\{\alpha^2,\beta^2\}\langle f, f\rangle.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{i\in\mathbb{N}}\langle \Lambda_i Jf, \Lambda_i Jf\rangle &\geqslant \min\{\alpha^2,\beta^2\}\sum_{i\in\mathbb{N}} f_i\bar{f}_i e_i \\
&= \min\{\alpha^2,\beta^2\}\{f_i\bar{f}_i\} = \min\{\alpha^2,\beta^2\}\langle f, f\rangle.
\end{aligned}$$

We show that L is close to K. We know that

$$T_K(\{e_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}}\Lambda_i^* e_i. \tag{6}$$

and

$$T_L(\{e_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}}(\Lambda_i J)^* e_i = \sum_{i\in\mathbb{N}} J\Lambda_i^* e_i \tag{7}$$

for every $\{e_i\}_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathcal{N}} V_i$. Therefore,

$$\begin{aligned}
|(T_L - T_K)\{e_i\}_{i\in\mathbb{N}}| &= |T_L(\{e_i\}_{i\in\mathbb{N}}) - T_K(\{e_i\}_{i\in\mathbb{N}})| = |\sum_{i\in\mathbb{N}} J\Lambda_i^* e_i - \Lambda_i^* e_i| \\
&= |\sum_{i\in\mathbb{N}}(J - I)\Lambda_i^* e_i| \leqslant \|J - I\||\sum_{i\in\mathbb{N}}\Lambda_i^* e_i| \\
&= \|J - I\||T_K(\{e_i\}_{i\in\mathbb{N}})|,
\end{aligned}$$

for every $\{e_i\}_{i\in\mathbb{N}} \in \bigoplus_{i\in\mathcal{N}} V_i$. Then, L is close to K.

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