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Measures of Noncompactness in $\left(\bar{N}^q_{\Delta^-}\right)$ Summable Difference Sequence Spaces

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Abstract. In this paper we first introduce $\bar{N}^q_{\Delta^-}$ summable difference sequence spaces and prove some properties of these spaces. We then obtain the necessary and sufficient conditions for infinite matrix A to map these sequence spaces on the spaces c, c_0 and ℓ_∞. Finally, the Hausdorff measure of noncompactness is used to obtain the necessary and sufficient conditions for the compactness of the linear operators defined on these spaces.

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1. Introduction and Preliminaries

We write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$ and ϕ, c_0, c and ℓ_∞ for the sets of all finite sequences, convergent to zero, and bounded sequences respectively. By e we denote the sequence of 1's, $e = (1, 1, 1, \ldots)$ and by $e^{(n)}$ the sequence with 1 as only nonzero term at the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. Further by cs and ℓ_1 we denote the spaces of all convergent series and absolutely

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convergent series respectively. For $x=(x_k)_{k=0}^{\infty}\in w$ $x^{[m]}=\sum_{k=0}^{m}x_k e^{(k)}$ denotes the m-th section of x.

A sequence space X is a linear subspace of w, such a subspace is called a BK space if it is a Banach space with continuous coordinates $P_n : X\to\mathbb{C}$ $(n=0,1,2,\ldots)$ where $P_n(x)=x_n,\ x=(x_k)_{k=0}^{\infty}\in X$. The BK space X is said to have AK if every $x=(x_k)_{k=0}^{\infty}\in X$ has a unique representation $x=\sum_{k=0}^{\infty}x_k e^{(k)}$, [12, Definition 1.18]. The spaces c_0, c and ℓ_∞ are BK spaces with respect to the norm $\|x\|_\infty=\sup_k\{|x_k| : k\in\mathbb{N}\}$. For any two sequences x and y in w the product xy is given by $xy=(x_k y_k)_{k=0}^{\infty}$.

The $\beta-$dual of a subset X of w is defined by

$$X^\beta=\{a\in w : ax=(a_k x_k)\in cs \ \text{ for all } x=(x_k)\in X\}.$$

If A is an infinite matrix with complex entries a_{nk} $n,k\in\mathbb{N}$, we write $A_n=(a_{nk})_{k=0}^{\infty}$ $n\in\mathbb{N}$ for the sequence in the nth row of A. The $A-$transform of any $x=(x_k)\in w$ is given by $Ax=(A_n(x))_{k=0}^{\infty}$, where

$$A_n(x)=\sum_{k=0}^{\infty}a_{nk}x_k \qquad n\in\mathbb{N},$$

provided the series on right converge for each n.

If X and Y are subsets of w, we denote by (X,Y), the class of all infinite matrices that map X into Y. So $A\in(X,Y)$ if and only if $A_n\in X^\beta$, $n=0,1,2,\ldots$ and $Ax\in Y$ for all $x\in X$. The matrix domain of an infinite matrix A in X is defined by

$$X_A=\{x\in w : Ax\in X\}.$$

If X and Y are Banach Spaces, then by $\mathcal{B}(X,Y)$ we denote the set of all bounded (continuous) linear operators $L : X\to Y$, which is itself a Banach space with the operator norm $\|L\|=\sup_x\{\|L(x)\|_Y : \|x\|=1\}$ for all $L\in\mathcal{B}(X,Y)$. The linear operator $L : X\to Y$ is said to be compact if its domain L is all of X and for every bounded sequence $(x_n)\in X$, the sequence $(L(x_n))$ has a sub-sequence which converges in Y. The operator

$L \in \mathcal{B}(X,Y)$ is said to be of finite rank if $\dim R(L) < \infty$, where $R(L)$ denotes the range space of L. A finite rank operator is clearly compact.

The concept of difference sequence spaces was first introduced by Kizmaz [7] and later several authors studied new sequence spaces defined by using difference operators like Mursaleen and Nouman [14], Mursaleen et. al. [13] , Jalal [5].

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of the difference operator, or of the matrices of the classical methods of summability in spaces such as ℓ_p, c_0, c, ℓ_∞ or others. For instance, some matrix domains of the difference operator were studied in [7, 15], of the Riesz matrices in [2] and so on.

In this paper, we first define three new sequence space as the matrix domains X_T of the product T of the triangles $\bar{N}^q$ and Δ^- and obtain basis for two of them and determine their β duals. We then find out the necessary and sufficient condition for the matrix transformations to map these spaces into c_0, c and ℓ_∞. Finally we characterize the classes of compact matrix operators from these spaces into c_0, c and ℓ_∞.

2. $\bar{N}^q_{\Delta^-}$ Summable Difference Sequence Spaces

The difference operator Δ^- is defined on ω by

$$\Delta^- x_k = x_{k-1} - x_k \quad , k = 0, 1, 2, \ldots \quad \text{where } x_{-1} = 0 \tag{1}$$

The $\Delta^- = (\delta_{nk})_{n,k=0}^{\infty}$ is a triangular matrix written as

$$\delta_{nk} = \begin{cases} -1 & k = n, \\ 1 & k = n-1, \\ 0 & k > n. \end{cases}$$

The inverse of this matrix is $S = (s_{nk})$ given as

$$s_{nk} = \begin{cases} -1 & 0 \leqslant k \leqslant n, \\ 0 & k > n. \end{cases}$$

Let $(q_k)_{k=0}^{\infty}$ be a given positive sequences and $(Q_n)_{n=0}^{\infty}$ be the sequence defined as $Q_n = \sum_{i=0}^{n} q_i$. The $(\bar{N}^q)$ transform of the sequence $(x_k)_{k=0}^{\infty}$

is the sequence $(t_n)_{n=0}^{\infty}$ defined as

$$t_n = \frac{1}{Q_n}\sum_{i=0}^{n} q_i x_i \quad \text{for all } n = 0, 1, \ldots .$$

The matrix $\bar{N}^q$ for this transformation is given by

$$(\bar{N}^q)_{nk} = \begin{cases} \frac{q_k}{Q_n} & 0 \leqslant k \leqslant n \\ 0 & k > n. \end{cases}$$

The inverse of this matrix (see [4]) is

$$(\bar{N}^q)^{-1}_{nk} = \begin{cases} (-1)^{n-k}\frac{Q_k}{q_n} & n-1 \leqslant k \leqslant n \\ 0 & 0 \leqslant k \leqslant n-2, k > n. \end{cases}$$

We define the spaces $\bar{N}^q_{\Delta^-}$ summable to zero, summable and bounded respectively as

$$(\bar{N}^q_{\Delta^-})_0 = (c_0, \Delta^-)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q\Delta^- x = \left(\frac{1}{Q_n}\sum_{k=0}^{n} q_k \Delta^- x_k\right)_{n=0}^{\infty} \in c_0 \right\},$$

$$(\bar{N}^q_{\Delta^-}) = (c, \Delta^-)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q\Delta^- x = \left(\frac{1}{Q_n}\sum_{k=0}^{n} q_k \Delta^- x_k\right)_{n=0}^{\infty} \in c \right\},$$

$$(\bar{N}^q_{\Delta^-})_\infty = (\ell_\infty, \Delta^-)_{\bar{N}^q} = \left\{ x \in w : \bar{N}^q\Delta^- x = \left(\frac{1}{Q_n}\sum_{k=0}^{n} q_k \Delta^- x_k\right)_{n=0}^{\infty} \in \ell_\infty \right\}.$$

For any sequence $x = (x_k)_{k=0}^{\infty}$, let $\tau = \tau(x) = (\tau_n(x))_{n=0}^{\infty}$ denote the sequence with nth term given by

$$\tau_n(x) = (\bar{N}^q_{\Delta^-})_n(x) = \frac{1}{Q_n}\sum_{k=0}^{n} q_k \Delta^- x_k \qquad (n = 0, 1, 2, \ldots) \tag{2}$$

2.1 Basis for the new sequence spaces

First we determine Schauder bases for the spaces $\left(\bar{N}^q_{\Delta^-}\right)_0$ and $\left(\bar{N}^q_{\Delta^-}\right)$. For the convenience of the reader, we state the following known results:

Proposition 2.1.1. *[17] Every triangle T has a unique inverse $S = (s_{nk})_{n,k=0}^{\infty}$ which is also a triangle, and $x = T(S(x)) = S(T(x))$ for all $x \in w$.*

Proposition 2.1.2. *[Theorem 2.1.3, 6] Let T be a triangle and S be its inverse, if $\left(b^{(n)}\right)_{n=0}^{\infty}$ is a basis of the linear metric space (X, d), then $\left(S(b^{(n)})\right)_{n=0}^{\infty}$ is a basis of $Z = X_T$ with the metric d_T defined by $d_T(z, \bar{z}) = d(T(z), T(\bar{z}))$ for all $z, \bar{z} \in Z$.*
It is obvious that $(c_0, \Delta^-)_{\bar{N}^q} = (c_0)_{\bar{N}^q \cdot \Delta^-}$, So the basis for new spaces are given by $\left(\bar{N}^q \cdot \Delta^-\right)^{-1}\left(e^{(n)}\right) = \left(\Delta^-\right)^{-1} \cdot \left(\bar{N}^q\right)^{-1}\left(e^{(n)}\right)$.

Theorem 2.1.3. *Let $\tau_k = \left(\left(\bar{N}^q_{\Delta^-}\right) x\right)_k$ for all $k \in \mathbb{N}$. Define the sequence $s^{(k)} = \left\{s_n^{(k)}\right\}_{n \in \mathbb{N}}$ of the elements of $(c_0, \Delta^-)_{\bar{N}^q}$ as*

$$s_n^k = \begin{cases} \sum_{j=1}^{k} Q_j \left(\frac{1}{q_{j+1}} - \frac{1}{q_j}\right) & 0 \leqslant k < n \\ -\frac{Q_k}{q_k} & k = n \\ 0 & k > n \end{cases}, \quad s_n^{(-1)} = \sum_{k=0}^{n}\left[\sum_{j=1}^{k-1} Q_j \left(\frac{1}{q_{j+1}} - \frac{1}{q_j}\right) + \frac{Q_k}{q_k}\right],$$

for every fixed $k \in \mathbb{N}$. Then

i) *The sequence $\left\{s^{(k)}\right\}_{k \in \mathbb{N}}$ is a basis for the space $(c_0, \Delta^-)_{\bar{N}^q}$ and any $x \in (c_0, \Delta^-)_{\bar{N}^q}$ can be uniquely represented in the form*

$$x = \sum_k \tau_k s^{(k)}.$$

ii) *The set $\left\{s^{(-1)}, s^{(k)}\right\}$ is a basis for the spaces $(c, \Delta^-)_{\bar{N}^q}$ and any $x \in (c, \Delta^-)_{\bar{N}^q}$ has a unique representation in the form*

$$x = l s^{(-1)} + \sum_k (\tau_k - l) s^{(k)}$$

where for all $k \in \mathbb{N}$, $l = \lim_{k \to \infty}\left(\left(\bar{N}^q_{\Delta^-}\right) x\right)_k$.

Proof. Since $(X, \Delta^-)_{\bar{N}^q} = (X)_{\bar{N}^q \cdot \Delta^-}$ for $X = c_0, c, \ell_\infty$, and $e = (e^{(k)})_{k=0}^{\infty}$ is the standard basis for c.

Also $\bar{N}^q$ is a triangle, Δ^- is triangle so $\bar{N}^q \cdot \Delta^-$ is also a triangle and

$$\left(\bar{N}^q \cdot \Delta^-\right)^{-1} = \left(\Delta^-\right)^{-1} \cdot \left(\bar{N}^q\right)^{-1} = \begin{cases} Q_k\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) & 0 \leqslant k < n \\ -\frac{Q_n}{q_n} & k = n \\ 0 & k > n. \end{cases}$$

Hence $\left\{s^{(k)}\right\}_{k\in\mathbb{N}}$ is a basis for the space $(c_0, \Delta^-)_{\bar{N}^q}$ and the results i) and ii) are obvious to follow. □

Theorem 2.1.4. *The sequence spaces* $\left(\bar{N}^q_{\Delta^-}\right)_0$, $\left(\bar{N}^q_{\Delta^-}\right)$ *and* $\left(\bar{N}^q_{\Delta^-}\right)_\infty$ *are BK-spaces with norm* $\| \|_{\bar{N}^q_{\Delta^-}}$ *given by*

$$\|x\|_{\bar{N}^q_{\Delta^-}} = \sup_n \left| \frac{1}{Q_n} \sum_{k=0}^{n} q_k \Delta^- x_k \right|.$$

If $Q_n \to \infty$ *as* $(n \to \infty)$*, then* $(\bar{N}^q_{\Delta^-})_0$ *has AK, and every sequence* $x = (x_k)_{k=0}^\infty \in (\bar{N}^q_{\Delta^-})$ *has unique representation*

$$x = le + \sum_k (\tau_k - l)e^{(k)}, \tag{3}$$

where $l \in \mathbb{C}$ *is such that* $x - le \in (\bar{N}^q_{\Delta^-})_0$.

Proof. Since $(X, \Delta^-)_{\bar{N}_q} = X_{\bar{N}_q \cdot \Delta^-}$ for all $X = c_0, c, \ell_\infty$ and the spaces c_0, c, ℓ_∞ are BK spaces with respect to natural norm [8, 217-218] and the matrix $\bar{N}_q \cdot \Delta^-$ is a triangle so by Theorem 4.3.12, [17], gives $\left(\bar{N}^q_{\Delta^-}\right)_0$, $\left(\bar{N}^q_{\Delta^-}\right)$ and $\left(\bar{N}^q_{\Delta^-}\right)_\infty$ are BK spaces

The space $(\bar{N}^q_{\Delta^-})_0$ has AK and the unique representation of elements of $\left(\bar{N}^q_{\Delta^-}\right)$ are simply followed from Theorem 2 of [1] and [11].

2.2 β Dual of the new spaces

In order to find the β dual we need the following results of [16]

Lemma 2.2.1. *If* $A = (a_{nk})_{n,k=0}^\infty$ *then* $A \in (c_0, l_1)$ *if and only if*

$$\sup_{K \in F} \left| \sum_{k \in K} a_{nk} \right| < \infty,$$

where F stands for the class of all finite subsets of $\mathbb{N}$.

Lemma 2.2.2. *If $A = (a_{nk})_{n,k=0}^{\infty}$ then $A \in (c_0, c)$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} a_{nk} - \alpha_k = 0.$$

Lemma 2.2.3. *If $A = (a_{nk})_{n,k=0}^{\infty}$ then $A \in (c_0, \ell_\infty)$ if and only if*

$$\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty,$$

Theorem 2.2.4. *Let $(q_k)_{k=0}^{\infty}$ be positive sequences, $Q_n = \sum_{i=0}^{n} q_i$ and $a = (a_k) \in w$ we define a matrix $C = (c_{nk})_{n,k=0}^{\infty}$ as*

$$c_{nk} = \begin{cases} Q_k \left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) \sum_{j=k+1}^{n} a_j & 0 \leqslant k < n \\ -\frac{Q_k a_k}{q_k} & k = n \\ 0 & k > n, \end{cases}$$

and consider the sets

$$c_1 = \left\{ a \in w : \sup_n \sum_k |c_{nk}| < \infty \right\} \quad ; c_2 = \left\{ a \in w : \lim_{n\to\infty} c_{nk} \text{ exists for each } \ k \in \mathbb{N} \right\}$$

$$c_3 = \left\{ a \in w : \lim_{n\to\infty} \sum_k |c_{nk}| = \sum_k \left| \lim_{n\to\infty} c_{nk} \right| \right\} \quad ; c_4 = \left\{ a \in w : \lim_{n\to\infty} \sum_k c_{nk} \text{ exists} \right\}.$$

Then $\left[\left(\bar{N}^q_{\Delta^-}\right)_0\right]^\beta = c_1 \cap c_2$, $\left[\left(\bar{N}^q_{\Delta^-}\right)\right]^\beta = c_1 \cap c_2 \cap c_4$ and $\left[\left(\bar{N}^q_{\Delta^-}\right)_\infty\right]^\beta = c_2 \cap c_3$.

Proof. We prove the result for $\left[\left(\bar{N}^q_{\Delta^-}\right)_0\right]^\beta$ for the other two same procedure can be followed. Let $x \in \left(\bar{N}^q_{\Delta^-}\right)_0$ then there exists a y such that $y = \bar{N}^q_{\Delta^-} x$.

Hence

$$\begin{aligned}\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} a_k \left(\bar{N}^q_{\Delta^-}\right)^{-1} y_k\\ &= \sum_{k=0}^{n} a_k \left[\sum_{j=0}^{k-1} Q_j \left(\frac{1}{q_{j+1}} - \frac{1}{q_j}\right) y_j - \frac{Q_k}{q_k} y_k\right]\\ &= \sum_{k=0}^{n} \left[Q_{k-1}\left(\frac{1}{q_k} - \frac{1}{q_{k-1}}\right) \sum_{j=k+1}^{n} a_j - \frac{Q_k a_k}{q_k}\right] y_k\\ &= (Cy)_n.\end{aligned}$$

So $ax = (a_n x_n) \in cs$ whenever $x \in \left(\bar{N}^q_{\Delta^-}\right)_0$ if and only if $Cy \in cs$ whenever $y \in c_0$.

Using Lemma 2.2.2 we get $\left[\left(\bar{N}^q_{\Delta^-}\right)_0\right]^\beta = c_1 \cap c_2$ In the same way we can show the other two results as well. □

By Theorem 7.2.9, [17] we know that if X is a BK-space and $a \in w$ then

$$\|a\|^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\},$$

provided the term on the right side exists and is finite, which is the case whenever $a \in X^\beta$.

Theorem 2.2.5. *For* $\left[\left(\bar{N}^q_{\Delta^-}\right)_0\right]^\beta$, $\left[\left(\bar{N}^q_{\Delta^-}\right)\right]^\beta$ *and* $\left[\left(\bar{N}^q_{\Delta^-}\right)_\infty\right]^\beta$ *the norm* $\| \ \|^*$ *is defined as*

$$\|a\|^* = \sup_n \left[\sum_{k=0}^{n-1} Q_k \left|\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right) \sum_{j=k+1}^{n} a_j\right| + \left|\frac{Q_n a_n}{q_n}\right|\right].$$

Proof. If $x^{[n]}$ denotes the nth section of the sequence $x \in \left(\bar{N}^q_{\Delta^-}\right)_0$ then using (2) we have

$$\tau_k^{[n]} = \tau_k(x^{[n]}) = \frac{1}{Q_k} \sum_{j=0}^{k} q_j \Delta^- x_j^{[n]}.$$

Let $a \in \left[\left(\bar{N}^q_{\Delta^-}\right)_0\right]^\beta$, then for any non-negative integer n define the sequence $d^{[n]}$ as

$$d_k^{[n]} = \begin{cases} Q_k\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right)\sum_{j=k+1}^n a_j & 0 \leqslant k < n \\ -\frac{Q_k a_k}{q_k} & k = n \\ 0 & k > n. \end{cases}$$

Let $\|a\|_\Pi = \sup_n \|d^{[n]}\|_1 = \sup_n \left(\sum_{k=0}^\infty |d_k^{[n]}|\right)$ where $\Pi = \left[\left(\bar{N}^q_{\Delta^-}\right)\right]^\beta$. The inequality $\|a\|_\Pi \leqslant \|a\|^*$ is obvious.

Also

$$\begin{aligned}
\left|\sum_{k=0}^\infty a_k x_k^{[n]}\right| &= \left|\sum_{k=0}^n a_k \left(\sum_{j=0}^k \frac{1}{q_j}(Q_j\tau_j^{[n]} - Q_{j-1}\tau_{j-1}^{[n]})\right)\right| \\
&\leqslant \left|\sum_{k=0}^{n-1} Q_k\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right)\left(\sum_{j=k+1}^n a_j\right)\tau_k^{[n]}\right| + \left|\frac{a_n Q_n}{q_n}\right| |\tau_n^{[n]}| \\
&\leqslant \sup_k |\tau_k^{[n]}| \cdot \left(\sum_{k=0}^{n-1} Q_k \left|\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right)\sum_{j=k+1}^n a_j\right| + \left|\frac{a_n Q_n}{q_n}\right|\right) \\
&= \|x^{[n]}\|_{\bar{N}^q_{\Delta^-}} \|d^{[n]}\|_1 \\
&= \|a\|_\Pi \|x^{[n]}\|_{\bar{N}^q_{\Delta^-}}.
\end{aligned}$$

Hence $\|a\|^* \leqslant \|a\|_\Pi$

From the above inequalities we get the required conclusion. □

Some well known results that are required for proving the compactness are.

Proposition 2.2.6. *[Theorem 7, 9] Let X and Y be BK spaces, then $(X, Y) \subset \mathcal{B}(X, Y)$ that is every matrix A from X into Y defines an element L_A of $\mathcal{B}(X, Y)$ where*

$$L_A(x) = A(x), \qquad \forall\ x \in X.$$

Also $A \in (X, \ell_\infty)$ if and only if

$$\|A\|^* = \sup_n \|A_n\|^* = \|L_A\| < \infty.$$

If $\left(b^{(k)}\right)_{k=0}^\infty$ is a basis of X, Y and Y_1 are FK spaces with Y_1 a closed subspace of Y, then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $A\left(b^{(k)}\right) \in Y_1$ for all $k = 0, 1, 2, \ldots$.

Proposition 2.2.7. *[Proposition 3.4, 10] Let T be a triangle*

(i) *If X and Y are subsets of w, then $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(ii) *If X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$\|L_A\| = \|L_B\|.$$

Using Proposition 2.2.6 and Theorem 2.2.5 we conclude the following corollary:

Corollary 2.2.8. *Let $(q_k)_{k=0}^\infty$ be a positive sequence, $Q_n = \sum_{k=0}^n q_k$ and Δ^- be the difference operator as defined in (1), then*

i) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right)_\infty, \ell_\infty\right)$ if and only if

$$\sup_{m,n}\left[\sum_{k=0}^{m-1} Q_k \left|\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right)\sum_{j=k+1}^{m} a_{nj}\right| + \left|\frac{Q_m a_{nm}}{q_m}\right|\right] < \infty, \quad (4)$$

and

$$\frac{A_n Q}{q} \in c_0, \qquad \forall\ n = 0, 1, \ldots. \quad (5)$$

ii) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right), \ell_\infty\right)$ if and only if condition (4) holds and

$$\frac{A_n Q}{q} \in c, \qquad \forall\ n = 0, 1, 2, \ldots. \quad (6)$$

iii) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right)_0, \ell_\infty \right)$ *if and only if condition* (4) *holds.*

iv) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right)_0, c_0 \right)$ *if and only if condition* (4) *holds and*

$$\lim_{n\to\infty} a_{nk} = 0, \qquad \textit{for all } k = 0, 1, 2 \ldots. \tag{7}$$

v) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right)_0, c \right)$ *if and only if condition* (4) *holds and*

$$\lim_{n\to\infty} a_{nk} = \alpha_k, \qquad \textit{for all } k = 0, 1, 2 \ldots. \tag{8}$$

vi) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right), c_0 \right)$ *if and only if conditions* (4), (5) *and* (7) *holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 0, \qquad \textit{for all } k = 0, 1, 2 \ldots. \tag{9}$$

vii) $A \in \left(\left(\bar{N}^q_{\Delta^-}\right), c \right)$ *if and only if conditions* (4), (5) *and* (8) *holds and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha, \qquad \textit{for all } k = 0, 1, 2 \ldots. \tag{10}$$

3. Hausdorff Measure of Noncompactness

Let S and M be the subsets of a metric space (X, d) and $\epsilon > 0$. Then S is called an $\epsilon-$net of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x, s) < \epsilon$. Further, if the set S is finite, then the $\epsilon-$net S of M is called *finite* $\epsilon-$*net* of M. A subset of a metric space is said to be *totally bounded* if it has a finite $\epsilon-$net for every $\epsilon > 0$.

If $\mathcal{M}_X$ denotes the collection of all bounded subsets of metric space (X, d). If $Q \in \mathcal{M}_X$ then the *Hausdorff Measure of Noncompactness* of the set Q is defined by

$$\chi(Q) = \inf \{\epsilon > 0 : Q \text{ has a finite } \epsilon - \text{net in } X\}.$$

The function $\chi : \mathcal{M}_X \to [0, \infty)$ is called *Hausdorff Measure of Noncompactness.*

The basic properties of *Hausdorff Measure of Noncompactness* can be found in ([3, 4, 12]). Some of those properties are
If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then

$$\begin{aligned}
\chi(Q) &= 0 \Leftrightarrow Q \text{ is totally bounded set}, \\
\chi(Q) &= \chi(\bar{Q}), \\
Q_1 \subset Q_2 &\Rightarrow \chi(Q_1) \leqslant \chi(Q_2), \\
\chi(Q_1 \cup Q_2) &= \max\left\{\chi(Q_1), \chi(Q_2)\right\}, \\
\chi(Q_1 \cap Q_2) &= \min\left\{\chi(Q_1), \chi(Q_2)\right\}.
\end{aligned}$$

Further if X is a normed space the χ has the additional properties connected with the linear structure.

$$\begin{aligned}
\chi(Q_1 + Q_2) &\leqslant \chi(Q_1) + \chi(Q_2) \\
\chi(\eta Q) &= |\eta|\chi(Q), \qquad\qquad \eta \in \mathbb{C}.
\end{aligned}$$

The most effective way of characterizing operators between Banach Spaces is by applying Hausdorff Measure of Noncompactness. If X and Y are Banach spaces, and $L \in \mathcal{B}(X, Y)$, then the Hausdorff Measure of Noncompactness of L, denoted by $\|L\|_\chi$ is defined as

$$\|L\|_\chi = \chi\left(L(S_X)\right).$$

Where $S_X = \{x \in X : \|x\| = 1\}$ is the unit ball in X.

From [Corollary 1.15, 16] we know that

$$L \text{ is compact if and only if } \|L\|_\chi = 0.$$

Proposition 3.1. *[Theorem 6.1.1 $X = c_0$, 3] Let $Q \in M_{c_0}$ and $P_r : c_0 \to c_0$ $r \in \mathbb{N}$. Then, we have*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{x\in Q}\|(I - P_r)(x)\|\right),$$

where I is the identity operator on c_0.

Proposition 3.2. *[Theorem 6.1.1, 3] Let X be a Banach space with a schauder basis $\{e_1, e_2, \ldots\}$, and $Q \in M_X$ and $P_n : X \to X$ $(n \in \mathbb{N}$ be the projector onto the linear span of $\{e_1, e_2, \ldots, e_n\}$. Then, we have*

$$\frac{1}{a} \lim_{n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \leqslant \chi(Q)$$
$$\leqslant \inf_n \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right) \leqslant \lim_{n\to\infty} \sup \left(\sup_{x\in Q} \|(I - P_n)(x)\| \right),$$

where $a = \lim\limits_{n\to\infty} \sup \|I - P_n\|$, and I is the identity operator on c.
If $X = c$ then $a = 2$. (see [3], p.22).

4. Compact Operators on the Spaces $\left(\bar{N}^q_{\Delta^-}\right)_0$, $\left(\bar{N}^q_{\Delta^-}\right)$ and $\left(\bar{N}^q_{\Delta^-}\right)_\infty$

Theorem 4.1. *Consider the matrix A as in Corollary 2.2.8, and for any integers n,s, $n > s$ set*

$$\|A\|^{(s)} = \sup_{n>s} \sup_m \left(\sum_{j=0}^{m-1} Q_j \left| \left(\frac{1}{q_{j+1}} - \frac{1}{q_j} \right) \sum_{i=j+1}^{m} a_{ni} \right| + \left| \frac{Q_m a_{nm}}{q_m} \right| \right). \tag{11}$$

If X be either $\left(\bar{N}^q_{\Delta^-}\right)_0$ or $\left(\bar{N}^q_{\Delta^-}\right)$ and $A \in (X, c_0)$. Then

$$\|L_A\|_\chi = \lim_{s\to\infty} \|A\|^{(s)}. \tag{12}$$

If X be either $\left(\bar{N}^q_{\Delta^-}\right)_0$ or $\left(\bar{N}^q_{\Delta^-}\right)$ and $A \in (X, c)$. Then

$$\frac{1}{2} \cdot \lim_{s\to\infty} \|A\|^{(s)} \leqslant \|L_A\|_\chi \leqslant \lim_{r\to\infty} \|A\|^{(s)}. \tag{13}$$

and if X be either $\left(\bar{N}^q_{\Delta^-}\right)_0$, $\left(\bar{N}^q_{\Delta^-}\right)$ or $\left(\bar{N}^q_{\Delta^-}\right)_\infty$ and $A \in (X, \ell_\infty)$. Then

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{s\to\infty} \|A\|^{(s)}. \tag{14}$$

Proof. Let $F = \{x \in X : \|x\| \leqslant 1\}$ if $A \in (X, c_0)$ and X is one of the spaces $\left(\bar{N}^q_{\Delta^-}\right)_0$ or $\left(\bar{N}^q_{\Delta^-}\right)$, then by Proposition 3.1.

$$\|L_A\|_\chi = \chi(AF) = \lim_{s\to\infty}\left[\sup_{x\in F}\|(I-P_s)Ax\|\right]. \tag{15}$$

Again using Proposition 2.2.6 and Corollary 2.2.8 we have

$$\|A\|^s = \sup_{x\in F}\|(I-P_s)Ax\|. \tag{16}$$

From (15) and (16) we get

$$\|L_A\|_\chi = \lim_{s\to\infty}\|A\|^{(s)}.$$

Since every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation
$x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)}$.
Where $l \in \mathbb{C}$ is such that $x - le \in c_0$.
We define $P_s : c \to c$ by $P_s(x) = le + \sum_{k=0}^{s}(x_k - l)e^{(k)}$, $s = 0, 1, 2, \ldots$.
Then $\|I - P_s\| = 2$ and using (16) and Proposition 3.2 we get

$$\frac{1}{2}\cdot\lim_{s\to\infty}\|A\|^{(s)} \leqslant \|L_A\|_\chi \leqslant \lim_{s\to\infty}\|A\|^{(s)}.$$

Finally we define $P_s : \ell_\infty \to \ell_\infty$ by $P_s(x) = (x_0, x_1, \ldots, x_s, 0, 0\ldots)$, $x = (x_k) \in \ell_\infty$.
Clearly $AF \subset P_s(AF) + (I - P_s)(AF)$
So using the properties of χ we get

$$\begin{aligned}\chi(AF) &\leqslant \chi[P_s(AF)] + \chi[(I-P_s)(AF)]\\ &= \chi[(I-P_s)(AF)]\\ &\leqslant \sup_{x\in F}\|(I-P_s)A(x)\|.\end{aligned}$$

Hence by Proposition 2.2.6 and and Corollary 2.2.8 we get

$$0 \leqslant \|L_A\|_\chi \leqslant \lim_{s\to\infty}\|A\|^{(s)}.$$

Which completes the proof. $\square$

A direct corollary of the above theorem is

Corollary 4.2. *Consider the matrix A as in Corollary 2.2.8, and $X = \left(\bar{N}^q_{\Delta^-}\right)_0$ or $X = \left(\bar{N}^q_{\Delta^-}\right)$ then if $A \in (X, c_0)$ or $A \in (X, c)$ we have*

$$L_A \text{ is compact if and only if } \lim_{s\to\infty} \|A\|^{(s)} = 0.$$

Further, for $X = \left(\bar{N}^q_{\Delta^-}\right)_0$, $X = \left(\bar{N}^q_{\Delta^-}\right)$ or $X = \left(\bar{N}^q_{\Delta^-}\right)_\infty$, if $A \in (X, \ell_\infty)$ then we have

$$L_A \text{ is compact if } \lim_{s\to\infty} \|A\|^{(s)} = 0. \tag{17}$$

In (17) it is possible for L_A to be compact although $\lim_{s\to\infty} \|A\|^{(s)} \neq 0$, that is the condition is only sufficient condition for L_A to be compact. For example, let the matrix A be defined as $A_n = e^{(1)} \quad n = 0, 1, 2, \dots$ and $q_n = 3^n$, $n = 0, 1, 2, \dots$.
Then by (4) we have

$$\sup_{m,n}\left[\sum_{k=0}^{m-1} Q_k \left|\left(\frac{1}{q_{k+1}} - \frac{1}{q_k}\right)\sum_{j=k+1}^{m} a_{nj}\right| + \left|\frac{Q_m a_{nm}}{q_m}\right|\right] = \sup_n \left(\frac{2}{3} + \frac{1}{2}(1 - 3^{-n})\right) < 2.$$

Now by Corollary 2.2.8 we know $A \in \left(\left(\bar{N}^q_{\Delta^-}\right)_\infty, \ell_\infty\right)$.
But

$$\|A\|^{(s)} = \sup_{n>s}\left[\frac{2}{3} + \frac{1}{2}(1 - 3^{-n})\right] = \frac{7}{6} - \frac{1}{2\cdot 3^{s+1}} \qquad \forall\ s.$$

Which gives $\lim_{s\to\infty} \|A\|^{(s)} = \frac{7}{6} \neq 0$.

Since $A(x) = x_1$ for all $x \in \left(\bar{N}^q_{\Delta^-}\right)_\infty$, so L_A is a compact operator.

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