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The Modified Three Step Iteration Process For $G-$Nonexpansive Mappings in Banach Spaces Involving a Graph

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Abstract. The purpose of this writing is to present strong convergence theorems of the modified three step iteration process for $G-$nonexpansive mappings in Banach spaces with a graph. The results presented in this study extend and improve a number of results in the literature.

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1. Introduction and Preliminaries

Jachymski [1] introduced a new concept of G-contraction, and showed that it was a real generalization for Banach contraction principle in a metric space involving a directed graph. Thereafter, many papers have been published on graph. For more detail see [2]-[9] and references therein.

Let (X,d) be a metric space, Δ be a diagonal of X^2, and G be a directed graph with no parallel edges such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. That is, G is determined by $(V(G), E(G))$. Furthermore,

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denote by G^{-1} the graph obtained from G by reversing the direction of the edges in G. Hence, $E\left(G^{-1}\right)=\left\{(x,y)\in X^2:(y,x)\in E\left(G\right)\right\}.$

Aleomrainejat et al. [11] gave some iterative scheme results for $G-$contractive and $G-$nonexpansive mappings on graphs by connecting graph theory&fixed point theory. Alfuraid and Khamsi [12] described the notion of $G-$monotone nonexpansive multivalued mappings on a metric space endowed with a graph. After that, Tiammee et al. [13] presented Browder's convergence theorem and Halpern iteration process for $G-$nonexpansive mappings in Hilbert space involving a directed graph. Tripak [14] studied two step iterative process for $G-$nonexpansive mappings in Banach space endowed with a graph. Recently, Suparatulatorn et al. [15] established convergence theorems for a modified $S-$iteration process for $G-$nonexpansive mappings in Banach space with a directed graph. In the sequel Hunde et al. [16] gave weak and strong convergence of finite step iteration sequences to common fixed point for $G-$nonexpansive mappings in Banach space with a digraph.

Inspired and motivated by this facts, we define and study the convergence theorems of three steps iterative sequences for G-nonexpansive mappings in Banach spaces involving a graph. The results of this paper can be viewed as an improvement and extension of the corresponding results of [10], [15] and others. The scheme (1) is defined as follows:

$$\begin{aligned} x_{n+1} &= \left(1-\eta_{n1}\right) f_3 y_n + \eta_{n1} f_2 z_n, \\ y_n &= \left(1-\eta_{n2}\right) f_1 x_n + \eta_{n2} f_2 z_n, \\ z_n &= \left(1-\eta_{n3}\right) x_n + \eta_{n3} f_1 x_n, \quad n\in\mathbb{N}, \end{aligned} \tag{1}$$

where $\{\eta_{ni}\}$ are sequences in $(0,1)$ for all $i\in\{1,2,3\}$.

For the beginning, some necessary definitions and lemma, which will be used in the sequel, are established here.

Definition 1.1. *[15] A self map $f:K\to K$ is called to be $G-$nonexpansive if it satisfies the conditions:*

1. *f preserves edges of G, videlicet, $(x,y)\in E\left(G\right)\Rightarrow(fx,fy)\in E\left(G\right)$,*
2. *f non-increases weights of edges of G in the following way:*

$$(x, y) \in E(G) \Rightarrow \|fx - fy\| \leqslant \|x - y\|.$$

Definition 1.2. *[15] Let $x_0 \in V(G)$ and Ξ a subset of $V(G)$. We say that*

1. *Ξ is dominated by x_0 if $(x_0, x) \in E(G)$ for all $x \in \Xi$.*
2. *Ξ dominates x_0 if for each $x \in \Xi$, $(x_0, x) \in E(G)$.*

Definition 1.3. *[15] (Property SG) Let K be a nonempty subset of a normed space X and let $G = (V(G), E(G))$ be a directed graph such that $V(G) = K$. Then, K is said to have Property SG if for each sequence $\{x_n\}$ in K converging strongly to $x \in K$ and $(x_n, x_{n+1}) \in E(G)$, there is a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $(x_{n_l}, x) \in E(G)$ for all $l \in \mathbb{N}$.*

Lemma 1.4. *[17] Let $q > 1$ and $D > 0$ be two fixed real numbers. Then a Banach space X is a uniformly convex if and only if there is a continuous, strictly increasing and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$ such that*

$$\|\gamma x + (1 - \gamma) y\|^q \leqslant \gamma \|x\|^q + (1 - \gamma) \|y\|^q - \omega_q(\gamma) g(\|x - y\|) \qquad (2)$$

for all $x, y \in B_D$ and $\gamma \in [0, 1]$, where B_D is the closed ball with center zero and radius D, $\omega_q(\gamma) = \gamma (1 - \gamma)^q + \gamma^p (1 - \gamma)$.

The main purpose of this paper is to study the convergence of the modified three steps iterative sequence $\{x_n\}$ identified by (1), under condition (C), semicompact conditions, respectively, for G-nonexpansive mappings in Banach spaces endowed with a directed graph. The results presented in this study extend and improve a number of results in the literature.

2. Main Results

From now onward, K express a nonempty subset of a Banach space X with $(V(G), E(G)) = G$ such that $V(G) = K$, convex of $E(G)$ and transitive of G.

Proposition 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{f_1, f_2, f_3\}$ be three $G-$nonexpansive mappings on K. Let $\theta_0 \in F = F(f_1) \cap F(f_2) \cap F(f_3)$ be such that (x_0, θ_0) and (θ_0, x_0) are in $E(G)$ for arbitrary $x_0 \in K$. Then, for a sequence $\{x_n\}$ generated by x_0 endowed with iterative scheme identified by (1), we possess (x_n, θ_0), (θ_0, x_n), (x_n, z_n), (z_n, x_n), (θ_0, z_n), (z_n, θ_0), (x_n, y_n), (y_n, x_n), (θ_0, y_n), (y_n, θ_0) and (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$.*

Proof. Let $(x_0, \theta_0) \in E(G)$. By edge-preserving of f_1, we get $(f_1 x_0, \theta_0) \in E(G)$. Using the convexity of $E(G)$, we have

$$(1-\eta_{03})(x_0, \theta_0) + \eta_{03}(f_1 x_0, \theta_0) = ((1-\eta_{03}) x_0 + \eta_{03} f_1 x_0, \theta_0) = (z_0, \theta_0) \in E(G).$$

Owing to edge-preserving of f_1 and f_2, we have $(f_1 x_0, \theta_0)$, $(f_2 z_0, \theta_0) \in E(G)$ and from the convexity of $E(G)$, we get

$$(1-\eta_{02})(f_1 x_0, \theta_0) + \eta_{02}(f_2 z_0, \theta_0) = ((1-\eta_{02}) f_1 x_0 + \eta_{03} f_2 z_0, \theta_0) = (y_0, \theta_0) \in E(G).$$

Due to the fact that f_2 and f_3 are edge-preserving, we get $(f_2 z_0, \theta_0)$, $(f_3 y_0, \theta_0) \in E(G)$ and again by the convexity of $E(G)$, we have

$$(1-\eta_{01})(f_3 y_0, \theta_0) + \eta_{01}(f_2 z_0, \theta_0) = ((1-\eta_{02}) f_3 y_0 + \eta_{02} f_2 z_0, \theta_0) = (x_1, \theta_0) \in E(G).$$

Continuing this process, we hold (z_1, θ_0), (y_1, θ_0), $(x_2, \theta_0) \in E(G)$. Now, we assume that $(x_l, \theta_0) \in E(G)$ on the score of edge-preserving of f_1, we get $(f_1 x_l, \theta_0) \in E(G)$, and therefore $(z_l, \theta_0) \in E(G)$ from the convexity of $E(G)$. As f_1 and f_2 are edge-preserving, we have $(f_1 x_l, \theta_0)$, $(f_2 z_l, \theta_0) \in E(G)$, so $(y_l, \theta_0) \in E(G)$ from the convexity of $E(G)$. On account of the fact that f_2 and f_3 are edge-preserving, we get $(f_2 z_l, \theta_0)$, $(f_3 y_l, \theta_0) \in E(G)$ and again by the convexity of $E(G)$, we obtain $(x_{l+1}, \theta_0) \in E(G)$. By repeating this process for $(x_{l+1}, \theta_0) \in E(G)$, we get (z_{l+1}, θ_0), $(y_{l+1}, \theta_0) \in E(G)$. Thereof, by induction, we deduce that (x_n, θ_0), (z_n, θ_0), (y_n, θ_0) are in $E(G)$ for all $n \in \mathbb{N}$. By use of a similar assertion, (θ_0, x_n), (θ_0, z_n), $(\theta_0, y_n) \in E(G)$ for all $n \in \mathbb{N}$ under the hypothesis that $(\theta_0, x_0) \in E(G)$. Using the transitivity of G, we obtain that (x_n, z_n), (z_n, x_n), (x_n, y_n), (y_n, x_n) and (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. This completes the proof. □

Lemma 2.2. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{f_1, f_2, f_3\}$ be three $G-$nonexpansive* *mappings on* K. *If* $0 < \liminf_{n\to\infty} \eta_{nj} \leqslant \limsup_{n\to\infty} \eta_{nj} < 1$ *for* $j = 1, 2, 3$ *and* (x_0, θ_0) *and* (θ_0, x_0) *are in* $E(G)$ *for arbitrary* $x_0 \in K$ *and* $\theta_0 \in F = F(f_1) \cap F(f_2) \cap F(f_3)$, *then for the sequence* $\{x_n\}$ *generated by (1), we possess*

(i) $\|x_{n+1} - \theta_0\| \leqslant \|x_n - \theta_0\|$, *for each* $n \in \mathbb{N}$, *and therefore* $\lim_{n\to\infty} \|x_n - \theta_0\|$ *exists;*

(ii) $\|x_n - f_i x_n\| \to 0$ *when* $n \to \infty$ *for* $i = 1, 2, 3$.

Proof. *(i)* Let $x_0 \in K$ and $\theta_0 \in F$. From Proposition 2.1, (x_n, θ_0), (θ_0, x_n), (x_n, z_n), (z_n, x_n), (x_n, y_n), (y_n, x_n) and (x_n, x_{n+1}) are in $E(G)$ for all $n \in \mathbb{N}$. Then, by (1) and $G-$nonexpansiveness of $\{f_1, f_2, f_3\}$, we get

$$\begin{aligned} \|z_n - \theta_0\| &= \|(1 - \eta_{n3}) x_n + \eta_{n3} f_1 x_n - \theta_0\| \\ &\leqslant (1 - \eta_{n3}) \|x_n - \theta_0\| + \eta_{n3} \|f_1 x_n - \theta_0\| \\ &\leqslant (1 - \eta_{n3}) \|x_n - \theta_0\| + \eta_{n3} \|x_n - \theta_0\| \\ &= \|x_n - \theta_0\|, \end{aligned} \tag{3}$$

and, by (1) and (3)

$$\begin{aligned} \|y_n - \theta_0\| &\leqslant (1 - \eta_{n2}) \|f_1 x_n - \theta_0\| + \eta_{n2} \|f_2 z_n - \theta_0\| \\ &\leqslant (1 - \eta_{n2}) \|x_n - \theta_0\| + \eta_{n2} \|z_n - \theta_0\| \\ &\leqslant \|x_n - \theta_0\|, \end{aligned} \tag{4}$$

herewith, from (1), (3) and (4),

$$\begin{aligned} \|x_{n+1} - \theta_0\| &\leqslant (1 - \eta_{n1}) \|f_3 y_n - \theta_0\| + \eta_{n1} \|f_2 z_n - \theta_0\| \\ &\leqslant (1 - \eta_{n1}) \|y_n - \theta_0\| + \eta_{n1} \|z_n - \theta_0\| \\ &\leqslant \|x_n - \theta_0\|. \end{aligned} \tag{5}$$

Thereby, $\lim_{n\to\infty} \|x_n - \theta_0\|$ exists.
(ii) By hypothesis *(i)*, $\{x_n - \theta_0\}$ is bounded for $\theta_0 \in F$. Thereby, it follows from (3) and (4) that $\{z_n - \theta_0\}$ and $\{y_n - \theta_0\}$ are also bounded sequences. Owing to $G-$nonexpansiveness of $\{f_1, f_2, f_3\}$, we can demostrate the sequences $\{f_1 x_n - \theta_0\}$, $\{f_2 z_n - \theta_0\}$ and $\{f_3 y_n - \theta_0\}$ are

all bounded. By (1) and Lemma 1.4, we have,

$$\begin{aligned}\|z_n-\theta_0\|^2 &= \|(1-\eta_{n3})\,x_n+\eta_{n3}f_1x_n-\theta_0\|^2\\ &\leqslant \|(1-\eta_{n3})\,(x_n-\theta_0)+\eta_{n3}\,(f_1x_n-\theta_0)\|^2\\ &\leqslant (1-\eta_{n3})\,\|x_n-\theta_0\|^2+\eta_{n3}\,\|f_1x_n-\theta_0\|^2\\ &\quad -\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\\ &\leqslant (1-\eta_{n3})\,\|x_n-\theta_0\|^2+\eta_{n3}\,\|x_n-\theta_0\|^2\\ &\quad -\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\\ &\leqslant \|x_n-\theta_0\|^2-\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\,, \end{aligned} \tag{6}$$

and, by (6)

$$\begin{aligned}&\|y_n-\theta_0\|^2\\ =\ &\|(1-\eta_{n2})\,f_1x_n+\eta_{n2}f_2z_n-\theta_0\|^2\\ \leqslant\ &(1-\eta_{n2})\,\|f_1x_n-\theta_0\|^2+\eta_{n2}\,\|f_2z_n-\theta_0\|^2\\ &-\eta_{n2}\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\\ \leqslant\ &(1-\eta_{n2})\,\|x_n-\theta_0\|^2+\eta_{n2}\,\|z_n-\theta_0\|^2\\ &-\eta_{n2}\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\\ \leqslant\ &(1-\eta_{n2})\,\|x_n-\theta_0\|^2\\ &+\eta_{n2}\left\{\|x_n-\theta_0\|^2-\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\right\}\\ &-\eta_{n2}\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\\ \leqslant\ &\|x_n-\theta_0\|^2-\eta_{n2}\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\\ &-\eta_{n2}\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\,, \end{aligned} \tag{7}$$

similarly, from (6) and (7), we have

$$\begin{aligned}&\|x_{n+1}-\theta_0\|^2\\ =\ &\|(1-\eta_{n1})\,f_3y_n+\eta_{n1}f_2z_n-\theta_0\|^2\\ \leqslant\ &(1-\eta_{n1})\,\|f_3y_n-\theta_0\|^2+\eta_{n1}\,\|f_2z_n-\theta_0\|^2\\ &-\eta_{n1}\,(1-\eta_{n1})\,g_3\,(\|f_3y_n-f_2z_n\|)\\ \leqslant\ &(1-\eta_{n1})\,\|y_n-\theta_0\|^2+\eta_{n1}\,\|z_n-\theta_0\|^2\\ &-\eta_{n1}\,(1-\eta_{n1})\,g_3\,(\|f_3y_n-f_2z_n\|)\\ \leqslant\ &(1-\eta_{n1})\left\{\begin{array}{c}\|x_n-\theta_0\|^2-\eta_{n2}\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\\ -\eta_{n2}\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\end{array}\right\}\\ &+\eta_{n1}\left\{\|x_n-\theta_0\|^2-\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\right\}\\ &-\eta_{n1}\,(1-\eta_{n1})\,g_3\,(\|f_3y_n-f_2z_n\|)\\ \leqslant\ &\|x_n-\theta_0\|^2-\eta_{n1}\eta_{n3}\,(1-\eta_{n3})\,g_1\,(\|x_n-f_1x_n\|)\\ &-\eta_{n2}\,(1-\eta_{n1})\,(1-\eta_{n2})\,g_2\,(\|f_1x_n-f_2z_n\|)\\ &-\eta_{n1}\,(1-\eta_{n1})\,g_3\,(\|f_3y_n-f_2z_n\|)\,. \end{aligned} \tag{8}$$

It follows from (8) that

$$\begin{aligned} & \eta_{n1}\left(1-\eta_{n1}\right) g_3\left(\left\|f_3 y_n-f_2 z_n\right\|\right) \\ \leqslant\ & \left\|x_n-\theta_0\right\|^2-\left\|x_{n+1}-\theta_0\right\|^2, \end{aligned} \tag{9}$$

$$\begin{aligned} & \eta_{n2}\left(1-\eta_{n1}\right)\left(1-\eta_{n2}\right) g_2\left(\left\|f_1 x_n-f_2 z_n\right\|\right) \\ \leqslant\ & \left\|x_n-\theta_0\right\|^2-\left\|x_{n+1}-\theta_0\right\|^2, \end{aligned} \tag{10}$$

$$\begin{aligned} & \eta_{n1}\eta_{n3}\left(1-\eta_{n3}\right) g_1\left(\left\|x_n-f_1 x_n\right\|\right) \\ \leqslant\ & \left\|x_n-\theta_0\right\|^2-\left\|x_{n+1}-\theta_0\right\|^2. \end{aligned} \tag{11}$$

If $0<\liminf_{n\to\infty}\eta_{nj}\leqslant\limsup_{n\to\infty}\eta_{nj}<1$ for $j=1,2,3$, there exist a positive integer n_0 and $\kappa\in(0,1)$ such that $0<\kappa<\eta_{nj}$ for $j=\overline{1,3}$ for all $n\geqslant n_0$. This implies by (9) that

$$\kappa\left(1-\kappa\right) g_3\left(\left\|f_3 y_n-f_2 z_n\right\|\right)\leqslant\left\|x_n-\theta_0\right\|^2-\left\|x_{n+1}-\theta_0\right\|^2 \quad \text{for all } n\geqslant n_0. \tag{12}$$

By (12) for $m\geqslant n_0$,

$$\begin{aligned} \sum_{n=n_0}^{m} g_3\left(\left\|f_3 y_n-f_2 z_n\right\|\right) & \leqslant \frac{1}{\kappa\left(1-\kappa\right)}\left(\sum_{n=n_0}^{m}\left\{\left\|x_n-\theta_0\right\|^2-\left\|x_{n+1}-\theta_0\right\|^2\right\}\right) \\ & \leqslant \frac{1}{\kappa\left(1-\kappa\right)}\left\|x_{n_0}-\theta_0\right\|^2. \end{aligned}$$

Then $\sum_{n=n_0}^{\infty} g_3\left(\left\|f_3 y_n-f_2 z_n\right\|\right)<\infty$, and so $\lim_{n\to\infty} g_3\left(\left\|f_3 y_n-f_2 z_n\right\|\right) = 0$. By virtue of the fact that g_3 is strictly increasing and continuous via $g_3(0)=0$, we get

$$\left\|f_3 y_n-f_2 z_n\right\|\to 0 \text{ when } n\to\infty. \tag{13}$$

From an analogue manner, allied with (10) and (11), it could be demonstrated that

$$\left\|f_1 x_n-f_2 z_n\right\| \to 0 \text{ when } n\to\infty, \tag{14}$$

$$\left\|x_n-f_1 x_n\right\| \to 0 \text{ when } n\to\infty. \tag{15}$$

It follows from (1) that

$$\begin{aligned} \left\|z_n-x_n\right\| & = \left\|\left(1-\eta_{n3}\right)x_n+\eta_{n3}f_1 x_n-x_n\right\| \\ & \leqslant \left\|f_1 x_n-x_n\right\| \\ & \to 0 \text{ when } n\to\infty. \text{ (by (15))} \end{aligned} \tag{16}$$

From (14) and (15), we have

$$\begin{aligned} \|f_2z_n - x_n\| &\leqslant \|f_1x_n - f_2z_n\| + \|x_n - f_1x_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{17}$$

It follows from (1) that

$$\begin{aligned} \|y_n - x_n\| &= \|(1-\eta_{n2}) f_1x_n + \eta_{n2}f_2z_n - x_n\| \\ &\leqslant (1-\eta_{n2}) \|f_1x_n - x_n\| + \eta_{n2} \|f_2z_n - x_n\| \\ &\to 0 \text{ when } n \to \infty. \text{(by (15) and (17))} \end{aligned} \tag{18}$$

Using (16) and (17), we get

$$\begin{aligned} \|f_2z_n - z_n\| &\leqslant \|f_2z_n - x_n\| + \|z_n - x_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{19}$$

By (14) and (19), we have

$$\begin{aligned} \|f_1x_n - z_n\| &\leqslant \|f_1x_n - f_2z_n\| + \|f_2z_n - z_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{20}$$

It follows from (1) that

$$\begin{aligned} \|y_n - z_n\| &= \|(1-\eta_{n2}) f_1x_n + \eta_{n2}f_2z_n - z_n\| \\ &\leqslant (1-\eta_{n2}) \|f_1x_n - z_n\| + \eta_{n2} \|f_2z_n - z_n\| \\ &\to 0 \text{ when } n \to \infty. \text{(by (19) and (20))} \end{aligned} \tag{21}$$

Because of that f_2 is $G-$nonexpansive mappings, it follows from (16), (19) and (21) that

$$\begin{aligned} \|x_n - f_2y_n\| &\leqslant \|x_n - z_n\| + \|z_n - f_2z_n\| + \|f_2z_n - f_2y_n\| \\ &\leqslant \|x_n - z_n\| + \|z_n - f_2z_n\| + \|z_n - y_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{22}$$

Again, by $G-$nonexpansive mappings of f_2, it follows from (18) and (22) that

$$\begin{aligned} \|x_n - f_2x_n\| &\leqslant \|x_n - f_2y_n\| + \|f_2y_n - f_2x_n\| \\ &\leqslant \|x_n - f_2y_n\| + \|y_n - x_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{23}$$

As f_2 is $G-$nonexpansive mappings, it follows from (13), (16) and (23) that

$$\begin{aligned}\|x_n - f_3 y_n\| &\leqslant \|x_n - f_2 x_n\| + \|f_2 x_n - f_2 z_n\| + \|f_2 z_n - f_3 y_n\| \\ &\leqslant \|x_n - f_2 x_n\| + \|x_n - z_n\| + \|f_2 z_n - f_3 y_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{24}$$

On account of the fact that f_3 is $G-$nonexpansive mappings, by (18) and (24), we have

$$\begin{aligned}\|x_n - f_3 x_n\| &\leqslant \|x_n - f_3 y_n\| + \|f_3 y_n - f_3 x_n\| \\ &\leqslant \|x_n - f_3 y_n\| + \|y_n - x_n\| \\ &\to 0 \text{ when } n \to \infty. \end{aligned} \tag{25}$$

This completes the proof. □

Theorem 2.3. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{f_1, f_2, f_3\}$ be three $G-$nonexpansive mappings on K. Suppose that $0 < \liminf_{n\to\infty} \eta_{nj} \leqslant \limsup_{n\to\infty} \eta_{nj} < 1$ for $j = 1, 2, 3$ and $\{x_n\}$ is a sequence generated by (1). If there is a nondecreasing function $g : R^+ \to R^+$ with $g(0) = 0$ and $g(a) > 0$ for all $a > 0$ such that for all $x \in K$, $\max_{1\leqslant\mu\leqslant3} \{\|x - f_\mu x\|\} \geqslant g(d(x, F))$ (condition (C)), $F = F(f_1) \cap F(f_2) \cap F(f_3)$ is dominated by x_0 and $F = F(f_1) \cap F(f_2) \cap F(f_3)$ dominates x_0, then $\{x_n\}$ converges strongly to a common fixed point of $\{f_1, f_2, f_3\}$.*

Proof. By (3), (23) and (25), $\|x - f_\mu x\| \to 0$ when $n \to \infty$ for $1 \leqslant \mu \leqslant 3$. Since $\max_{1\leqslant\mu\leqslant3} \{\|x_n - f_\mu x_n\|\} \geqslant g(d(x_n, F))$, we have $g(d(x_n, F)) \to 0$ as $n \to \infty$ which implies $\lim_{n\to\infty} d(x_n, F) = 0$ by definition of the function g. We shall show that $\{x_n\}$ is a Cauchy sequence. By virtue of $\lim_{n\to\infty} d(x_n, F) = 0$, for given $\varepsilon > 0$, there exists n_0 in N such that $\frac{\varepsilon}{2} > d(x_n, F)$ for all $n \geqslant n_0$. Hence, we get $\frac{\varepsilon}{2} > d(x_{n_0}, F)$. This means that there exists $\theta^* \in F$ such that $\frac{\varepsilon}{2} > \|x_{n_0} - \theta^*\|$. Next, for $m, n \geqslant n_0$,

$$\|x_{n+m} - x_n\| \leqslant \|x_{n+m} - \theta^*\| + \|\theta^* - x_n\| \leqslant 2\|x_{n_0} - \theta^*\|,$$

taking the infimum in the above inequalities for all $\theta^* \in F$, we obtain

$$\|x_{n+m} - x_n\| \leqslant 2d(x_{n_0}, F) < \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Hereby, owing to the completeness of X, there exists a $\theta \in K$ such that $\lim_{n\to\infty} x_n = \theta$, and so $\lim_{n\to\infty} d(x_n, F) = 0$ yields that $d(\theta, F) = 0$, viz $\theta \in F$. This completes the proof. □

A mapping $f : K \to K$ is called *semicompact* if for a sequence $\{x_n\}$ in K with $\|x_n - f x_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \vartheta \in K$.

Theorem 2.4. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $\{f_1, f_2, f_3\}$ be three $G-$nonexpansive mappings on K. Suppose that $0 < \liminf_{n\to\infty} \eta_{nj} \leqslant \limsup_{n\to\infty} \eta_{nj} < 1$ for $j = 1, 2, 3$, K has property SG and $\{x_n\}$ is a sequence generated by (1). Assume that one of f_1, f_2, f_3 is semicompact (without loss of generality, we assume f_1 is semicompact) , $F = F(f_1) \cap F(f_2) \cap F(f_3)$ is dominated by x_0 and $F = F(f_1) \cap F(f_2) \cap F(f_3)$ dominates x_0, then $\{x_n\}$ converges strongly to a common fixed point of $\{f_1, f_2, f_3\}$.*

Proof. In connection with semi-compactness of f_1, by the fact that $\|x_n - f_1 x_n\| \to 0$ when $n \to \infty$ and $\{x_n\}$ is bounded, there exists a $\vartheta \in K$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = \vartheta$. Now by the hypothesis of the theorem, we attain $(x_{n_k}, \vartheta) \in E(G)$. In the present case, we find

$$\|\vartheta - f_\mu \vartheta\| = \lim_{k\to\infty} \|x_{n_k} - f_\mu x_{n_k}\| = 0 \text{ for } 1 \leqslant \mu \leqslant 3.$$

This shows that $\vartheta \in F$. Due to the fact that $\|x_n - \vartheta\| \to 0$ as $n \to \infty$ exists, we also have

$$\lim_{n\to\infty} \|x_n - \vartheta\| = \lim_{k\to\infty} \|x_{n_k} - \vartheta\| = 0,$$

which means that $\{x_n\}$ converges to $\vartheta \in F$. Herewith, $\{x_n\}$ converges strongly to a common fixed point of $\{f_1, f_2, f_3\}$. This completes the proof. □

Remark 2.5. *(i) If $\eta_{n1} \equiv 0$ for all $n \geqslant 1$, then Theorem 2.3 and 2.4 extend and improve the results of Suparatulatorn et al. [15, Theorem 2 and 3].*

(ii) If we take $f_1 = f_2 = f_3 = f$, then the results of this study are improvement and extension of the corresponding results of Abbas and Nazir [10].

(iii) If $\eta_{n1} = \eta_{n2} \equiv 0$ for all $n \geqslant 1$, then we get the strong convergence theorems of Mann iteration process for $G-$nonexpansive mappings in the framework of Banach space with graph.

Now, we give the numerical example to support our main theorem in a dimensional space of real numbers. In this example illustrates the efficiency of approximation of common fixed points for $G-$nonexpansive mappings in Banach spaces with a graph.

Example 2.6. Let $X = \mathbb{R}$ be endowed with standard norm $\|.\| = |.|$, $K = [1, 3]$ and $(V(G), E(G)) = G$ such that $V(G) = K$ and $(x, y) \in E(G)$ iff $1 \leqslant x, y \leqslant 1.90$ or $x = y$. Define three mappings $\{f_1, f_2, f_3\}$: $K \to K$ by $f_1 x = \sin(x - 1) + 1$, $f_2 x = 3^{\frac{2(x-1)}{41}}$, $f_3 x = 5^{\frac{1}{20}(x-1)}$ for any $x \in K$. Let

$$\eta_{n1} = \frac{n}{7n + 2}, \eta_{n2} = \frac{n}{3n + 11}, \eta_{n3} = \frac{n + 1}{6n + 5} \text{ for } n \geqslant 1.$$

It is easy to see that f_1, f_2, f_3 are $G-$nonexpansive mappings. It is also clear that $F = F(f_1) \cap F(f_2) \cap F(f_3) = \{1\}$.

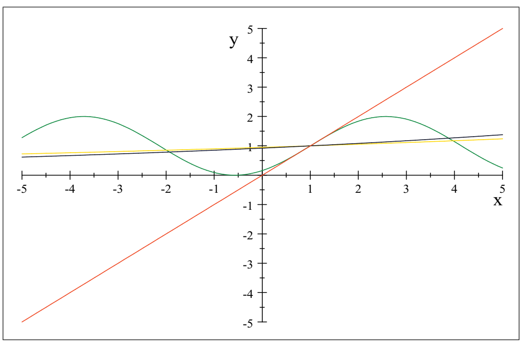

Figure 1. Plot showing fixed point of f_1 (green line), f_2 (yellow line), f_3 (blue line)

3. Conclusion

Our theorems improve the common fixed point theorems for $G-$nonexpansive mappings in Abbas and Nazir [10] and Suparatulatorn et al. [15]. Within the future scope of the idea, reader may prove the convergence theorems of the following iterative processes to a common fixed point of asymptotically nonexpansive mappings (or totally asymptotically nonexpansive mappings, shortly TAN), identified on a nonempty closed convex subset of a Banach space.

1. Let K be a nonempty closed convex subset of a uniformly convex Banach space X. Let $I_1, I_2 : K \to K$ be asymptotically nonexpansive mappings (or TAN), $f_3 : K \to K$ be $I-$asymptotically nonexpansive mappings (or $I-$TAN). Then for three given sequences $\{\eta_{ni}\}$ are sequences in $[0,1]$ for all $i \in \{1,2,3\}$ and $x_0 \in K$, $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= (1-\eta_{n1}) f_3^n y_n + \eta_{n1} I_2^n z_n, \\ y_n &= (1-\eta_{n2}) I_1^n x_n + \eta_{n2} I_2^n z_n, \\ z_n &= (1-\eta_{n3}) x_n + \eta_{n3} I_1^n x_n, \quad n \in N. \end{aligned} \tag{26}$$

2. Let K be a nonempty closed convex subset of a real normed linear space X with retraction P. Let $f_1, f_2, f_3 : K \to X$ be three nonself asymptotically nonexpansive mappings with reference to P:

$$\begin{aligned} x_{n+1} &= \eta_{n1} (Pf_3)^n y_n + \eta_{n2} (Pf_2)^n z_n + \eta_{n3} \sigma_n \\ y_n &= \widehat{\eta_{n1}} (Pf_1)^n x_n + \widehat{\eta_{n2}} (Pf_2)^n z_n + \widehat{\eta_{n3}} \omega_n \\ z_n &= \widetilde{\eta_{n1}} x_n + \widetilde{\eta_{n2}} (Pf_1)^n x_n + \widetilde{\eta_{n3}} \nu_n, \quad n \in N, \end{aligned} \tag{27}$$

where $\{\eta_{n1}\}, \{\eta_{n2}\}, \{\eta_{n3}\}, \{\widehat{\eta_{n1}}\}, \{\widehat{\eta_{n2}}\}, \{\widehat{\eta_{n3}}\}, \{\widetilde{\eta_{n1}}\}, \{\widetilde{\eta_{n2}}\}, \{\widetilde{\eta_{n3}}\}$ are sequences in $[0,1]$ satisfying

$$\eta_{n1} + \eta_{n2} + \eta_{n3} = \widehat{\eta_{n1}} + \widehat{\eta_{n2}} + \widehat{\eta_{n3}} = \widetilde{\eta_{n1}} + \widetilde{\eta_{n2}} + \widetilde{\eta_{n3}} = 1,$$

and $\{\nu_n\}, \{\omega_n\}, \{\sigma_n\}$ are bounded sequences in K.

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