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Fixed Point Theorem Via Measure of Noncompactness and Application to Volterra Integral Equations in Banach Algebras

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Abstract. We propose a new notion of contraction mappings for two class of functions involving the measure of noncompactness in Banach space and derive some basic Darbo type fixed and coupled fixed point results. This work includes and extends the results of Falset and Latrach [J. G. Falset and K. Latrach, On Darbo-Sadovskii's fixed point theorems type for abstract measures of (weak) noncompactness, Bull. Belg. Math. Soc. Simon Stevin 22 (2015), 797-812]. The results are also correlated with the classical generalized Banach fixed point theorems. Finally, we will discuss the applicability of obtained results to the Volterra integral equations in Banach algebras with an illustration.

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1. Introduction and preliminaries

To understand the work in the underlying area, we start with listing some notations and preliminaries that we shall need to express our results.

Throughout the paper,

$\mathbb{R}$ = the set of real numbers,

$\mathbb{N}$ = the set of natural numbers,

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$\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.
Let $(E, \|.\|)$ be a real Banach space with zero element θ. Let $\mathcal{B}(x, r)$ denote the closed ball centered at x with radius r. The symbol $\mathcal{B}_r$ stands for the ball $\mathcal{B}(\theta, r)$. For X, a nonempty subset of E, we denote by $\overline{X}$ and $ConvX$ the closure and the convex closure of X, respectively. Moreover, let us denote by $\mathfrak{M}_E$ the family of nonempty bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We use the following definition of the measure of noncompactness (MNC) given in [10].

Definition 1.1. *A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ is said to be a MNC in E if it satisfies the following conditions:*

(1^0) The family $ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathfrak{N}_E$,

(2^0) (Monotonicity) $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$,

(3^0) (Invariance under closure) $\mu(\overline{X}) = \mu(X)$,

(4^0) (invariance under passage to the convex hull) $\mu(ConvX) = \mu(X)$,

(5^0) (convexity) $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$,

(6^0) (Cantor's generalized intersection property) If (X_n) is a decreasing sequence of nonempty, closed sets in $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ $(n = 1, 2, \ldots)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is non-empty and compact.

The family $ker\mu$ defined in axiom (1^0)is called the kernel of the MNC μ.

One of the properties of the MNC is $X_\infty \in ker\mu$. Indeed, from the inequality $\mu(X_\infty) \leqslant \mu(X_n)$ for $n = 1, 2, 3, \ldots$, we infer that $\mu(X_\infty) = 0$.

In 1930, Kuratowski [18] opened up a new direction of research with the introduction of a measure of noncompactness, denoted by α

The Kuratowski MNC is the map $\alpha : \mathfrak{M}_E \to \mathbb{R}^+$ with

$$\alpha(\mathcal{Q}) = \inf \left\{ \epsilon > 0 : \mathcal{Q} \subset \bigcup_{k=1}^{n} S_k, S_k \subset E, diam(S_k) < \epsilon \, (k \in \mathbb{N}) \right\}. \quad (1)$$

In 1955, Darbo [11] used the notion of Kuratowski measure of non-compactness α to prove fixed point theorem and generalized topological Schauder fixed point theorem [10] and classical Banach fixed point theorem [6].

Theorem 1.2. *[10] Let $\mathcal{X}$ be a closed, convex subset of a Banach space E. Then every compact, continuous map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ has at least one fixed point.*

Theorem 1.3. *[11] Let $\mathcal{X}$ be a nonempty, bounded, closed and convex subset of a Banach space E, μ be the Kuratowski MNC on E. Let $\mathcal{T} : \Omega \to \Omega$ be a continuous and μ-set contraction operator, that is, there exists a constant $k \in [0, 1)$ with*

$$\mu(\mathcal{T}\mathcal{M}) \leqslant k\mu(\mathcal{M})$$

for any nonempty subset $\mathcal{M}$ of $\mathcal{X}$. Then $\mathcal{T}$ has a fixed point.

Following this result, many authors proved several Darbo-type fixed point and coupled theorems by using different types of control functions. Here we mention the paper discussed in ([1, 23]).

With the above discussion in mind, we establish some results of Darbo's type which generalizes and include work mentioned in [3, 4, 5, 9, 11, 12, 13, 17] as well (see Remark 2.4). In the final section, we apply the obtained result to solve the Volterra integral equations in Banach algebras and justify with an example.

2. Fixed Point Theorems

We start the section with the following notion:

Definition 2.1. *[23] Let $\mathfrak{F}$ be the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:*

(F_1) *F is continuous and strictly increasing;*

(F_2) *for each sequence $\{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$.*

(F_3) *There exists* $k \in (0,1)$ *such that* $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

The first main result is:

Theorem 2.2. *Let* $\mathcal{X}$ *be a nonempty, bounded, closed, and convex subset of a Banach space* E, *and* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is continuous operator. If there exist* $\tau > 0$, $F \in \mathfrak{F}$ *and a continuous and strictly increasing mapping* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\mu(\mathcal{T}\mathcal{M}) > 0 \Rightarrow \tau + F(\mu(\mathcal{T}\mathcal{M}) + \varphi(\mu(\mathcal{T}\mathcal{M}))) \leqslant F(\mu(\mathcal{M}) + \varphi(\mu(\mathcal{M}))), \tag{2}$$

for all $\mathcal{M} \subseteq \mathcal{X}$, *where* μ *is an arbitrary MNC. Then* $\mathcal{T}$ *has at least one fixed point in* $\mathcal{X}$.

Proof. Starting with the assumption $\mathcal{X}_0 = \mathcal{X}$, we define a sequence $\{\mathcal{X}_n\}$ such that $\mathcal{X}_{n+1} = Conv(\mathcal{T}\mathcal{X}_n)$, for $n \in \mathbb{N}^*$. If $\mu(\mathcal{X}_{n_0}) + \varphi(\mu(\mathcal{X}_{n_0})) = 0$, that is, $\mu(\mathcal{X}_{n_0}) = 0$ for some natural number $n_0 \in \mathbb{N}$, then $\mathcal{X}_{n_0}$ is compact and since $\mathcal{T}(\mathcal{X}_{n_0}) \subseteq Conv(\mathcal{T}\mathcal{X}_{n_0}) = \mathcal{X}_{n_0+1} \subseteq \mathcal{X}_{n_0}$. Thus we conclude the result from Theorem 1.2, hence we assume that $\mu(\mathcal{X}_n) + \varphi(\mu(\mathcal{X}_n)) > 0$, for all $n \in \mathbb{N}^*$. From (4), and (4^0) of Definition 1.1, we have

$$\begin{aligned}
F(\mu(\mathcal{X}_{n+1}) + \varphi(\mu(\mathcal{X}_{n+1}))) &= F(\mu(Conv(\mathcal{T}\mathcal{X}_n)) + \varphi(\mu(Conv(\mathcal{T}\mathcal{X}_n)))) \\
&= F(\mu(\mathcal{T}\mathcal{X}_n) + \varphi(\mu(\mathcal{T}\mathcal{X}_n))) \\
&\leqslant F(\mu(\mathcal{X}_n) + \varphi(\mu(\mathcal{X}_n))) - \tau \\
&\leqslant F(\mu(\mathcal{X}_{n-1}) + \varphi(\mu(\mathcal{X}_{n-1}))) - 2\tau \\
&\vdots \\
&\leqslant F(\mu(\mathcal{X}_0) + \varphi(\mu(\mathcal{X}_0))) - (n+1)\tau,
\end{aligned}$$

that is,

$$F(\mu(\mathcal{X}_{n+1}) + \varphi(\mu(\mathcal{X}_{n+1}))) \leqslant F(\mu(\mathcal{X}_0) + \varphi(\mu(\mathcal{X}_0))) - (n+1)\tau, \text{ for all } n \in \mathbb{N}. \tag{3}$$

Therefore by (3), we get $F(\mu(\mathcal{X}_{n+1}) + \varphi(\mu(\mathcal{X}_{n+1}))) \to -\infty$ as $n \to \infty$. Thus, from the property (F_2), we have

$$\lim_{n \to \infty} \mu(\mathcal{X}_n) + \varphi(\mu(\mathcal{X}_n)) = 0,$$

therefore

$$\lim_{n\to\infty} \mu(\mathcal{X}_n) = 0.$$

Now from (6^0) of Definition 1.1, we have $\mathcal{X}_\infty = \bigcap_{n=1}^{\infty} \mathcal{X}_n$ is nonempty, closed, convex set and $\mathcal{X}_\infty \subseteq \mathcal{X}_n$ for all $n \in \mathbb{N}$. Also $\mathcal{T}(\mathcal{X}_\infty) \subset \mathcal{X}_\infty$ and $\mathcal{X}_\infty \in ker\mu$. Therefore, by Theorem 1.2, $\mathcal{T}$ has a fixed point u in the set $\mathcal{X}_\infty$ and hence $u \in \mathcal{X}$. $\square$

Remark 2.3. *If $\varphi(t) = 0$ in Theorem 2.2, then we get Theorem 3.2 [13].*

Remark 2.4. *Taking various concrete functions $F \in \mathfrak{F}$ in the condition* (2) *of Theorems 2.2, we can get various classes of μ-set contractive conditions. We state just a few examples which include results existing in the literature:*

(A1) Taking $F(t) = \ln t$ $(t > 0)$, $\tau = ln(\frac{1}{\lambda})$ where $\lambda \in (0,1)$, we have condition

$$\mu(\mathcal{T}\mathcal{M}) > 0 \Rightarrow \mu(\mathcal{T}\mathcal{M}) + \varphi(\mu(\mathcal{T}\mathcal{M})) \leqslant \lambda[\mu(\mathcal{M}) + \varphi(\mu(\mathcal{M}))].$$

(A2) Taking $F(t) = \ln t + t$ $(t > 0)$, $\tau = ln(\frac{1}{\lambda})$ where $\lambda \in (0,1)$, we have condition

$$\begin{aligned} \mu(\mathcal{T}\mathcal{M}) &> 0 \Rightarrow [\mu(\mathcal{T}\mathcal{M}) + \varphi(\mu(\mathcal{T}\mathcal{M}))]e^{\mu(\mathcal{T}\mathcal{M})+\varphi(\mu(\mathcal{T}\mathcal{M}))-[\mu(\mathcal{M})+\varphi(\mu(\mathcal{M}))]} \\ &\leqslant \lambda[\mu(\mathcal{M}) + \varphi(\mu(\mathcal{M}))]. \end{aligned}$$

(A3) Taking $F(t) = \ln t$ $(t > 0)$ and $\tau = \ln(\frac{1}{\lambda})$, $\varphi(t) = t$ where $\lambda \in (0,1)$, we have Darbo's μ-set contraction condition

$$\mu(\mathcal{T}\mathcal{M}) > 0 \Rightarrow \mu(\mathcal{T}\mathcal{M}) \leqslant \lambda\mu(\mathcal{M}).$$

(A4) Taking $F(t) = -\frac{1}{\sqrt{t}}$ $(t > 0)$, $\tau = \lambda$ $(\lambda > 0)$, the condition is

$$\mu(\mathcal{T}\mathcal{M}) > 0 \Rightarrow \mu(\mathcal{T}\mathcal{M}) + \varphi(\mu(\mathcal{T}\mathcal{M})) \leqslant \frac{\mu(\mathcal{M}) + \varphi(\mu(\mathcal{M}))}{[1 + \lambda\sqrt{\mu(\mathcal{M}) + \varphi(\mu(\mathcal{M}))}]^2}.$$

Proposition 2.5. *Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of a Banach space E and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is continuous operator. If there exist $\tau > 0$, $F \in \mathfrak{F}$ and a continuous mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\mu(\mathcal{T}\mathcal{M}) > 0 \Rightarrow \tau + F(diam(\mathcal{T}\mathcal{M}) + \varphi(diam(\mathcal{T}\mathcal{M}))) \leqslant F(diam(\mathcal{M}) + \varphi(diam(\mathcal{M}))),$$

for all $\mathcal{M} \subseteq \mathcal{X}$. Then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.

Proof. Following Theorem 2.2 and argument of Proposition 3.2 [13], $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

To prove uniqueness, we suppose that there exist two distinct fixed points $\zeta, \xi \in \mathcal{X}$, then we may define the set $\Upsilon := \{\zeta, \xi\}$. In this case $diam(\Upsilon) = diam(\mathcal{T}(\Upsilon)) = \|\xi - \zeta\| > 0$. Then using (4), we get

$$diam(\mathcal{T}(\Upsilon)) > 0 \Rightarrow \tau + F(diam(\mathcal{T}(\Upsilon)) + \varphi(diam(\mathcal{T}(\Upsilon)))) \leqslant F(diam(\Upsilon) + \varphi(diam(\Upsilon))), \quad (4)$$

a contradiction and hence $\xi = \zeta$. □

Following is the generalized classical fixed point result derived from Proposition 2.5.

Corollary 2.6. *Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of a Banach space E and let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be an operator. It there exist $\tau > 0$, $F \in \mathfrak{F}$ and a continuous and strictly increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|\mathcal{T}u - \mathcal{T}v\| > 0 \Rightarrow \tau + F(\|\mathcal{T}u - \mathcal{T}v\| + \varphi(\|\mathcal{T}u - \mathcal{T}v\|)) \leqslant F(\|u - v\| + \varphi(\|u - v\|)) \quad (5)$$

for all $u, v \in \mathcal{X}$. Then $\mathcal{T}$ has a unique fixed point.

Proof. Let $\mu : \mathfrak{M}_E \to \mathbb{R}^+$ be a set quantity defined by the formula $\mu(\mathcal{X}) = diam\mathcal{X}$, where $diam\mathcal{X} = \sup\{\|u - v\| : u, v \in \mathcal{X}\}$ stands for the diameter of $\mathcal{X}$. It is easily seen that μ is a MNC in a space E in the sense of Definition 1.1. Therefore from (5) we have

$$\sup_{u,v\in\mathcal{X}} \|\mathcal{T}u - \mathcal{T}v\| > 0 \Rightarrow$$
$$\begin{aligned}\tau + F(\sup_{u,v\in\mathcal{X}} \|\mathcal{T}u - \mathcal{T}v\| + \varphi(\sup_{u,v\in\mathcal{X}} \|\mathcal{T}u - \mathcal{T}v\|)) &= \tau + \sup_{u,v\in\mathcal{X}} F(\|\mathcal{T}u - \mathcal{T}v\| + \varphi(\|\mathcal{T}u - \mathcal{T}v\|)) \\ &\leqslant \sup_{u,v\in\mathcal{X}} F(\|u - v\| + \varphi(\|u - v\|)) \\ &\leqslant F(\sup_{u,v\in\mathcal{X}} \|u - v\| + \varphi(\sup_{u,v\in\mathcal{X}} \|u - v\|))\end{aligned}$$

which implies that

$$diam(\mathcal{T}(\mathcal{X})) > 0 \Rightarrow \tau + F(diam(\mathcal{T}(\mathcal{X})) + \varphi(diam(\mathcal{T}(\mathcal{X})))) \leqslant F(diam(\mathcal{X}) + \varphi(diam(\mathcal{X})).$$

Thus following Proposition 2.5, $\mathcal{T}$ has an unique fixed point. □

Corollary 2.7. *Let $(E, \|\cdot\|)$ be a Banach space and let $\mathcal{X}$ be a closed convex subsets of E. Let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{X} \to \mathcal{X}$ be two operators satisfying the following conditions:*

(I) $(\mathcal{T}_1 + \mathcal{T}_2)(\mathcal{X}) \subseteq \mathcal{X}$;

(II) there exist $\tau > 0$, $F \in \mathfrak{F}$ and a continuous and increasing mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\mathcal{T}_1 u - \mathcal{T}_1 v\| > 0 \Rightarrow \tau + F(\|\mathcal{T}_1 u - \mathcal{T}_1 v\| + \varphi(\|\mathcal{T}_1 u - \mathcal{T}_1 v)\|)) \leqslant F(\|u - v\| + \varphi(\|u - v\|))$$

(III) $\mathcal{T}_2$ is a continuous and compact operator.

Then $\mathcal{J} := \mathcal{T}_1 + \mathcal{T}_2 : \mathcal{X} \to \mathcal{X}$ has a fixed point $u \in \mathcal{X}$.

Proof. Suppose that $\mathcal{M}$ is a subset of X with $\alpha(\mathcal{M}) > 0$. By the notion of Kuratowski MNC, for each $n \in \mathbb{N}$, there exist $\mathcal{C}_1, \ldots, \mathcal{C}_{m(n)}$ bounded subsets such that $\mathcal{M} \subseteq \bigcup_{i=1}^{m(n)} \mathcal{C}_i$ and $diam(\mathcal{C}_i) \leqslant \alpha(\mathcal{M}) + \frac{1}{n}$. Suppose that $\alpha(\mathcal{T}_1(\mathcal{M})) > 0$. Since $\mathcal{T}_1(\mathcal{M}) \subseteq \bigcup_{i=1}^{m(n)} \mathcal{T}_1(\mathcal{C}_i)$, there exists $i_0 \in \{1, 2, \ldots, m(n)\}$ such that $\alpha(\mathcal{T}_1(\mathcal{M})) \leqslant diam(\mathcal{T}_1(\mathcal{C}_{i_0}))$. Using (6) condition of $\mathcal{T}_1$ with discussed arguments, we have

$$\begin{aligned} \tau + F(\alpha(\mathcal{T}_1(\mathcal{M})) + \varphi(\alpha(\mathcal{T}_1(\mathcal{M})))) &\leqslant \tau + F(diam(\mathcal{T}_1(\mathcal{C}_{i_0})) + \varphi(diam(\mathcal{T}_1(\mathcal{C}_{i_0})))) \\ &\leqslant F(diam(\mathcal{C}_{i_0}) + \varphi(diam(\mathcal{C}_{i_0})) \\ &\leqslant F\left(\alpha(\mathcal{M}) + \frac{1}{n} + \varphi\left(\alpha(\mathcal{M}) + \frac{1}{n}\right)\right). \end{aligned} \quad (7)$$

Passing to the limit in (7) as $n \to \infty$, we get

$$\tau + F(\alpha(\mathcal{T}_1(\mathcal{M})) + \varphi(\alpha(\mathcal{T}_1(\mathcal{M})))) \leqslant F(\alpha(\mathcal{M}) + \varphi(\alpha(\mathcal{M}))).$$

Using (III) hypothesis, we have by the notion of α that

$$\begin{aligned} \tau + F(\alpha(\mathcal{J}(\mathcal{M})) + \varphi(\alpha(\mathcal{J}(\mathcal{M})))) &= \tau + F(\alpha(\mathcal{T}_1(\mathcal{M}) + \mathcal{T}_2(\mathcal{M})) + \varphi(\alpha(\mathcal{T}_1(\mathcal{M}) + \mathcal{T}_2(\mathcal{M})))) \\ &\leqslant \tau + F(\alpha(\mathcal{T}_1(\mathcal{M})) + \alpha(\mathcal{T}_2(\mathcal{M})) + \varphi(\alpha(\mathcal{T}_1(\mathcal{M})) + \alpha(\mathcal{T}_2(\mathcal{M})))) \\ &= \tau + F(\alpha(\mathcal{T}_1(\mathcal{M})) + \varphi(\alpha(\mathcal{T}_1(\mathcal{M})))) \\ &\leqslant F(\alpha(\mathcal{M}) + \varphi(\alpha(\mathcal{M}))). \end{aligned}$$

Thus by Theorem 2.2, $\mathcal{J}$ has a fixed point $u \in \mathcal{X}$. □

Remark 2.8. If $\varphi(t) = 0$ in Proposition 2.5-Corollary 2.7, we get result given in [13, Proposition 3.2-Corollary 3.4].

2.1 Coupled fixed point theorems

This section is concern with the coupled fixed point theorems of Section 2.

Definition 2.1.1. *[15] An element $(u, v) \in E^2$ is called a coupled fixed point of a mapping $\mathcal{G} : E^2 \to E$ if $\mathcal{G}(u, v) = u$ and $\mathcal{G}(v, u) = v$.*

The first coupled fixed point result is:

Theorem 2.1.2. *Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of a Banach space E. Suppose that $\mathcal{G} : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ is continuous operator. If there exist $\tau > 0$, $F \in \mathfrak{F}$, and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, increasing and sub-additive mapping such that*

$$\left.\begin{array}{c} \text{for } i, j \in \{1, 2\}, i \neq j,\ \mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j)) > 0 \\ \Rightarrow \tau + F\left(\begin{array}{c} \mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_i \times \mathcal{X}_j))) + \\ \mu(\mathcal{G}(\mathcal{X}_j \times \mathcal{X}_i)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_j \times \mathcal{X}_i))) \end{array}\right) \\ \leqslant F(\mu(\mathcal{X}_i) + \mu(\mathcal{X}_j) + \varphi(\mu(\mathcal{X}_i) + \mu(\mathcal{X}_j))) \end{array}\right\} \quad (8)$$

for all $\mathcal{X}_i, \mathcal{X}_j \subseteq \mathcal{X}$*, where* μ *is an arbitrary MNC. Then* $\mathcal{G}$ *has at least a coupled fixed point* $(u, v) \in \mathcal{X} \times \mathcal{X}$.

Proof. Consider the map $\widehat{\mathcal{G}} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ defined by the formula

$$\widehat{\mathcal{G}}(u, v) = (\mathcal{G}(u, v), \mathcal{G}(v, u)).$$

$\widehat{\mathcal{G}}$ is continuous due to continuity of $\mathcal{G}$. Following [4], we define a new MNC in the space $\mathcal{X} \times \mathcal{X}$ as

$$\widehat{\mu}(\mathcal{M}) = \mu(\mathcal{X}_1) + \mu(\mathcal{X}_2),$$

where $\mathcal{X}_i$, $i = 1, 2$ denote the natural projections of $\mathcal{X}$. Without loss of generality, we can assume $\mathcal{M}$ is a nonempty subset of $\mathcal{X}^2$. Hence, by the condition (8) and using (2^0) of Definition 1.1 we conclude that

$$\begin{aligned} \widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) &\leqslant \widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(X_2 \times \mathcal{X}_1)) \\ &= \mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)) + \mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)), \end{aligned}$$

therefore by the assumption, we have

$$\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) > 0,$$

that implies

$$\begin{aligned} & \tau + F(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) + \varphi(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})))) \\ \leqslant\ & \tau + F(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)) + \varphi(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)))) \\ \leqslant\ & \tau + F(\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2)) + \mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)) + \varphi(\mu(\mathcal{G}(\mathcal{X}_1 \times \mathcal{X}_2))) + \varphi(\mu(\mathcal{G}(\mathcal{X}_2 \times \mathcal{X}_1)))) \\ \leqslant\ & F(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2)) + \varphi(\mu(\mathcal{X}_1) + \mu(\mathcal{X}_2))) \\ =\ & F(\widehat{\mu}(\mathcal{M}) + \varphi(\widehat{\mu}(\mathcal{M}))), \end{aligned}$$

that is,

$$\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) > 0 \Rightarrow \tau + F(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) + \varphi(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})))) \leqslant F(\widehat{\mu}(\mathcal{M}) + \varphi(\widehat{\mu}(\mathcal{M}))).$$

Therefore from Theorem 2.2, we get $\widehat{\mathcal{G}}$ has at least one fixed point in $\mathcal{X}^2$, and hence $\mathcal{G}$ has a coupled fixed point. □

The second result is as follows:

Theorem 2.1.3. *Let $\mathcal{X}$ be a nonempty, bounded, closed, and convex subset of a Banach space E. Suppose that $\mathcal{G}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}$ is continuous operator. If there exist $\tau>0$, $F\in\mathfrak{F}$ and a continuous and increasing mapping $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ such that*

$$\left.\begin{array}{r} \text{for } i,j\in\{1,2\}, i\neq j,\ \mu\left(\mathcal{G}\left(\mathcal{X}_i\times\mathcal{X}_j\right)\right)>0 \\ \Rightarrow \tau+F\left(\ \mu\left(\mathcal{G}\left(\mathcal{X}_i\times\mathcal{X}_j\right)\right)+\varphi\left(\ \mu\left(\mathcal{G}\left(\mathcal{X}_i\times\mathcal{X}_j\right)\right)\right)\right) \\ \leqslant F\left(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\}+\varphi\left(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\}\right)\right) \end{array}\right\}, \tag{9}$$

for all $\mathcal{X}_i,\mathcal{X}_j\subseteq\mathcal{X}$, where μ is an arbitrary MNC. Then $\mathcal{G}$ has at least a coupled fixed point $(u,v)\in\mathcal{X}\times\mathcal{X}$.

Proof. Consider the map $\widehat{\mathcal{G}}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ defined by the formula

$$\widehat{\mathcal{G}}(u,v)=(\mathcal{G}(u,v),\mathcal{G}(v,u)).$$

$\widehat{\mathcal{G}}$ is continuous due to continuity of $\mathcal{G}$. Following [4], we define a new MNC in the space $\mathcal{X}\times\mathcal{X}$ as

$$\widehat{\mu}(\mathcal{M})=\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\}$$

where $\mathcal{X}_i$, $i=1,2$ denote the natural projections of $\mathcal{M}$. Without loss of generality, we can assume $\mathcal{M}$ is a nonempty subset of $\mathcal{X}\times\mathcal{X}$. Following previous theorem, we have

$$\begin{aligned} \widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})) &\leqslant \widehat{\mu}(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)\times\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1)) \\ &= \max\{\mu(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)),\mu(\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))\}, \end{aligned}$$

which is, by the assumption,

$$\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M}))>0.$$

Hence, by the condition (9), and using (2^0) of Definition 1.1 we conclude that

$$\begin{aligned} &\tau+F(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M}))+\varphi(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M})))) \\ \leqslant\ &\tau+F(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)\times\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))+\varphi(\widehat{\mu}(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)\times\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1)))) \\ =\ &\tau+F(\max\{\mu(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)),\mu(\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))\}+\varphi(\max\{\mu(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)),\mu(\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))\})) \\ =\ &\tau+\max\{F(\mu(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2))+\varphi(\mu(\mathcal{G}(\mathcal{X}_1\times\mathcal{X}_2)))),F(\mu(\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))+\varphi(\mu(\mathcal{G}(\mathcal{X}_2\times\mathcal{X}_1))))\} \\ \leqslant\ &\max\left\{\begin{array}{l} F(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\}+\varphi(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\})), \\ F(\max\{\mu(\mathcal{X}_2),\mu(\mathcal{X}_1)\}+\varphi(\max\{\mu(\mathcal{X}_2),\mu(\mathcal{X}_1)\})) \end{array}\right\} \\ =\ &F(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\}+\varphi(\max\{\mu(\mathcal{X}_1),\mu(\mathcal{X}_2)\})) \\ =\ &F(\widehat{\mu}(\mathcal{M})+\varphi(\widehat{\mu}(\mathcal{M}))), \end{aligned}$$

that is,

$$\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M}) > 0 \Rightarrow \tau + F(\widehat{\mu}(\widehat{\mathcal{G}}(\mathcal{M}))) \leqslant F(\widehat{\mu}(\mathcal{M}) + \varphi(\widehat{\mu}(\mathcal{M}))).$$

Hence by Theorem 2.2, we reached that $\widehat{\mathcal{G}}$ has at least one fixed point in $\mathcal{X}^2$, and thus $\mathcal{G}$ has a coupled fixed point. □

Remark 2.1.4. *In view of the Remark 2.4 (A1-(A3)), some new coupled fixed point results can be derived from Theorems 2.1.2 and Theorems 2.1.3.*

3. Application to the Volterra Integral Equations in Banach Algebras

Let $(X, \|.\|)$ be a real Banach algebra and the symbol $C([0,T], X)$ stands for the space consisting of all continuous mappings $f : [0,T] \to X$, with $T > 0$. In this section inspired by Theorem 4.1 of J. Garcia-Falset *et al.* [13], we will consider the existence of a solution $x \in C([0,T], X)$ to the following nonlinear Volterra integral equation

$$x(t) = f(t, x(t)) + Gx(t) \int_0^t g(s, x(s))\, Qx(s)\, ds. \qquad (10)$$

We will assume that the following conditions are satisfied:

(a) $f : [0,T] \times X \to X$ is a continuous mapping such that there exists a bijective, strictly increasing function $F : (0, \infty) \to (-\infty, 0)$ and

$$\|f(t,x) - f(t,y)\| \;\breve{>}\; 0 \implies \acute{\tau} + F(\|f(t,x) - f(t,y)\| + \phi(\|f(t,x) - f(t,y)\|)) \leqslant F(\|x-y\| + \phi(\|x-y\|)). \qquad (11)$$

(b) G and Q are some operators acting continuously from the space $C([0,T], X)$ into itself and there are increasing functions $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} \|G(x)\| &\leqslant \varphi(\|x\|) \\ \|Q(x)\| &\leqslant \psi(\|x\|), \end{aligned}$$

for any $x \in C([0,T], X)$.

(c) $g : [0,T] \times X \to X$ is a continuous mapping such that there exists an increasing function $u \in L^1([0,T], \mathbb{R}^+)$ and an increasing continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|g(t,x)\| \leqslant u(t)\,\theta(\|x\|),$$

for any $x \in X$ and a.e. $t \in [0,T]$. Moreover, for any fixed $x \in X$ the mapping $t \longrightarrow g(t,x)$ is measurable over the interval $[0,T]$ and the mapping $x \longrightarrow g(t,x)$ is continuous for a.e. $t \in [0,T]$,

(d) $\liminf_{\gamma\to\infty} \dfrac{\varphi(\gamma)\,\psi(\gamma)\,\theta(\gamma)\,\|u\|_1}{\gamma} < 1.$

Theorem 3.1. *Under assumptions (a)-(d), Eq.(10) has at least one solution in the space $x \in C([0,T], X)$.*

Proof. Define an integral operator $\mathcal{J} : C([0,T], X) \to C([0,T], X)$ by

$$\mathcal{J}x(t) = f(t, x(t)) + Gx(t)\int_0^t g(s, x(s))\,Qx(s)\,ds.$$

We will show that the operator $\mathcal{J}$ has a one fixed point. To this end we define the following two mappings $\mathcal{T}_1, \mathcal{T}_2 : C([0,T], X) \to C([0,T], X)$ by:

$$\begin{aligned} \mathcal{T}_1 x(t) &= f(t, x(t)), \\ \mathcal{T}_2 x(t) &= Gx(t)\int_0^t g(s, x(s))\,Qx(s)\,ds, \end{aligned}$$

where $\mathcal{J} = \mathcal{T}_1 + \mathcal{T}_2$.

It is easy to see that $\mathcal{T}_1$ is well-defined. Now we show that $\mathcal{T}_2$ is well-defined, let $\varepsilon > 0$ arbitrarily and $x \in C([0,T], X)$ be given and fixed. Next let $M_1 = \sup_{t\in[0,T]} \|g(t, x(t))\|$, since Gx is uniformly continuous on $[0,T]$, there exists $\delta_1(\varepsilon) > 0$ such that for all $t_1, t_2 \in [0,T]$, with $|t_2 - t_1| < \delta_1(\varepsilon)$ we have

$$\|Gx(t_2) - Gx(t_1)\| < \frac{\varepsilon}{2(1 + TM_1\|Qx\|_\infty)}.$$

Put $\delta(\varepsilon) = \min\left\{\delta_1(\varepsilon), \dfrac{\varepsilon}{2\left(1 + M_1 \|Gx\|_\infty \|Qx\|_\infty\right)}\right\}$. Without loss of generality we may assume that $t_1 < t_2$ and $t_2 - t_1 < \delta(\varepsilon)$ and we obtain

$$\begin{aligned}
\|\mathcal{T}_2 x(t_2) - \mathcal{T}_2 x(t_1)\| &= \left\| \begin{array}{c} (Gx(t_2) - Gx(t_1)) \int_0^{t_1} g(s, x(s))\, Qx(s)\, ds \\ +Gx(t_2) \int_{t_1}^{t_2} g(s, x(s))\, Qx(s)\, ds \end{array} \right\| \\
&\leqslant \|Gx(t_2) - Gx(t_1)\| \,.\, \int_0^{t_1} \|g(s, x(s))\| \,.\, \|Qx(s)\|\, ds + \|Gx(t_2)\| \,. \\
&\quad \int_{t_1}^{t_2} \|g(s, x(s))\| \,.\, \|Qx(s)\|\, ds \\
&< \frac{\varepsilon}{2\left(1 + TM_1 \|Qx\|_\infty\right)} TM_1 \|Qx\|_\infty + M_1 \|Gx\|_\infty \,.\, \|Qx\|_\infty (t_2 - t_1) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Next, we show that $\mathcal{T}_2$ is a continuous operator. Fix $y \in C([0, T], X)$ and $\varepsilon > 0$ be given, since G and Q are some operators acting continuously from the space $C([0, T], X)$ into itself, so there exist $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$\begin{aligned}
\forall x &\in C([0, T], X), \ \left(\|x - y\|_\infty < \delta_1 \Longrightarrow \|Gx - Gy\|_\infty < \varepsilon_1\right) \\
\forall x &\in C([0, T], X), \ \left(\|x - y\|_\infty < \delta_2 \Longrightarrow \|Qx - Qy\|_\infty < \varepsilon_2\right),
\end{aligned}$$

where ε_1 and ε_2 depend on ε and will be given.

On the other hand, since the mapping $x \longrightarrow g(t, x)$ is continuous for a.e. $t \in [0, T]$, there exits a $\delta_3 > 0$ such that for a.e. $t \in [0, T]$ we have

$$\forall x \in C([0, T], X), \ \left(\|x(t) - y(t)\| < \delta_3 \Longrightarrow \|g(t, x) - g(t, y)\| < \varepsilon_3\right),$$

where ε_3 depends on ε and will be given.

Now if we put $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, then for any $x \in C([0, T], X)$ that $\|x - y\|_\infty < \delta$, by the triangle inequality we obtain

$$
\begin{aligned}
\left\|\mathcal{T}_2x(t)-\mathcal{T}_2y(t)\right\| &= \left\|Gx(t)\int_0^t g(s,x(s))\,Qx(s)\,ds - Gy(t)\int_0^t g(s,y(s))\,Qy(s)\,ds\right\| \\
&\leqslant \|Gx(t)-Gy(t)\|\,.\int_0^t \|g(s,x(s))\|\,.\,\|Qx(s)\|\,ds \\
&\quad +\|Qx-Qy\|_\infty\,.\,\|Gy(t)\|\,.\int_0^t \|g(s,x(s))\|\,ds \\
&\quad +\|Gy\|_\infty\,.\,\|Qy\|_\infty\,.\int_0^t \|g(s,x(s))-g(s,y(s))\|\,ds \\
&\leqslant \|Gx-Gy\|_\infty\,.\,\|Qx\|_\infty\,.\int_0^T \|g(s,x(s))\|\,ds \\
&\quad +\|Qx-Qy\|_\infty\,.\,\|Gy\|_\infty\,.\int_0^T \|g(s,x(s))\|\,ds \\
&\quad +\|Gy\|_\infty\,.\,\|Qy\|_\infty\,.\int_0^T \|g(s,x(s))-g(s,y(s))\|\,ds \\
&\leqslant \varepsilon_1.\psi(\|x\|_\infty)\,.\int_0^T u(s)\,\theta(\|x(s)\|)\,ds+\varepsilon_2\varphi(\|y\|_\infty)\int_0^T u(s)\,\theta(\|x(s)\|)\,ds \\
&\quad +\varepsilon_3T\,\|Gy\|_\infty\,.\,\|Qy\|_\infty \\
&\leqslant \varepsilon_1.\psi(\|y\|_\infty+\delta)\,.\,\|u\|_1\,.\theta(\|y\|_\infty+\delta)+\varepsilon_2\varphi(\|y\|_\infty)\,.\,\|u\|_1\,.\theta(\|y\|_\infty+\delta) \\
&\quad +T\varepsilon_3\varphi(\|y\|_\infty)\,.\psi(\|y\|_\infty) \\
&< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_1 &= \frac{\varepsilon}{3\left(1+\psi(\|y\|_\infty+\delta)\,\|u\|_1\,\theta(\|y\|_\infty+\delta)\right)}, \\
\varepsilon_2 &= \frac{\varepsilon}{3\left(1+\varphi(\|y\|_\infty)\,\|u\|_1\,\theta(\|y\|_\infty+\delta)\right)}, \\
\varepsilon_3 &= \frac{\varepsilon}{3\left(1+T\varepsilon_3\varphi(\|y\|_\infty)\,.\psi(\|y\|_\infty)\right)}.
\end{aligned}
$$

Now we show that $\mathcal{T}_2$ is a compact operator.

If $B=\{x\in C([0,T],X):\|x\|_\infty<1\}$ is the open unit ball of $C([0,T],X)$, then we claim that $\overline{\mathcal{T}_2(B)}$ is a compact subset of $C([0,T],X)$. To see this, by the Arzelà–Ascoli's theorem, we need only to show that $\mathcal{T}_2(B)$ is an uniformly bounded and equi-continuous subset of $C([0,T],X)$. First we show that $\mathcal{T}_2(B) = \{\mathcal{T}_2 x : x \in B\}$ is uniformly bounded. By the conditions (a) and (b) for any $x \in B$, we have the following estimates

$$\begin{aligned}
\|\mathcal{T}_2 x(t)\| &= \left\| Gx(t) \int_0^t g(s, x(s))\, Qx(s)\, ds \right\| \\
&\leqslant \|Gx(t)\| \,.\, \left\| \int_0^t g(s, x(s))\, Qx(s)\, ds \right\| \\
&\leqslant \|Gx\|_\infty \int_0^t \|g(s, x(s))\, Qx(s)\|\, ds \\
&\leqslant \|Gx\|_\infty \int_0^t \|g(s, x(s))\| \,.\, \|Qx(s)\|\, ds \\
&\leqslant \|Gx\|_\infty \,.\, \|Qx\|_\infty \int_0^T \|g(s, x(s))\|\, ds \\
&\leqslant \|Gx\|_\infty \,.\, \|Qx\|_\infty \,.\, \int_0^T u(t)\, \theta(\|x(s)\|)\, ds \\
&\leqslant \varphi(\|x\|_\infty) \,.\varphi(\|x\|_\infty) \,.\theta(\|x\|_\infty) \,.\, \|u\|_1 \\
&\leqslant \varphi(1) \,.\psi(1) \,.\theta(1) \,.\, \|u\|_1 \,.
\end{aligned}$$

Hence, putting $M_2 := \varphi(1) \,.\psi(1) \,.\theta(1) \,.\, \|u\|_1$, we conclude that, $\mathcal{T}_2(B)$ is uniformly bounded. Now we show that, $\mathcal{T}_2(B)$ is an uniformly equicontinuous subset of $C([0, T], X)$. To see this, let $x \in B$ be arbitrary, and let $\varepsilon > 0$. Since Gx is uniformly continuous, there exists some $\delta_1(\varepsilon) > 0$ such that

$$\forall t_1, t_2 \in [0, T], (|t_2 - t_1| < \delta_1(\varepsilon) \Longrightarrow \|Gx(t_2) - Gx(t_1)\| < \varepsilon_1),$$

where ε_1 depends on ε and will be given. Without loss of generality we may assume that $t_1 < t_2$. Now by the Mean Value Theorem for Integrals, there exists some $c_x \in (t_1, t_2)$ such that

$$\int_{t_1}^{t_2} \|g(s, x(s))\|\, ds = (t_2 - t_1) \,.\, \|g(c_x, x(c_x))\| \,.$$

Let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \varepsilon_2\}$, where ε_2 is depends on ε and will be given. Therefore, if $t_1, t_2 \in [0, T]$ satisfies $0 < t_2 - t_1 < \delta(\varepsilon)$ and $x \in B$, then we have the following estimates

$$\begin{aligned}
\|\mathcal{T}_2x(t_2)-\mathcal{T}_2x(t_1)\| &= \left\|Gx(t_2)\int_0^{t_2}g(s,x(s))\,Qx(s)\,ds-Gx(t_1)\int_0^{t_1}g(s,x(s))\,Qx(s)\,ds\right\| \\
&\leqslant \|Gx(t_2)-Gx(t_1)\|\,.\int_0^{t_1}\|g(s,x(s))\|\,.\,\|Qx(s)\|\,ds \\
&\quad +\|Gx(t_2)\|\,.\int_{t_1}^{t_2}\|g(s,x(s))\|\,.\,\|Qx(s)\|\,ds \\
&\leqslant \varepsilon_1\|Qx\|_\infty\,.\int_0^T\|g(s,x(s))\|\,ds+\|Gx\|_\infty\,.\,\|Qx\|_\infty\,.\int_{t_1}^{t_2}\|g(s,x(s))\|\,ds \\
&\leqslant \varepsilon_1\psi(\|x\|_\infty)\,.\int_0^T u(s)\,\theta(\|x(s)\|)\,ds \\
&\quad +\varphi(\|x\|_\infty)\,.\psi(\|x\|_\infty)(t_2-t_1)\,.\,\|g(c_x,x(c_x))\| \\
&\leqslant \varepsilon_1\psi(\|x\|_\infty)\,.\,\|u\|_1\,.\theta(\|x\|_\infty)+\varepsilon_2\varphi(\|x\|_\infty)\,.\psi(\|x\|_\infty)\,u(c_x)\,\theta(\|x(c_x)\|) \\
&\leqslant \varepsilon_1\psi(1)\,.\theta(1)\,\|u\|_1+\varepsilon_2\varphi(1)\,.\psi(1)\,u(T)\,\theta(1) \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_1 &= \frac{\varepsilon}{2(1+\psi(1)\,.\theta(1)\,.\,\|u\|_1)}, \\
\varepsilon_2 &= \frac{\varepsilon}{2(1+\varphi(1)\,.\psi(1)\,.u(T)\,.\theta(1))}.
\end{aligned}$$

Therefore $\mathcal{T}_2$ is a compact operator.

Next, we show that $\mathcal{T}_1$ is a F-contraction. Let $x,y\in C([0,T],X)$, and $\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty>0$. By applying the fact that every continuous function attains it's maximum on a compact set, there exists $t\in[0,T]$ such that $0<\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty=\|f(t,x(t))-f(t,y(t))\|$. By (a) and using the fact that F and ϕ are strictly increasing functions we obtain

$$\begin{aligned}
\tau+F(\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty+\phi(\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty)) &= \tau+F\begin{pmatrix}\|f(t,x(t))-f(t,y(t))\| \\ +\phi(\|f(t,x(t))-f(t,y(t))\|)\end{pmatrix} \\
&\leqslant F(\|x(t)-y(t)\|+\phi(\|x(t)-y(t)\|)) \\
&\leqslant F(\|x-y\|_\infty+\phi(\|x-y\|_\infty)).
\end{aligned}$$

Hence $\mathcal{T}_1$ is a F-contraction. Now we show that there exists some $M_3>0$ such that $\|\mathcal{T}_1x\|_\infty\leqslant M_3$ holds for each $x\in C([0,T],X)$. Since

F is bijective and strictly increasing we have

$\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty+\phi\left(\|\mathcal{T}_1x-\mathcal{T}_1y\|_\infty\right)\leqslant F^{-1}\left[F\left(\|x-y\|_\infty+\phi\left(\|x-y\|_\infty\right)\right)-\tau\right].$

Let $0<\|x\|_\infty+\varphi\left(\|x\|_\infty\right)$, since $F\left(\|x\|_\infty+\varphi\left(\|x\|_\infty\right)\right)<0$, the above inequality implies that

$$\begin{aligned}\|\mathcal{T}_1x\|_\infty &\leqslant \|\mathcal{T}_1x-\mathcal{T}_10\|_\infty+\|\mathcal{T}_10\|_\infty\\ &\leqslant \|\mathcal{T}_1x-\mathcal{T}_10\|_\infty+\phi\left(\|\mathcal{T}_1x-\mathcal{T}_10\|_\infty\right)+\|\mathcal{T}_10\|_\infty\\ &\leqslant F^{-1}\left[F\left(\|x\|_\infty+\phi\left(\|x\|_\infty\right)\right)-\tau\right]+\|\mathcal{T}_10\|_\infty\\ &\leqslant F^{-1}\left[-\tau\right]+\|\mathcal{T}_10\|_\infty .\end{aligned}$$

Therefore

$$\exists\ M_3>0,\forall x\ \left(x\in C\left([0,T],X\right)\implies\|\mathcal{T}_1x\|_\infty\leqslant M_3\right),$$

where, $M_3:=F^{-1}\left[-\tau\right]+\|\mathcal{T}_10\|_\infty$.

Finally, we claim that there exists some $r>0$, such that $\mathcal{J}\left(B_r\left(0\right)\right)\subseteq B_r\left(0\right)$ with $B_r\left(0\right)=\left\{x\in C\left([0,T],X\right):\|x\|_\infty\leqslant r\right\}$. On the contrary, for any $\gamma>0$ there exists some $x_\gamma\in B_r\left(0\right)$ such that $\|\mathcal{J}\left(x_\gamma\right)\|>\gamma$. This implies that $\liminf_{\gamma\to\infty}\frac{1}{\gamma}\|\mathcal{J}\left(x_\gamma\right)\|\geqslant1$. On the other hand, we have

$$\begin{aligned}\|\mathcal{J}x_\gamma\left(t\right)\| &= \|f\left(t,x_\gamma\left(t\right)\right)\|+\left\|Gx_\gamma\left(t\right)\int_0^t g\left(s,x_\gamma\left(s\right)\right)Qx_\gamma\left(s\right)ds\right\|\\ &\leqslant \|\mathcal{T}_1x_\gamma\|_\infty+\|Gx_\gamma\left(t\right)\|.\left\|\int_0^t g\left(s,x_\gamma\left(s\right)\right)Qx_\gamma\left(s\right)ds\right\|\\ &\leqslant M_3+\|Gx_\gamma\|_\infty.\|Qx_\gamma\|_\infty\int_0^t\|g\left(s,x_\gamma\left(s\right)\right)\|ds\\ &\leqslant M_3+\varphi\left(\|x_\gamma\|_\infty\right).\psi\left(\|x_\gamma\|_\infty\right).\int_0^T u\left(t\right)\theta\left(\|x_\gamma\left(s\right)\|\right)ds\\ &\leqslant M_3+\varphi\left(\gamma\right).\psi\left(\gamma\right).\theta\left(\gamma\right).\|u\|_1 .\end{aligned}$$

Hence, by the above estimate and condition (d) we get

$$\liminf_{\gamma\to\infty}\frac{1}{\gamma}\|\mathcal{J}\left(x_\gamma\right)\|_\infty\leqslant\liminf_{\gamma\to\infty}\frac{\varphi\left(\gamma\right).\psi\left(\gamma\right).\theta\left(\gamma\right).\|u\|_1}{\gamma}<1$$

which is a contradiction. Thus in view of the above discussions and

Corollary 2.7 we conclude that Eq.(10) has at least one solution in $B_r(0) \subseteq C([0,T], X)$. $\square$

Example 3.2. Consider the Volterra integral equation of the form

$$x(t) = \arctan\left(e^{3t}\right) + \frac{|x(t)|}{\left(1+7\sqrt[5]{8|x(t)|}\right)^5} + \frac{1}{4}\frac{\sqrt[3]{|x(t)|}}{2+\cos^4 x(t)} \int_0^t \frac{s^2\sqrt[7]{|x(s)|}e^{-x(s)} \sin x(s)}{(1+s)(1+x^2(s))} \ln\left(1+\frac{\sqrt[5]{|x(s)|}}{3}\right) ds. \tag{12}$$

Let $X := \mathbb{R}$, $t \in [0,1]$, $T := 1$ and $\tau := 7$. Now in order to investigate the conditions of Theorem 3.1 we have

(a) Define the functions $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ given by $f(t,x) = \arctan\left(e^{3t}\right) + \frac{|x(t)|}{\left(1+7\sqrt[5]{8|x(t)|}\right)^5}$, f is continuous and

$$|f(t,x) - f(t,y)| \leqslant \left| \frac{|x(t)|}{\left(1+7\sqrt[5]{8|x(t)|}\right)^5} - \frac{|y(t)|}{\left(1+7\sqrt[5]{8|y(t)|}\right)^5} \right|.$$

Consider the function $h : [0,\infty) \to [0,\infty)$, $h(t) = \frac{t}{\left(1+7\sqrt[5]{8t}\right)^5}$.

Since $h'(t) = \frac{1}{\left(1+7\sqrt[5]{8t}\right)^6} > 0$ and $h''(t) = \frac{-336}{\sqrt[5]{(8t)^4}\left(1+7\sqrt[5]{8t}\right)^7} < 0$, so h is strictly increasing and concave. Moreover since $h(0) = 0$ and h is concave, then $h(t+s) \leqslant h(t) + h(s)$.

Without loss of generality, we can assume that $|x| \geqslant |y|$. In this case, we obtain

$$\begin{aligned} |f(t,x) - f(t,y)| &\leqslant h(|x|) - h(|y|) \\ &\leqslant h(|x|-|y|) \leqslant h(|x-y|) \\ &= \frac{|x-y|}{\left(1+7\sqrt[5]{8|x-y|}\right)^5}. \end{aligned} \tag{13}$$

Now, by choosing the function $F : (0,\infty) \to (-\infty, 0)$ given by $F(t) = \frac{-1}{\sqrt[5]{t}}$, and the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\phi(t) = 7t$, it is easy to see that the inequality (13) implies that the condition (11) holds.

Indeed we have

$$
\begin{aligned}
\tau + F\left(\left|f(t,x)-f(t,y)\right| + \phi\left(\left|f(t,x)-f(t,y)\right|\right)\right) &\leqslant F\left(\left|x-y\right| + \phi\left(\left|x-y\right|\right)\right)\\
&\Longleftrightarrow 7 + F\left(8\left|f(t,x)-f(t,y)\right|\right) \leqslant F\left(8\left|x-y\right|\right)\\
&\Longleftrightarrow 7 - \frac{1}{\sqrt[5]{8\left|f(t,x)-f(t,y)\right|}} \leqslant \frac{-1}{\sqrt[5]{8\left|x-y\right|}}\\
&\Longleftrightarrow \frac{1+7\sqrt[5]{8\left|x-y\right|}}{\sqrt[5]{8\left|x-y\right|}} \leqslant \frac{1}{\sqrt[5]{8\left|f(t,x)-f(t,y)\right|}}\\
&\Longleftrightarrow \sqrt[5]{8\left|f(t,x)-f(t,y)\right|} \leqslant \frac{\sqrt[5]{8\left|x-y\right|}}{1+7\sqrt[5]{8\left|x-y\right|}}\\
&\Longleftrightarrow \left|f(t,x)-f(t,y)\right| \leqslant \frac{\left|x-y\right|}{\left(1+7\sqrt[5]{8\left|x-y\right|}\right)^{5}}.
\end{aligned}
$$

(b) Define the continuous operators G , $Q : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$
\begin{aligned}
Gx &= \frac{\sqrt[3]{|x|}}{4\left(2+\cos^4 x\right)},\\
Qx &= \ln\left(1+\frac{\sqrt[5]{|x|}}{3}\right).
\end{aligned}
$$

By choosing the strictly continuous functions $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\varphi(t) = \frac{\sqrt[3]{t}}{4}$ and $\psi(t) = \frac{\sqrt[5]{t}}{3}$, we have

$$
\begin{aligned}
|Gx| &\leqslant \varphi(|x|),\\
|Qx| &\leqslant \psi(|x|).
\end{aligned}
$$

(c) Define the continuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ given by

$$
g(t,x) = \frac{t^2\sqrt[7]{|x|}e^{-x}\sin x}{(1+t)\left(1+x^2\right)}.
$$

Considering the increasing function $u \in L^1([0,1],\mathbb{R}^+)$ given by $u(t) = \frac{t^2}{1+t}$, and the increasing and continuous function $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\theta(x) = \sqrt[7]{x}$, and we have

$$|g(t,x)| \leqslant \frac{t^2 \sqrt[7]{|x|}}{(1+t)} = u(t).\theta(|x|).$$

(d) Since $\int_0^1 \frac{t^2}{1+t}dt = \ln 2 - \frac{1}{2}$, so we have

$$\begin{aligned}
\lim_{\gamma\to\infty} \frac{\varphi(\gamma)\psi(\gamma)\theta(\gamma)\|u\|_1}{\gamma} &= \lim_{\gamma\to\infty} \frac{\frac{\sqrt[3]{\gamma}}{4}\frac{\sqrt[5]{\gamma}}{3}\sqrt[7]{\gamma}\int_0^1 \frac{t^2}{1+t}dt}{\gamma} \\
&= \frac{1}{12}\left(\ln 2 - \frac{1}{2}\right)\lim_{\gamma\to\infty}\left(\frac{1}{\gamma}\right)^{\frac{34}{105}} = 0 < 1.
\end{aligned}$$

So all the conditions of Theorem 3.1 are satisfied and Eq.(12) has at least one solution in $C([0,1],\mathbb{R})$.

Finally, we present another example.

Example 3.3. We consider the following Volterra integral equation

$$x(t) = 4\cos t + \frac{|x(t)|}{\left(1+\frac{1}{5}\sqrt[4]{|x(t)|}\right)^4} + \frac{12e^{x(t)}\sqrt{|x(t)|}}{1+e^{4x(t)}}\int_0^t \frac{\cos s\sqrt[4]{|x(s)|}\ln\left(1+200\sqrt[6]{|x(s)|}\right)}{(1+s)(1+x^2(s))}ds, \tag{14}$$

where, $X := [-5,5]$, $t \in [0,2]$, $T := 2$ and $\tau = \frac{1}{5}$. Now we examine all the conditions of Theorem 3.1:

(a) Define the function $f : [0,2]\times[-5,5] \to [-5,5]$ given by $f(t,x) = 4\cos t + \frac{|x(t)|}{\left(1+\frac{1}{5}\sqrt[4]{|x(t)|}\right)^4}$, the graph of this function depicted in Figure 1.

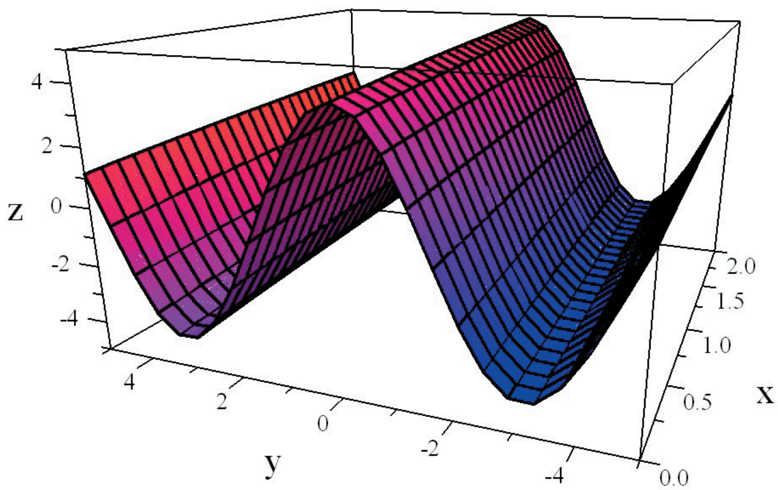

Figure 1. The graph of $f(t,x) = 4\cos t + \frac{|x|}{\left(1+\frac{1}{5}\sqrt[4]{|x|}\right)^4}$

It is easy to see that f is continuous and

$$|f(t,x) - f(t,y)| \leqslant \left| \frac{|x(t)|}{\left(1+\frac{1}{5}\sqrt[4]{|x(t)|}\right)^4} - \frac{|y(t)|}{\left(1+\frac{1}{5}\sqrt[4]{|y(t)|}\right)^4} \right|.$$

Now we define the function $k : [0,\infty) \to [0,\infty), k(t) = \frac{t}{\left(1+\frac{1}{5}\sqrt[4]{t}\right)^4}$. With the same calculations as the previous example, it can be shown that k is strictly increasing and concave, so $k(t+s) \leqslant k(t) + k(s)$ .Without loss of generality, we can assume that $|x| \geqslant |y|$. In this case, we obtain

$$\begin{aligned} |f(t,x) - f(t,y)| &\leqslant k(|x|) - k(|y|) \\ &\leqslant k(|x| - |y|) \\ &\leqslant k(|x-y|) \\ &= \frac{|x-y|}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|}\right)^4}. \end{aligned}$$

So we have

$$|f(t,x)-f(t,y)|+|f(t,x)-f(t,y)|^2 \leqslant \frac{|x-y|}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|}\right)^4}+\frac{|x-y|^2}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|}\right)^8}.$$

Now we show that

$$\frac{|x-y|}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|}\right)^4}+\frac{|x-y|^2}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|}\right)^8} \leqslant \frac{|x-y|+|x-y|^2}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|+|x-y|^2}\right)^4}.$$

For this purpose, we show that for each $a \in [0,5]$ the following inequality is established,

$$P(a) := \frac{a+a^2}{\left(1+\frac{1}{5}\sqrt[4]{a+a^2}\right)^4} - \frac{a}{\left(1+\frac{1}{5}\sqrt[4]{a}\right)^4} - \frac{a^2}{\left(1+\frac{1}{5}\sqrt[4]{a}\right)^8} \geqslant 0.$$

Indeed, by plotting the function of P, it is observed that the above inequality is always true. See Figure 2.

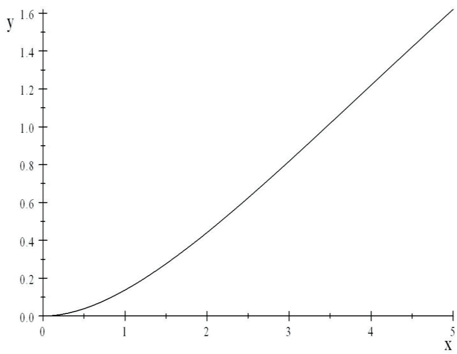

Figure 2. The graph of $P(a)$

Therefore, with the above discussion, we have the following inequality

$$|f(t,x)-f(t,y)|+|f(t,x)-f(t,y)|^2 \leqslant \frac{|x-y|+|x-y|^2}{\left(1+\frac{1}{5}\sqrt[4]{|x-y|+|x-y|^2}\right)^4}.$$

Hence

$$\tau+F\left(|f(t,x)-f(t,y)|+\phi\left(|f(t,x)-f(t,y)|\right)\right) \leqslant F\left(|x-y|+\phi\left(|x-y|\right)\right),$$

where the function $F:(0,\infty)\to(-\infty,0)$ given by $F(t)=-\frac{1}{\sqrt[4]{t}}$, and the function $\phi:\mathbb{R}^+\to\mathbb{R}^+$ given by $\phi(t)=t^2$, so the condition (11) holds.

(b) Define the continuous operators $G, Q : C([0,2],[-5,5]) \to C([0,2],[-5,5])$ given by

$$\begin{aligned} Gx &= \frac{12e^x\sqrt{|x|}}{1+e^{4x}}, \\ Qx &= \frac{\ln\left(1+200\sqrt[6]{|x|}\right)}{1+x^2}. \end{aligned}$$

By choosing the strictly continuous functions $\varphi,\psi:\mathbb{R}^+\to\mathbb{R}^+$ given by $\varphi(t)=12\sqrt{t}$ and $\psi(t)=200\sqrt[6]{t}$, we have

$$\begin{aligned} |Gx| &\leqslant \varphi(|x|), \\ |Qx| &\leqslant \psi(|x|). \end{aligned}$$

(c) Define the continuous function $g:[0,2]\times[-5,5]\to[-5,5]$ given by

$$g(t,x)=\cos t\frac{\sqrt[4]{|x|}}{(1+t)}.$$

Considering the increasing function $u\in L^1([0,1],\mathbb{R}^+)$ given by $u(t)=\dfrac{1}{1+t}$, and the increasing and continuous function $\theta:\mathbb{R}^+\to\mathbb{R}^+$ given by $\theta(x)=\sqrt[4]{x}$, and we have

$$|g(t,x)| \leqslant \frac{\sqrt[4]{|x|}}{(1+t)} = u(t).\theta(|x|).$$

(d) Since $\int_0^2 \dfrac{1}{1+t}dt = \ln 3$, so we have

$$\begin{aligned}\lim_{\gamma\to\infty}\frac{\varphi(\gamma)\,\psi(\gamma)\,\theta(\gamma)\,\|u\|_1}{\gamma} &= \lim_{\gamma\to\infty}\frac{2400\sqrt{\gamma}\sqrt[6]{\gamma}\sqrt[4]{\gamma}\ln 3}{\gamma}\\ &= 2400\times\ln 3\lim_{\gamma\to\infty}\frac{1}{\sqrt[12]{\gamma}} = 0 < 1.\end{aligned}$$

So the all conditions of Theorem 3.1 are satisfied and Eq.(14) has at least one solution in $C([0,2],[-5,5])$.

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