Some Common Fixed Point of Two Families of Weakly Compatible Self-Maps on Quasi-Metric Spaces

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Abstract. In this paper, we find the conditions guaranteeing the existence of a unique common fixed point of two families of weakly compatible self-maps on quasi-metric spaces.

1. Introduction

Through out this paper, ρ denotes a quasi-metric on a nonempty set X; that is, a real valued function ρ on $X \times X$ such that for every $x, y, z \in X$,

(i) $\rho(x, y) \geqslant 0$;

(ii) $x = y$ if and only if $\rho(x, y) = \rho(y, x) = 0$;

(iii) $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$.

A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is called $\rho-convergence$ at a point $x \in X$ if for every $\varepsilon > 0$ there is an integer n_0 such that $n \geqslant n_0$ implies that $\rho(x, x_n) < \varepsilon$. It is said to be $\rho-Cauchy$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ if $n_0 \leqslant n \leqslant m$. A quasi-metric space (X, ρ) is called $\rho-complete$ if every $\rho-$Cauchy sequence in X is $\rho-$convergent. A point $x_0 \in X$ is called a *limit point* of set $E \subseteq X$ if there exists a sequence $\{x_n\}$ in E such that

$$\lim_{n \to \infty} \rho(x_0, x_n) = 0.$$

We denote by E' the set of all limit points of E in X, and set

$$\overline{E} = E \cup E'.$$

A self-mapping A on a quasi-metric space (X, ρ) is called $\rho-continuous$ at $x_0 \in X$ if

$$\lim_{n \to \infty} \rho(A(x_0), A(x_n)) = \lim_{n \to \infty} \rho(A(x_n), A(x_0)) = 0,$$

when for any sequence $\{x_n\}$ in X

$$\lim_{n \to \infty} \rho(x_0, x_n) = \lim_{n \to \infty} \rho(x_n, x_0) = 0.$$

Also, self-mappings A and S of a quasi-metric space (X, ρ) is said to be $\rho-compatible$ if

$$\lim_{n \to \infty} \rho(SAx_n, ASx_n) = \lim_{n \to \infty} \rho(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} \rho(x_0, Ax_n) = \lim_{n \to \infty} \rho(x_0, Sx_n) = 0$$

for some $x_0 \in X$. In particular, the pair (A, S) is said to be *weakly compatible* if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Schellekens [18] introduced the concept of quasi-metric spaces as a generalization of the concept of metric spaces. Quasi-metric spaces have some applications in the study of computer science; for example see [7,

9, 17] for the applications of this theory to the asymptotic complexity analysis of Divide and Conquer algorithms. Some other authors extended the fixed point theorems in metric spaces to quasi-metric spaces [4, 6, 10, 11, 15, 16]. For instance, Hick [10] proved if there exists $0 \leqslant \gamma < 1$ such that

$$\rho(Ax, Ay) \leqslant \gamma \max\{\rho(x, y), \rho(x, Ax), \rho(y, Ay), 1/2[\rho(x, Ay) + \rho(y, Ax)]\},$$

then A has a fixed point. He also proved a fixed point theorem for self-mappings A of a $\rho-$complete quasi-metric (X, ρ) which satisfying the following condition.

$$\rho(y, Ay) \leqslant \phi(y) - \phi(Ay),$$

where ϕ is a positive function on X. Ciric [4] generalized this result by proving the following common fixed point theorem.

Theorem 1.1. *Suppose $A, S : X \to X$ and $\phi : X \to [0, \infty)$, where X is a complete quasi-metric space. Let there is $x_0 \in X$ such that*

$$\rho(y, Ay) + \rho(Ay, SAy) \leqslant \phi(y) - \phi(SAy)$$

for all $y \in \{x_0, Ax_0, SAx_0, A(SA)x_0, ..., (SA)^n x_0, A(SA)^n x_0, ...\}$.
If $G_1(x) = \rho(x, Ax)$ and $G_2(x) = \rho(x, Sx)$ are $(S, A)-$orbitally weak lower semi-continuous relative to x_0, then $Ap = p = Sp$ for some $p \in X$.

Jungck [12] and Jungck and Rhoades [13] introduced the notions of compatible and weakly compatible mappings on metric spaces. These notions are a generalization of the notion of commuting self-mappings. Using concepts of compatible and weakly compatible mappings on metric spaces, Singh and Jain [19] proved the following result.

Theorem 1.2. *Let P_i and Q_j be self-mappings of a complete metric space $(\mathcal{X}, d)$ for $i = 1, ..., 4$ and $j = 0, 1$. If*

(i) $Q_0(\mathcal{X}) \subseteq P_1 P_3(\mathcal{X}), Q_1(\mathcal{X}) \subseteq P_2 P_4(\mathcal{X})$.

(ii) $P_2 P_4 = P_4 P_2, P_1 P_3 = P_3 P_1, Q_0 P_4 = P_4 Q_0, Q_1 P_3 = P_3 Q_1$.

(iii) *for every* $x, y \in \mathcal{X}$ *and for some* $0 < \gamma < 1$

$$
\begin{aligned}
(Q_0 x, Q_1 y) \leqslant \ & \gamma \max\{d(Q_0 x, P_2 P_4 x), d(Q_1 y, P_1 P_3 y), d(P_2 P_4 x, P_1 P_3 y), \\
& 1/2[d(Q_0 x, P_1 P_3 y) + d(Q_1 y, P_2 P_4 x)]\}
\end{aligned}
\tag{1}
$$

 (iv) *the pair* $(Q_0, P_2 P_4)$ *is compatible and the pair* $(Q_1, P_1 P_3)$ *is weakly compatible.*

(v) *either* $P_2 P_4$ *or* Q_0 *is continuous.*

Then P_i *and* Q_j *have a unique common fixed point for* $i = 1, ..., 4$ *and* $j = 0, 1$.

Ciric et al.[5] obtained an extension of Theorem 1.2. In fact, they proved the theorem for a countable family of compatible self-mappings of a complete metric space by replacing relation (1) by

$$
\begin{aligned}
d(Q_0 x, Q_1 y) \leqslant \ & \max\{\varphi(d(Q_0 x, \pi_{i=1}^n P_{2i} x)), \varphi(d(Q_1 y, \pi_{i=1}^n P_{2i-1} y)), \\
& \varphi(d(\pi_{i=1}^n P_{2i} x, \pi_{i=1}^n P_{2i-1} y)), \varphi(1/2[d(Q_0 x, \pi_{i=1}^n P_{2i-1} y)) \\
+ \ & \varphi(d(Q_1 y, \pi_{i=1}^n P_{2i} x)])\},
\end{aligned}
\tag{2}
$$

where $\pi_{i=\ell}^m P_i = P_\ell P_{\ell+1} ... P_m$ and φ is an element of Φ, the set of continuous non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

In this paper, we investigate the question and prove an analogue of Ciric et al. [5] for quasi-metric spaces.

2.   Main Results

We commence this section with the main result of the paper.

Theorem 2.1. *Let* $S_1, S_2, \ldots, S_{2n}, A_0$ *and* A_1 *be self-mappings of a* $\rho-$*complete quasi-metric space* (X, d) *such that*

(i) $A_0(X) \subseteq \pi_{i=1}^n S_{2i-1}(X)$ and $A_1(X) \subseteq \pi_{i=1}^n S_{2i}(X)$;

(ii) $\pi_{i=1}^\ell S_{2i} \pi_{i=\ell+1}^n S_{2i} = \pi_{i=\ell+1}^n S_{2i} \pi_{i=1}^\ell S_{2i}$ *for* $\ell = 1, ..., n-1$;

(iii) $A_0(\pi_{i=\ell}^n S_{2i}) = (\pi_{i=\ell}^n S_{2i}) A_0$ *for* $\ell = 2, ..., n$;

(iv) $\pi_{i=1}^{\ell} S_{2i-1} \pi_{i=\ell+1}^{n} S_{2i-1} = \pi_{i=\ell+1}^{n} S_{2i-1} \pi_{i=1}^{\ell} S_{2i-1}$ *for $\ell = 1, ..., n-1$;*

(v) $A_1(\pi_{i=\ell}^{n} S_{2i-1}) = (\pi_{i=\ell}^{n} S_{2i-1}) A_1$ *for $\ell = 2, ..., n$;*

(vi) $\pi_{i=1}^{n} S_{2i}$ *or A_0 is $\rho-$continuous;*

(vii) *the pair $(A_0, \pi_{i=1}^{n} S_{2i})$ is $\rho-$compatible and pair $(A_1, \pi_{i=1}^{n} S_{2i-1})$ is weakly compatible;*

(viii) *there exists $\varphi \in \Phi$ such that for every $u, v \in X$, $x \in \overline{\pi_{i=1}^{n} S_{2i-1}(X)}$ and $y \in \overline{\pi_{i=1}^{n} S_{2i}(X)}$,*

$$
\begin{aligned}
\rho(A_0 u, y) + \rho(A_1 v, x) \quad \leqslant \quad &\max\{\varphi(\rho(x, A_0 u)), \varphi(\rho(y, A_1 v)), \varphi(\rho(x, y)), \\
&\varphi(1/2[\rho(x, \pi_{i=1}^{n} S_{2i-1} v)) + \rho(y, \pi_{i=1}^{n} S_{2i} u)])\}. \quad (3)
\end{aligned}
$$

Then $S_1, S_2, \ldots, S_{2n}, A_0, A_1$ have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Choose $x_1, x_2 \in X$ such that

$$
A_0 x_0 = \pi_{i=1}^{n} S_{2i-1} x_1 := y_0 \quad \text{and} \quad A_1 x_1 = \pi_{i=1}^{n} S_{2i} x_2 := y_1.
$$

For any $k \in \mathbb{N}$, set

$$
A_0 x_{2k} = \pi_{i=1}^{n} S_{2i-1} x_{2k+1} := y_{2k} \quad \text{and} \quad A_1 x_{2k+1} = \pi_{i=1}^{n} S_{2i} x_{2k+2} := y_{2k+1}.
$$

From properties of φ and condition (viii) we see that

$$
\begin{aligned}
\rho(y_{2k}, y_{2k+1}) \quad + \quad &\rho(y_{2k+1}, y_{2k}) \\
\leqslant \quad &\max\{\varphi(\rho(A_0 x_{2k}, A_0 x_{2k})), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})), \\
&\varphi(\rho(A_0 x_{2k}, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0 x_{2k}, \pi_{i=1}^{n} S_{2i-1} x_{2k+1}) \\
+ \quad &\rho(A_1 x_{2k+1}, \pi_{i=1}^{n} S_{2i} x_{2k})])\} \\
= \quad &\max\{\varphi(\rho(y_{2k}, y_{2k+1})), \varphi(1/2[\rho(y_{2k}, y_{2k}) + \rho(y_{2k+1}, y_{2k-1})])\} \\
\leqslant \quad &\max\{\varphi(\rho(y_{2k}, y_{2k+1})), \varphi(1/2[\rho(y_{2k+1}, y_{2k}) + \rho(y_{2k}, y_{2k-1})])\} \\
\leqslant \quad &\varphi(\max\{\rho(y_{2k}, y_{2k+1}), \rho(y_{2k+1}, y_{2k}), \rho(y_{2k}, y_{2k-1})\}) \\
\leqslant \quad &\varphi(\rho(y_{2k}, y_{2k-1})).
\end{aligned}
$$

This shows that

$$\rho(y_{2k+1}, y_{2k}) \leqslant \varphi(\rho(y_{2k}, y_{2k-1})) \leqslant \rho(y_{2k}, y_{2k-1}) \tag{4}$$

and

$$\rho(y_{2k}, y_{2k+1}) \leqslant \rho(y_{2k}, y_{2k-1}). \tag{5}$$

A similar argument shows that

$$\rho(y_{2k+2}, y_{2k+1}) \leqslant \varphi(\rho(y_{2k+1}, y_{2k})) \leqslant \rho(y_{2k+1}, y_{2k}) \tag{6}$$

and

$$\rho(y_{2k+1}, y_{2k+2}) \leqslant \rho(y_{2k+1}, y_{2k}). \tag{7}$$

By relation (6)–(7), we have

$$0 \leqslant \rho(y_{n+1}, y_n) \leqslant \varphi(\rho(y_n, y_{n-1})) \leqslant \rho(y_n, y_{n-1}) \tag{8}$$

and

$$0 \leqslant \rho(y_n, y_{n+1}) \leqslant \rho(y_n, y_{n-1}) \tag{9}$$

for all $n \in \mathbb{N}$. Hence $\{\rho(y_{n+1}, y_n)\}$ is a non-increasing sequence. Thus there exists $\alpha \geqslant 0$ such that $\lim_{n\to\infty} \rho(y_{n+1}, y_n) = \alpha$. This together with (8) and continuity of ϕ shows that

$$\alpha = \lim_{n\to\infty} \varphi(\rho(y_{n+1}, y_n)) = \varphi(\alpha).$$

So $\alpha = 0$. Thus

$$\lim_{n\to\infty} \rho(y_{n+1}, y_n) = \lim_{n\to\infty} \rho(y_n, y_{n-1}) = 0.$$

From (9) we see that

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

Let ε and δ be positive numbers with $\delta < (\varepsilon - \varphi(\varepsilon))/3$. By

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = \lim_{n\to\infty} \rho(y_{n+1}, y_n) = 0,$$

choose $N \in \mathbb{N}$ such that $\rho(y_n, y_{n+1}) < \delta$ and $\rho(y_{n+1}, y_n) < \delta$ for all $n \geqslant N$. If $k, q \in \mathbb{N}$, then by (viii) we have

$$
\begin{aligned}
\rho(y_{2q+1}, y_{2k+1}) &\leqslant \rho(A_1 x_{2q+1}, A_0 x_{2k+2}) + \rho(A_0 x_{2k+2}, y_{2k+1}) \\
&\leqslant \max\{\varphi(\rho(A_0 x_{2k+2}, A_0 x_{2k+2})), \varphi(\rho(y_{2k+1}, A_1 x_{2q+1})), \\
&\quad \varphi(\rho(A_0 x_{2k+2}, y_{2k+1})), \varphi(1/2[\rho(A_0 x_{2k+2}, \pi_{i=1}^n S_{2i-1} x_{2q+1}) \\
&\quad + \rho(y_{2k+1}, \pi_{i=1}^n S_{2i} x_{2k+2})])\} \\
&= \max\{\varphi(\rho(y_{2k+1}, y_{2q+1})), \varphi(\rho(y_{2k+2}, y_{2k+1})), \\
&\quad \varphi(1/2[\rho(y_{2k+2}, y_{2q}) + \rho(y_{2k+1}, y_{2k+1})])\} \qquad (10) \\
&\leqslant \max\{\varphi(\rho(y_{2k+1}, y_{2q+1})), \varphi(\rho(y_{2k+2}, y_{2k+1})), \\
&\quad \varphi(\rho(y_{2k+2}, y_{2k+1}) + \rho(y_{2k+1}, y_{2q+1}) + \rho(y_{2q+1}, y_{2q})\} \\
&\leqslant \varphi(\rho(y_{2k+2}, y_{2k+1}) + \rho(y_{2k+1}, y_{2q+1}) + \rho(y_{2q+1}, y_{2q})) \\
&\leqslant 2\delta + \rho(y_{2k+1}, y_{2q+1}).
\end{aligned}
$$

From properties of φ and (viii) with $x = y_{2k}, y = A_1 x_{2q+1}, u = x_{2k}$ and $v = x_{2q+1}$, we infer that

$$
\begin{aligned}
\rho(y_{2k}, y_{2q+1}) &\leqslant \rho(A_0 x_{2k}, A_1 x_{2q+1}) + \rho(A_1 x_{2k+1}, y_{2k}) \\
&\leqslant \max\{\varphi(\rho(y_{2k}, A_0 x_{2k}), \varphi(\rho(A_1 x_{2q+1}, A_1 x_{2k+1})), \\
&\quad \varphi(\rho(y_{2k}, A_1 x_{2q+1})), \varphi(1/2[\rho(y_{2k}, \pi_{i=1}^n S_{2i-1} x_{2k+1}) \\
&\quad + \rho(A_1 x_{2q+1}, \pi_{i=1}^n S_{2i} x_{2k})])\} \\
&= \max\{\varphi(\rho(y_{2q+1}, y_{2k+1})), \varphi(\rho(y_{2k}, y_{2q+1})), \\
&\quad \varphi(1/2[\rho(y_{2k}, y_{2k}) + \rho(y_{2q+1}, y_{2k-1})])\} \\
&\leqslant \varphi(t_{n,m}),
\end{aligned}
$$

where

$$
t_{n,m} = \max\{\rho(y_{2q+1}, y_{2k+1}), 1/2(\rho(y_{2q+1}, y_{2k-1})\}.
$$

In view of (10), we conclude that

$$
\begin{aligned}
t_{n,m} &\leqslant \max\{\rho(y_{2q+1}, y_{2k+1}), \\
&\quad \max\{\rho(y_{2q+1}, y_{2k+1}), \rho(y_{2k+1}, y_{2k-1})\}\} \\
&= \max\{\rho(y_{2q+1}, y_{2k+1}), \rho(y_{2k+1}, y_{2k-1})\} \\
&\leqslant \max\{2\delta + \rho(y_{2k+1}, y_{2q+1}), 2\delta\} \\
&= 2\delta + \rho(y_{2k+1}, y_{2q+1}).
\end{aligned}
$$

Now, we prove that if

$$\rho(y_n, y_m) < \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 2\delta \qquad (11)$$

for any $m \geqslant n \geqslant N$, then $t_{n,m} < \varepsilon + 6\delta$. For this end, we consider the following cases.

Case 1. Let $n = 2r$ and $m = 2s$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}
\rho(y_{2r+1}, y_{2s+1}) &\leqslant \rho(y_{2r+1}, y_{2r}) + \rho(y_{2r}, y_{2s}) + \rho(y_{2s}, y_{2s+1}) \\
&\leqslant \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 4\delta.
\end{aligned}$$

Hence $t_{n,m} < \varepsilon + 6\delta$.

Case 2. Let $n = 2r$ and $m = 2s + 1$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}
\rho(y_{2r+1}, y_{2s+1}) &\leqslant \rho(y_{2r+1}, y_{2r}) + \rho(y_{2r}, y_{2s+1}) \\
&\leqslant \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 3\delta.
\end{aligned}$$

So $t_{n,m} < \varepsilon + 6\delta$.

Case 3. Let $n = 2r + 1$ and $m = 2s$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}
\rho(y_{2r+1}, y_{2s+1}) &\leqslant \rho(y_{2r+1}, y_{2s}) + \rho(y_{2s}, y_{2s+1}) \\
&\leqslant \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 3\delta.
\end{aligned}$$

Thus $t_{n,m} < \varepsilon + 6\delta$.

Case 4. Let $n = 2r + 1$ and $m = 2s + 1$ for some $r, s \in \mathbb{N}$. According to (2.), we get $t_{n,m} < \varepsilon + 6\delta$.

By a similar argument as given in [5], we can show that the sequence $\{y_n\}$ is $\rho-$Cauchy. Hence from the $\rho-$completeness of X, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} \rho(z, y_n) = 0$. Hence

$$\begin{aligned}
\lim_{k\to\infty} \rho(z, A_1 x_{2k+1}) &= \lim_{k\to\infty} \rho(z, \pi_{i=1}^n S_{2i-1} x_{2k+1}) \\
&= \lim_{k\to\infty} \rho(z, A_0 x_{2k}) = \lim_{k\to\infty} \rho(z, \pi_{i=1}^n S_{2i} x_{2k}) = 0 \quad (12)
\end{aligned}$$

and so

$$\lim_{k\to\infty} (\rho(A_0 x_{2k}, z) \;+\; \rho(A_1 x_{2k+1}, z))$$

$$\leqslant \lim_{k\to\infty} (\max\{\varphi(\rho(z, A_0 x_{2k})), \varphi(\rho(z, A_1 x_{2k+1})), \varphi(\rho(z, z)),$$

$$\varphi(1/2[\rho(z, \pi_{i=1}^n S_{2i-1} x_{2k+1}) + \rho(z, \pi_{i=1}^n S_{2i} x_{2k})])\})$$

$$=\;\; 0.$$

Thus

$$\lim_{k\to\infty} \rho(A_1 x_{2k+1}, z) \;=\; \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i-1} x_{2k+1}, z)$$

$$=\;\; \lim_{k\to\infty} \rho(A_0 x_{2k}, z) = \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} x_{2k}, z) \quad (13)$$

$$=\;\; 0.$$

Now, we consider the following cases.

Case I. Let $\pi_{i=1}^n S_{2i}$ is $\rho-$continuous. From (12) and (13) we see that

$$\lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} \pi_{i=1}^n S_{2i} x_{2k}, \pi_{i=1}^n S_{2i} z) = \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} z, \pi_{i=1}^n S_{2i} \pi_{i=1}^n S_{2i} x_{2k}) = 0$$

and

$$\lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} A_0 x_{2k}, \pi_{i=1}^n S_{2i} z) = \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} z, \pi_{i=1}^n S_{2i} A_0 x_{2k}) = 0.$$

Since $(A_0, \pi_{i=1}^n S_{2i})$ is $\rho-$compatible, we have

$$\lim_{k\to\infty} \rho(A_0 \pi_{i=1}^n S_{2i} x_{2k}, \pi_{i=1}^n S_{2i} z) = \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} z, A_0 \pi_{i=1}^n S_{2i} x_{2k}) = 0.$$

Step 1. From (viii) with $u = \pi_{i=1}^n S_{2i} x_{2k}$, $v = x_{2k+1}$, $x = A_0 \pi_{i=1}^n S_{2i} x_{2k}$ and $y = A_1 x_{2k+1}$, we have

$$\rho(A_0 \pi_{i=1}^n S_{2i} x_{2k}, A_1 x_{2k+1}) \;+\; \rho(A_1 x_{2k+1}, A_0 \pi_{i=1}^n S_{2i} x_{2k})$$

$$\leqslant \max\{\varphi(\rho(A_0 \pi_{i=1}^n S_{2i} x_{2k}, A_0 \pi_{i=1}^n S_{2i} x_{2k})),$$

$$\varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})),$$

$$\varphi(\rho(A_0 \pi_{i=1}^n S_{2i} x_{2k}, A_1 x_{2k+1})),$$

$$\varphi(1/2[\rho(A_0 \pi_{i=1}^n S_{2i} x_{2k}, \pi_{i=1}^n S_{2i-1} x_{2k+1})$$

$$+\;\; \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i} \pi_{i=1}^n S_{2i} x_{2k})])\}.$$

Letting $k \to \infty$, we see that

$$
\begin{aligned}
\rho(\pi_{i=1}^n S_{2i}z, z) \;+\;& \rho(z, \pi_{i=1}^n S_{2i}z) \\
\leqslant\;& \max\{\varphi(\rho(\pi_{i=1}^n S_{2i}z, z)), \\
& \varphi(1/2[\rho(\pi_{i=1}^n S_{2i}z, z) + \rho(z, \pi_{i=1}^n S_{2i}z)])\} \\
\leqslant\;& \varphi(\max\{\rho(z, \pi_{i=1}^n S_{2i}z), \rho(\pi_{i=1}^n S_{2i}z, z)\}).
\end{aligned}
$$

If

$$
\max\{\rho(z, \pi_{i=1}^n S_{2i}z), \rho(\pi_{i=1}^n S_{2i}z, z)\} = \rho(z, \pi_{i=1}^n S_{2i}z), \qquad (14)
$$

then

$$
\rho(z, \pi_{i=1}^n S_{2i}z) \leqslant \rho(\pi_{i=1}^n S_{2i}z, z) + \rho(z, \pi_{i=1}^n S_{2i}z) \leqslant \varphi(\rho(z, \pi_{i=1}^n S_{2i}z)).
$$

So $\rho(z, \pi_{i=1}^n S_{2i}z) = 0$. By (14), we have

$$
0 \leqslant \rho(\pi_{i=1}^n S_{2i}z, z) \leqslant \rho(z, \pi_{i=1}^n S_{2i}z) = 0.
$$

It follows that

$$
\rho(\pi_{i=1}^n S_{2i}z, z) = \rho(z, \pi_{i=1}^n S_{2i}z) = 0.
$$

Thus $\pi_{i=1}^n S_{2i}z = z$. Similarly, if

$$
\max\{\rho(z, \pi_{i=1}^n S_{2i}z), \rho(\pi_{i=1}^n S_{2i}z, z)\} = \rho(\pi_{i=1}^n S_{2i}z, z),
$$

then $\pi_{i=1}^n S_{2i}z = z$.

Step 2. Put $u = z$, $v = x_{2k+1}$, $x = A_0 z$ and $y = A_1 x_{2k+1}$ in condition (viii). Then

$$
\begin{aligned}
\rho(A_0 z, A_1 x_{2k+1}) \;+\;& \rho(A_1 x_{2k+1}, A_0 z) \\
\leqslant\;& \max\{\varphi(\rho(A_0 z, A_0 z)), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})), \\
& \varphi(\rho(A_0 z, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0 z, \pi_{i=1}^n S_{2i-1} x_{2k+1}) \\
\;+\;& \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i}z)])\}.
\end{aligned}
$$

Letting $k \to \infty$, we obtain

$$
\begin{aligned}
\rho(A_0 z, z) \;+\;& \rho(z, A_0 z) \\
\leqslant\;& \max\{\varphi(\rho(A_0 z, z)), \varphi(1/2[\rho(A_0 z, z) + \rho(z, \pi_{i=1}^n S_{2i}z)])\}.
\end{aligned}
$$

Since $\pi_{i=1}^{n} S_{2i} z = z$ and φ is non-decreasing, it follows that

$$\rho(A_0 z, z) + \rho(z, A_0 z) \leqslant \varphi(\rho(A_0 z, z)). \qquad (15)$$

This implies that $\rho(A_0 z, z) = 0$. From (3) and the fact that $\varphi(0) = 0$ we see that $\rho(z, A_0 z) = 0$. Therefore,

$$A_0 z = \pi_{i=1}^{n} S_{2i} z = z.$$

Step 3. From (viii) with $u = \pi_{i=2}^{n} S_{2i} z, v = x_{2k+1}, x = A_0 \pi_{i=2}^{n} S_{2i} z$ and $y = A_1 x_{2k+1}$, we see that

$$
\begin{aligned}
\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_1 x_{2k+1}) \quad + \quad & \rho(A_1 x_{2k+1}, A_0 \pi_{i=2}^{n} S_{2i} z) \\
\leqslant \quad & \max\{\varphi(\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_0 \pi_{i=2}^{n} S_{2i} z)), \\
& \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})), \\
& \varphi(\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_1 x_{2k+1})), \\
& \varphi(1/2[\rho(A_0 \pi_{i=2}^{n} S_{2i} z, \pi_{i=1}^{n} S_{2i-1} x_{2k+1}) \\
+ \quad & \rho(A_1 x_{2k+1}, \pi_{i=1}^{n} S_{2i} \pi_{i=2}^{n} S_{2i} z)])\}.
\end{aligned}
$$

Since $A_0 z = z$, by letting $k \to \infty$, we get

$$
\begin{aligned}
\rho(\pi_{i=2}^{n} S_{2i} z, z) \quad + \quad & \rho(z, \pi_{i=2}^{n} S_{2i} z) \\
\leqslant \quad & \max\{\varphi(\rho(\pi_{i=2}^{n} S_{2i} z, z)), \varphi(1/2[\rho(\pi_{i=2}^{n} S_{2i} z, z) \\
+ \quad & \rho(z, \pi_{i=2}^{n} S_{2i} z)])\} \\
\leqslant \quad & \varphi(\max\{\rho(\pi_{i=2}^{n} S_{2i} z, z), \rho(z, \pi_{i=2}^{n} S_{2i} z)\}).
\end{aligned}
$$

This shows that $\pi_{i=2}^{n} S_{2i} z = z$. Thus $S_2(\pi_{i=2}^{n} S_{2i} z) = S_2 z$ and so $S_2 z = \pi_{i=1}^{n} S_{2i} z = z$. Continuing this procedure, we obtain $A_0 z = S_{2i} z = z$ for $i = 1, ..., n$.

Step 4. By condition (i), there exists $v \in X$ such that

$$z = A_0 z = \pi_{i=1}^{n} S_{2i-1} v.$$

Putting $u = x_{2k}, x = A_0 x_{2k}$ and $y = A_1 v$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0 x_{2k}, A_1 v) \quad + \quad & \rho(A_1 v, A_0 x_{2k}) \\
\leqslant \quad & \max\{\varphi(\rho(A_0 x_{2k}, A_0 x_{2k})), \varphi(\rho(A_1 v, A_1 v)), \\
& \varphi(\rho(A_0 x_{2k}, A_1 v)), \varphi(1/2[\rho(A_0 x_{2k}, \pi_{i=1}^{n} S_{2i-1} v) \\
+ \quad & \rho(A_1 v, \pi_{i=1}^{n} S_{2i} x_{2k})])\}.
\end{aligned}
$$

Letting $k \to \infty$, we find

$$
\begin{aligned}
\rho(z, A_1 v) \quad + \quad & \rho(A_1 v, z) \\
\leqslant \quad & \max\{\varphi(\rho(z, A_1 v)), \varphi(1/2[\rho(z, \pi_{i=1}^n S_{2i-1} v) + \rho(A_1 v, z)])\} \\
= \quad & \max\{\varphi(\rho(z, A_1 v)), \varphi(1/2[\rho(z, z) + \rho(A_1 v, z)])\} \\
\leqslant \quad & \varphi(\max\{\rho(z, A_1 v), \rho(A_1 v, z)\}).
\end{aligned}
$$

Hence $A_1 v = z$ and therefore

$$
\pi_{i=1}^n S_{2i-1} v = A_1 v = z.
$$

As $(A_1, \pi_{i=1}^n S_{2i-1})$ is weakly compatible, we have

$$
\pi_{i=1}^n S_{2i-1} A_1 v = A_1 \pi_{i=1}^n S_{2i-1} v.
$$

Thus $\pi_{i=1}^n S_{2i-1} z = A_1 z$.

Step 5. Putting $u = x_{2k}$, $v = z$, $x = A_0 x_{2k}$ and $y = A_1 z$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0 x_{2k}, A_1 z) \quad + \quad & \rho(A_1 z, A_0 x_{2k}) \\
\leqslant \quad & \max\{\varphi(\rho(A_0 x_{2k}, A_0 x_{2k})), \varphi(\rho(A_1 z, A_1 z)), \\
& \varphi(\rho(A_0 x_{2k}, A_1 z)), \varphi(1/2[\rho(A_0 x_{2k}, \pi_{i=1}^n S_{2i-1} z) \\
+ \quad & \rho(A_1 z, \pi_{i=1}^n S_{2i} x_{2k})])\}.
\end{aligned}
$$

Letting $k \to \infty$, we get

$$
\begin{aligned}
\rho(z, A_1 z) \quad + \quad & \rho(A_1 z, z) \\
\leqslant \quad & \max\{\varphi(\rho(z, A_1 z)), \varphi(1/2[\rho(z, A_1 z) + \rho(A_1 z, z)])\} \\
\leqslant \quad & \varphi(\max\{\rho(z, A_1 z), \rho(A_1 z, z)\}).
\end{aligned}
$$

So $\pi_{i=1}^n S_{2i-1} z = A_1 z = z$.

Step 6. Putting $u = x_{2k}, v = \pi_{i=2}^n S_{2i-1} z, x = A_0 x_{2k}$ and $y = A_1 \pi_{i=2}^n S_{2i-1} z$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0 x_{2k}, A_1 \pi_{i=2}^n S_{2i-1} z) \quad + \quad & \rho(A_1 \pi_{i=2}^n S_{2i-1} z, A_0 x_{2k}) \\
\leqslant \quad & \max\{\varphi(\rho(A_0 x_{2k}, A_0 x_{2k})), \varphi(\rho(A_1 \pi_{i=2}^n S_{2i-1} z, \\
& A_1 \pi_{i=2}^n S_{2i-1} z)), \varphi(\rho(A_0 x_{2k}, A_1 \pi_{i=2}^n S_{2i-1} z)), \\
& \varphi(1/2[\rho(A_0 x_{2k}, \pi_{i=1}^n S_{2i-1} \pi_{i=2}^n S_{2i-1} z) \\
+ \quad & \rho(A_1 \pi_{i=2}^n S_{2i-1} z, \pi_{i=1}^n S_{2i} x_{2k})])\}.
\end{aligned}
$$

Letting $k \to \infty$ shows that

$$
\begin{aligned}
\rho(z, \pi_{i=2}^n S_{2i-1}z) \quad + \quad & \rho(\pi_{i=2}^n S_{2i-1}z, z) \leqslant \max\{\varphi(\rho(z, \pi_{i=2}^n S_{2i-1}z)), \\
& \varphi(1/2[\rho(z, \pi_{i=2}^n S_{2i-1}z) + \rho(\pi_{i=2}^n S_{2i-1}z, z)])\} \\
\leqslant \quad & \varphi(\max\{\rho(z, \pi_{i=2}^n S_{2i-1}z), \rho(\pi_{i=2}^n S_{2i-1}z, z)\}).
\end{aligned}
$$

So $\pi_{i=2}^n S_{2i-1}z = z$ and hence $S_3 z = z$. Continuing this procedure, we have $A_1 z = S_{2i-1}z$ for $i = 1, ..., n$. Thus $A_0 z = A_1 z = S_i z = z$ for $i = 1, ..., 2n$. That is, z is a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$.

Case II. Let A_0 be ρ−continuous. By (12) and (13),

$$
\lim_{k \to \infty} \rho(A_0{}^2 x_{2k}, A_0 z) = \lim_{k \to \infty} \rho(A_0 z, A_0{}^2 x_{2k}) = 0.
$$

Since $(A_0, \pi_{i=1}^n S_{2i})$ is ρ−compatible, we have

$$
\lim_{k \to \infty} \rho(\pi_{i=1}^n S_{2i} A_0 x_{2k}, A_0 z) = \lim_{k \to \infty} \rho(A_0 z, \pi_{i=1}^n S_{2i} A_0 x_{2k}) = 0.
$$

Step 7. Putting $u = A_0 x_{2k}, v = x_{2k+1}, x = A_0^2 x_{2k}$ and $y = A_1 x_{2k+1}$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0^2 x_{2k}, A_1 x_{2k+1}) \quad + \quad & \rho(A_1 x_{2k+1}, A_0^2 x_{2k}) \\
\leqslant \quad & \max\{\varphi(\rho(A_0^2 x_{2k}, A_0^2 x_{2k})), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})), \\
& \varphi(\rho(A_0^2 x_{2k}, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0^2 x_{2k}, \pi_{i=1}^n S_{2i-1} x_{2k+1}) \\
+ \quad & \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i} A_0 x_{2k})])\}.
\end{aligned}
$$

Letting $k \to \infty$ gives that

$$
\begin{aligned}
\rho(A_0 z, z) \quad + \quad & \rho(z, A_0 z) \\
\leqslant \quad & \max\{\varphi(\rho(A_0 z, z)), \varphi(1/2[\rho(A_0 z, z) + \rho(z, A_0 z)])\} \\
\leqslant \quad & \varphi(\max\{\rho(A_0 z, z), \rho(z, A_0 z)\}).
\end{aligned}
$$

Hence $A_0 z = z$ and therefore, for every $i = 1, ..., n$

$$
A_1 z = S_{2i-1}z = z,
$$

by the same argument that was used in Steps 4–6.

Step 8. By condition (i), there exists $u \in X$ such that

$$z = A_1 z = \pi_{i=1}^n S_{2i} u.$$

Putting $v = x_{2k+1}$, $x = A_0 u$ and $y = A_1 x_{2k+1}$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0 u, A_1 x_{2k+1}) \;\; &+ \;\; \rho(A_1 x_{2k+1}, A_0 u) \\
&\leqslant \;\; \max\{\varphi(\rho(A_0 u, A_0 u)), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})), \\
&\qquad \varphi(\rho(A_0 u, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0 u, \pi_{i=1}^n S_{2i-1} x_{2k+1}) \\
&+ \;\; \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i} u)])\}.
\end{aligned}
$$

Letting $k \to \infty$ and using Step 7, we see that

$$
\begin{aligned}
\rho(A_0 u, z) \;\; &+ \;\; \rho(z, A_0 u) \\
&\leqslant \;\; \max\{\varphi(\rho(A_0 u, z)), \varphi(1/2[\rho(A_0 u, z) + \rho(z, z)])\} \\
&= \;\; \varphi(\rho(A_0 u, z)).
\end{aligned}
$$

This implies that

$$A_0 u = z = \pi_{i=1}^n S_{2i} u.$$

As $(A_0, \pi_{i=2}^n S_{2i})$ is weakly compatible, we have

$$A_0 z = \pi_{i=1}^n S_{2i} z = z.$$

A discussion similar to Step 3 shows that $S_{2i} z = A_0 z = z$ for $i = 1, ..., n$. Thus $A_0 z = A_1 z = S_i z = z$ for $i = 1, ..., 2n$. That is, z is a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$.

To prove the uniqueness theorem, let w be a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$. Hence

$$A_0 w = A_1 w = S_i w = w$$

for $i = 1, ..., 2n$. Putting $u = z, v = w, x = A_0 z$ and $y = A_1 w$ in condition (viii), we have

$$
\begin{aligned}
\rho(A_0 z, A_1 w) \;\; &+ \;\; \rho(A_1 w, A_0 z) \\
&\leqslant \;\; \max\{\varphi(\rho(A_0 z, A_0 z)), \varphi(\rho(A_1 w, A_1 w)), \varphi(\rho(A_0 z, A_1 w)), \\
&\qquad \varphi(1/2[\rho(A_0 z, \pi_{i=1}^n S_{2i-1} w) + \rho(A_1 w, \pi_{i=1}^n S_{2i} z)])\} \\
&\leqslant \;\; \varphi(\max\{\rho(z, w), \rho(w, z)\}).
\end{aligned}
$$

Therefore, $z = w$. That is, z is a unique common fixed point of the mappings. $\square$

We conclude the paper with the following result.

Proposition 2.2. *Let $\{S_{2i}\}_{i=1}^n$ and $\{T_\alpha\}_{\alpha\in J}$ be two families of self-mappings of a $\rho-$complete quasi-metric space (X,d). If there exists $\beta \in J$ such that*

(i) $T_\beta(X) \subseteq \pi_{i=1}^n S_{2i-1}(X)$ and $T_\alpha(X) \subseteq \pi_{i=1}^n S_{2i}(X)$ *for all $\alpha \in J$.*

(ii) $\pi_{i=1}^\ell S_{2i}\pi_{i=\ell+1}^n S_{2i} = \pi_{i=\ell+1}^n S_{2i}\pi_{i=1}^\ell S_{2i}$ *for $\ell = 1, ..., n-1$;*

(iii) $T_\beta(\pi_{i=\ell}^n S_{2i}) = (\pi_{i=\ell}^n S_{2i})T_\beta$ *for $\ell = 2, ..., n$;*

(iv) $\pi_{i=1}^\ell S_{2i-1}\pi_{i=\ell+1}^n S_{2i-1} = \pi_{i=\ell+1}^n S_{2i-1}\pi_{i=1}^\ell S_{2i-1}$ *for $\ell = 1, ..., n-1$;*

(v) $T_\alpha(\pi_{i=\ell}^n S_{2i-1}) = (\pi_{i=\ell}^n S_{2i-1})T_\alpha$ *for $\ell = 2, ..., n$;*

(vi) $\pi_{i=1}^n S_{2i}$ *or T_β is $\rho-$continuous;*

(vii) *the pair $(T_\beta, \pi_{i=1}^n S_{2i})$ is $\rho-$compatible and pair $(T_\alpha, \pi_{i=1}^n S_{2i-1})$ is weakly compatible;*

(viii) *there exists $\varphi \in \Phi$ such that for every $u, v \in X$, $x \in \overline{\pi_{i=1}^n S_{2i-1}(X)}, y \in \overline{\pi_{i=1}^n S_{2i}(X)}$ and $\alpha \in J$*

$$\rho(T_\beta u, y) + \rho(T_\alpha v, x) \leqslant \max\{\varphi(\rho(x, T_\beta u)), \varphi(\rho(y, T_\alpha v)), \varphi(\rho(x, y)),$$
$$\varphi(1/2[\rho(x, \pi_{i=1}^n S_{2i-1}v)) + \rho(y, \pi_{i=1}^n S_{2i}u)])\},$$

then $\{T_\alpha\}$ and $\{S_{2i}\}_{i=1}^n$ have a unique common fixed point in X.

Proof. Let $\alpha_0 \in J$. In Theorem 2.1, set $A_0 = T_\beta$ and $A_1 = T_{\alpha_0}$. Then $T_{\alpha_0}, T_\beta, S_1, ..., S_{2n}$ have a unique fixed point, say z. Now, let $\alpha \in J$. Then

$$\rho(T_\beta z, T_\alpha z) + \rho(T_\alpha z, T_\beta z)$$
$$\leqslant \max\{\varphi(\rho(T_\beta z, T_\beta z)), \varphi(\rho(T_\alpha z, T_\alpha z)), \varphi(\rho(T_\beta z, T_\alpha z)),$$
$$\varphi(1/2[\rho(T_\beta z, \pi_{i=1}^n S_{2i-1}z) + \rho(T_\alpha z, \pi_{i=1}^n S_{2i}z)])\}$$
$$\leqslant \varphi(\max\{\rho(z, T_\alpha z), \rho(T_\alpha z, z)\}).$$

Since $T_\beta z = z$, it follows that $T_\alpha z = z$ for all $\alpha \in J$. $\square$

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