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Bounds for the Dimension of Lie Algebras

H. Arabyani
Neyshabur Branch, Islamic Azad University

Abstract. In 1993, Moneyhun showed that if L is a Lie algebra such that $\dim(L/Z(L)) = n$, then $\dim(L^2) \leqslant \frac{1}{2}n(n-1)$. The author and Saeedi investigated the converse of Moneyhun's result under some conditions. In this paper, We extend their results to obtain several upper bounds for the dimension of a Lie algebra L in terms of dimension of L^2, where L^2 is the derived subalgebra. Moreover, we give an upper bound for the dimension of the c-nilpotent multiplier of a pair of Lie algebras.

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1. Introduction

Let G be an arbitrary group. A theorem of Schur [20] asserts that if the center of a group G has finite index, then the derived subgroup of G is finite. Also, some bounds for the order of the derived subgroup in terms of the index of the center were given by some authors. The best bound was given by Wiegold [21] which shows that if $[G : Z(G)] = n$, then $|G'| \leqslant n^{\frac{1}{2}\log_2^n}$. Several authors have investigated the converse of Schur's theorem. Macdonald [10] gave an explicit bound for $[G : Z_2(G)]$. Also, Podoski and Szegedy [16] improved the Macdonald's bound.

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Neumann [13], proved that the converse of Schur's theorem is true for finitely generated groups. Niroomand [14] showed that the converse of Schur's theorem holds when $G/Z(G)$ considered to be finitely generated and in such case he could obtain a bound for $[G : Z(G)]$ as $[G : Z(G)] \leqslant |G'|^{d(G/Z(G))}$, where $d(G/Z(G))$ denotes the minimal number of generators of $G/Z(G)$ (See [7, 8, 9, 17] for more information).

The Lie algebra analogue of Schur's theorem is well-known. In 1993 K. Moneyhun [12] proved that if L is a Lie algebra such that $\dim L/Z(L) = n$, then $\dim L^2 \leqslant \frac{1}{2}n(n-1)$. In [6], the authors discussed some results concerning the converse of Moneyhun's theorem. In this paper, we give some upper bounds for the dimension of a Lie algebra L in terms of dimension of L^2. Moreover, we obtain an inequality for the dimension of the c-nilpotent multiplier of a pair of Lie algebras.

2. Some Inequalities for the Dimension of Lie Algebras

In this section, we give some upper bounds for the dimensions of Lie algebras. First we discuss some preliminaries which are needed for the proof of our main results.

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. We recall that a Lie algebra L is called a Heisenberg algebra provided that $L^2 = Z(L)$ and $\dim L^2 = 1$. A Heisenberg Lie algebra has odd dimension with a basis $e, e_1, ..., e_{2m}$ subject to the relations $[e_{2i-1}, e_{2i}] = e$ for $i = 1, ..., m$. The Heisenberg Lie algebra of dimension $2m + 1$ is denoted by $H(m)$.

A Lie algebra L is said to abelian, if $[x, y] = 0$, for all $x, y \in L$ and $A(n)$ will denote the abelian Lie algebra of dimension n. Also $\Phi(L)$ denotes the Frattini subalgebra. The Frattini subalgebra $\Phi(L)$ of a Lie algebra L is the intersection of all maximal subalgebras of L.

Recall that a Lie algebra L is called capable, if there exists a Lie algebra K such that $L \cong K/Z(K)$. The following results are proved by the author and Saeedi in [6] which have an important role in the proof of our results.

Theorem 2.1. [6, Corollary 3.9] *Let L be a capable Lie algebra such that* $\dim L^2 = n$. *Then* $\dim(L/Z(L)) \leqslant 2n^2$.

Theorem 2.2.[6, Corollary 3.13] *Suppose that L is a Lie algebra such that* $\dim L^2 = n$ *and $Z_2(L)$ is abelian, then*

$$\dim(L/Z(L)) \leqslant 2(n^3 + n^2).$$

In the following result, we extend [9, Theorem C].

Corollary 2.3. *Let L be a capable nilpotent Lie algebra of finite dimensional such that* $\dim L^2 = 1$. *Then* $\dim L/Z(L) = 2$.

Proof. Let $\dim L = n$. Since L is a nilpotent Lie algebra and $\dim L^2 = 1$, then using [15, Lemma 3.3] we have

$$L \cong H(m) \oplus A(n - 2m - 1), \quad (m \geqslant 1). \tag{1}$$

On the other hand, L is a capable Lie algebra so, $\dim L/Z(L) = 2m \leqslant 2$, by (1) and Theorem 2.1. So, $m = 1$. □

In the following corollary, we generalize [7, Theorem 1.1].

Corollary 2.4. *Let L be a finite dimensional Lie algebra over a field of characteristic zero such that* $\dim L^2 = n$ *and* $\Phi(L) = 0$. *Then*

$$\dim(L/Z(L)) \leqslant 2(n^3 + n^2).$$

Proof. By [11], we have $L^2 \cap Z(L) \leqslant \Phi(L) = 0$. This implies that $Z(L/Z(L)) = 0$. Hence, by Theorem 2.2 we have

$$\dim(L/Z(L)) \leqslant 2(n^3 + n^2). \quad \square$$

The next lemma is useful in the proof of our results.

Lemma 2.5. *Let L be a Lie algebra such that* $\dim L^2 = n$. *Then*

$$\dim(L/C_L(L^2)) \leqslant n^2.$$

Proof. Suppose that $A = \{l_1, l_2, \ldots, l_n\}$ is a basis for L^2. Define the following mapping

$$\begin{aligned} \psi :& L/C_L(L^2) \to \oplus_{i=1}^{n} L^2 \\ &\psi(l + C_L(L^2)) = ([l, l_1], \cdots, [l, l_n]) \end{aligned}$$

where $C_L(L^2) = \{x \in L : [x, y] = 0, \textit{for all } y \in L^2\}$ is the centralizer of L^2 in L. Let $x, y \in L$. $[x, l_i] = [y, l_i]$ if and only if $[x - y, l_i] = 0$ if and only if $x - y \in C_L(L^2)$. Hence, ψ is a well-defined and one-to-one linear transformation. Thus,

$$\dim(L/C_L(L^2)) \leqslant n^2. \quad \square$$

By Lemma 2.5, we prove the following results

Corollary 2.6. *Let L be a Lie algebra such that dimension of L^2 is finite and $C_L(L^2) \leqslant L^2$, where $C_L(L^2)$ is the centralizer of L^2 in L. Then dimension of L is finite. Moreover,*

$$\dim L \leqslant (\dim L^2)(\dim L^2 + 1).$$

Proof. Suppose that $\{l_1, l_2, ..., l_m\}$ is a basis for L^2. We can see that $\dim L/C_L(L^2) \leqslant m^2$, by Lemma 2.5. Since $C_L(L^2) \leqslant L^2$, we have

$$\dim L/L^2 \leqslant \dim L/C_L(L^2) \leqslant m^2. \tag{2}$$

Hence, L/L^2 is a finite dimensional Lie algebra. On the other hand, L^2 is a finite dimensional Lie algebra so, dimension of L is finite. Moreover, using 2, we obtain

$$\dim\ L \leqslant (\dim L^2)(\dim L^2 + 1). \quad \square$$

Example 2.7. Let $L = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_1, x_4] = x_5,\ [x_2, x_5] = x_6,\ [x_3, x_4] = -x_6 \rangle$. A simple observation shows that $C_L(L^2) = \langle x_5, x_6 \rangle \lneqq L^2$ and $\dim L < (\dim L^2)(\dim L^2 + 1)$. Also, if $L = \langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_3,\ [x_1, x_3] = x_4,\ [x_2, x_3] = x_5 \rangle$, then we can see that $C_L(L^2) = L^2$ and

$$\dim L < (\dim L^2)(\dim L^2 + 1).$$

Isaacs in [9] proved that if G is a group such that $Z(G) = 1$ and $|G^{'}| = n$. Then $|G| \leqslant F(n)$, where $F(n)$ is a function. Here, we extend this result.

Theorem 2.8. *Let L be a Lie algebra such that dimension of L^2 is finite and $Z(L) = 0$. Then*

$$\dim\, L \leqslant (\dim L^2)^3 + (\dim L^2)^2.$$

Proof. Let A be an abelian subalgebra of L such that $\dim(L/A) = m$ and $\dim\, L^2 = n$, by [6, Lemma 3.12] we have

$$\dim(L/Z(L)) \leqslant mn + m. \tag{3}$$

On the other hand, $[C_L(L^2), C_L(L^2), L] = 0$, hence; $[C_L(L^2), C_L(L^2)] \leqslant Z(L)$. since $Z(L) = 0$, so $C_L(L^2)$ is an abelian Lie algebra. Also, $\dim(L/C_L(L^2)) \leqslant n^2$ by Lemma 2.5. Therefore, by 3 we have

$$\dim L \leqslant n^2(n+1). \quad \square$$

Let L be a finite dimensional Lie algebra such that $\dim L^2 = 1$ and the set $\{[a, b]\}$ be a basis of L^2 for some $a, b \in L$. Using Jacobi identity, it is easy to see that

$$C_L(a) \cap C_L(b) \subseteq C_L(L^2). \tag{4}$$

In the following theorem we show that for a Lie algebra L with $\dim L^2 = 1$, the equality in 4 does not hold.

Theorem 2.9. *Let L be a finite dimensional Lie algebra with* $\dim L^2 = 1$. *Then $C_L(a) \cap C_L(b)$ is a proper subalgebra of $C_L(L^2)$ for all $a, b \in L$, where $\{[a, b]\}$ is a basis of L^2.*

Proof. If L is a nilpotent Lie algebra of dimension n, then by [15, Lemma 3.3] we have

$$L \cong H(m) \oplus A(n - 2m - 1) \quad (m \geqslant 1).$$

Clearly, $C_L(L^2) = L$. Since $H(m)$ has a basis $\{e_1, \ldots, e_{2m}, e\}$ such that $[e_{2i-1}, e_{2i}] = e$ for $i = 1, \ldots, m$, we have

$$C_L(e_{2i-1}) \cap C_L(e_{2i}) = K \oplus A(n - 2m - 1),$$

where K is a subalgebra of $H(m)$ with basis $\{e_1, \ldots, e_{2m}, e\} \setminus \{e_{2i-1}, e_{2i}\}$ and the result follows.

Now, let L be an n-dimensional non-nilpotent Lie agebra and suppose $\{[a, b]\}$ is a basis of L^2 and $C_L(L^2) = C_L(a) \cap C_L(b)$. Consider a map φ as follows:

$$\begin{array}{rcl} \varphi : L/C_L(L^2) & \longrightarrow & L^2 \oplus L^2 \\ x + C_L(L^2) & \longmapsto & ([a, x], [x, b]). \end{array}$$

Since $C_L(L^2) = C_L(a) \cap C_L(b)$, it is easy to see that $x \in C_L(L^2)$ if and only if $[a, x] = [x, b] = 0$. Therefore, φ is a well-defined and one-to-one linear transformation. Also, we have

$$\varphi((\beta a + \alpha b) + C_L(L^2)) = (\alpha[a, b], \beta[a, b]).$$

Hence, φ is onto. Thus $\dim L/C_L(L^2) = 2$, which is a contradiction by Lemma 2.5. This completes the proof of the theorem. $\square$

Now, we prove the last theorem. Let (N, L) be a pair of Lie algebras, in which N is an ideal in L, if N has a complement in L, then for each free presentation $0 \to R \to F \to L \to 0$ of L, $\mathcal{M}^{(c)}(N, L) = R \cap [S,_c F]/[R,_c F]$ is the c-nilpotent multiplier of the pair (N, L). $(c \geqslant 1)$, where S is an ideal in free Lie algebra F such that $N \cong S/R$. see [1, 2, 3, 4, 18] for more information.

We recall that the subalgebras $Z_c(N, L)$ and $[N,_c L]$, for all $c \geqslant 1$ as follows:

$$Z_c(N, L) = \{n \in N \mid [n, l_1, \ldots, l_c] = 0, \forall l_1, , l_c \in L\},$$

$$[N,_c L] = \langle [n, l_1, \ldots, l_c] \mid n \in N, l_1, \ldots, l_c \in L \rangle,$$

where $[n, l_1, \ldots, l_c] = [\ldots[n, l_1], l_2], \ldots, l_c]$. Moreover, a pair (N, L) is called nilpotent of class k, if $[N,_k L] = 0$ and $[N,_{k-1} L] \neq 0$, for some positive integer k.

In [5], the authors proved the following results.

Lemma 2.10. *Let L and K be two Lie algebras with central subalgebras N and M, respectively. If $\theta : L \to K$ is an epimorphism with $\theta(N) = M$, then*

$$\dim \mathcal{M}^{(c)}(M, K) \leqslant \dim \mathcal{M}^{(c)}(N, L).$$

Lemma 2.11. *Let (M, K) be a pair of nilpotent Lie algebras. Then for each pair (N, L) of finite dimensional Lie algebras with $L/Z_c(N, L) \cong K$ and $N/Z_c(N, L) \cong M$,*

$$\dim([N,_c L] \cap Z_c(N, L)) \leqslant \dim \mathcal{M}^{(c)}(M/[M,_c K], K/[M,_c K]) + \dim([M,_c K])(d(K/Z_c(M, K)) - 1).$$

In the following theorem, we extend a result of Salemkar and Niri [19].

Theorem 2.12. *Let (N, L) be a pair of finite dimensional nilpotent Lie algebras such that N has a complement in L. Then*

$$\dim \mathcal{M}^{(c)}(N, L) \leqslant \dim(N/[N,_c L], L/[N,_c L])$$

$$+(\dim[N,_c L])(d(L/Z_c(N, L)) - 1),$$

where $d(X)$ is the minimal number of generators of a Lie algebra X.

Proof. Let $\sigma : M \to L$ be a c-cover of the pair (N, L). By [19], there exists a Lie algebra H containing M such that

$$\mathcal{M}^{(c)}(N, L) \cong Ker\sigma \subseteq [M,_c H] \cap Z_c(M, H),$$

and

$$(N, L) \cong (M/Ker\sigma, H/Ker\sigma).$$

On the other hand, (M, H) is a pair of nilpotent Lie algebras. Set

$$(P, K) = (\frac{M}{[M,_c H] \cap Z_c(M, H)}, \frac{H}{[M,_c H] \cap Z_c(M, H)}).$$

Using Lemma 2.10 and Lemma 2.11, we obtain

$$\begin{aligned} \dim\mathcal{M}^{(c)}(N, L) &\leqslant \dim([M,_c H] \cap Z_c(M, H)) \\ &\leqslant \dim(\mathcal{M}^{(c)}(P/[M,_c K], K/[M,_c K])) \\ &+ (\dim[P,_c K])(d(K/Z_c(P, K)) - 1) \\ &\leqslant \dim \mathcal{M}^{(c)}(N/[N,_c L], L/[N,_c L]) \\ &+ (\dim[N,_c L])(d(L/Z_c(N, L)) - 1), \end{aligned}$$

as required. $\square$

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Homayoon Arabyani
Assistant Professor of Mathematics
Department of Mathematics
Neyshabur Branch, Islamic Azad University
Neyshabur, Iran
E-mail: h.arabyani@iau-neyshabur.ac.ir
E-mail: arabyani.h@gmail.com