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Approximate Solutions For a Fractional Q-Integro-Difference Equation

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Abstract. We investigate approximate solutions for a nonlinear fractional q-integro-difference equation with some boundary value conditions including the q-derivative of the Caputo type derivation. By providing two examples, we illustrate our main result.

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1. Introduction

The subject of q-difference equations introduced in 1910 by Jackson ([17]). Later, it published many works on fractional q-differential equations (see for example, [7]-[8], [10], [14]-[19] and [21]). It is known that fractional calculus has numerous applications in different sciences such as mechanics, electricity, biology, control theory, signal and image processing ([1]-[6]).

Let $q \in (0,1)$, $a \in \mathbb{R}$ and α be a non-zero real number. Define $[a]_q = \dfrac{1-q^a}{1-q}$ ([17]) and $(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a-bq^k}{a-bq^{\alpha+k}}$. If $b=0$, then it is clear that $a^{(\alpha)} =$

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a^α. The q-Gamma function is defined by $\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$ where $x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$ ([17]). Note that, $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$. The q-derivative of a function f is given by $(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_q f)(0) = \lim_{x\to 0}(D_q f)(x)$ ([1]). Also, the q-derivative of higher order of a function f is defined by $(D_q^0 f)(x) = f(x)$ and $(D_q^n f)(x) = D_q(D_q^{n-1} f)(x)$ for all $n \geqslant 1$ ([1]). Also, the q-integral of a function f defined in the interval $[0, b]$ is given by $I_q f(x) = \int_0^x f(s) d_q s = x(1-q)\sum_{k=0}^{\infty} f(xq^k) q^k$ for $x \in [0, b]$, provided that the sum converges absolutely ([1]). If $a \in [0, b]$, then

$$\int_a^b f(s) d_q s = I_q f(b) - I_q f(a) = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s$$

$$= (1-q)\sum_{k=0}^{\infty}[b f(bq^k) - a f(aq^k)] q^k$$

whenever the series exists. The operator I_q^n is defined by $I_q^0 f(x) = f(x)$ and $I_q^n f(x) = I_q(I_q^{n-1} f)(x)$ for all $n \geqslant 1$ ([1]). It has been proved that $(D_q I_q f)(x) = f(x)$ and $(I_q D_q f)(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ ([1]). Let $\alpha \geqslant 0$ and f be a function on $[0, 1]$. The fractional Riemann-Liouville type q-integral of the function f is defined by $(I_q^0 f)(x) = f(x)$ and $(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha-1)} f(s) d_q s$ for $x \in [0, 1]$ and $\alpha > 0$ ([9] and[13]). Also, the fractional Caputo type q-derivative of the function f is defined by

$$({}^c D_q^\alpha f)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f)(x) = \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^x (x - qs)^{([\alpha]-\alpha-1)} (D_q^{[\alpha]} f)(s) d_q s$$

for $x \in [0, 1]$ and $\alpha > 0$ ([9] and[13]). It has been proved that $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$ and $(D_q^\alpha I_q^\alpha f)(x) = f(x)$, where $\alpha, \beta \geqslant 0$ ([14]). Also, $(I_q^\alpha D_q^n f)(x) = (D_q^n I_q^\alpha f)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k f)(0)$, where $\alpha > 0$ and $n \geqslant 1$ ([14]).

In 2012, Ahmad, Ntouyas and Purnaras investigated the q-difference equation $({}^c D_q^\alpha u)(t) = f(t, u(t))$ with nonlocal boundary conditions $\alpha_1 u(0) - \beta_1 D_q u(0) = \gamma_1 u(\eta_1)$ and $\alpha_2 u(1) + \beta_2 D_q u(1) = \gamma_2 u(\eta_2)$, where $0 \leqslant t \leqslant 1$, $1 < \alpha \leqslant 2$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ for all i ([10]). In 2013, Zhao, Chen and Zhang reviewed the nonlinear fractional q-difference equation $(D_q^\alpha u)(t) + f(t, u(t)) = 0$ with the nonlocal q-integral boundary value conditions $u(0) = 0$ and $u(1) = \mu I_q^\beta u(\eta)$, where $0 < t < 1$, $1 < \alpha \leqslant 2$, $0 < \beta \leqslant 2$ and $\mu > 0$ ([21]). In 2015, Etemad, Ettefagh and Rezapour investigated the q-differential equation $({}^c D_q^\alpha)(t) = f(t, u(t), D_q u(t))$ with boundary conditions $\lambda_1 u(0) + \mu_1 D_q u(0) = \eta_1 I_q^\beta u(\xi_1)$ and $\lambda_2 u(1) + \mu_2 D_q u(1) = \eta_2 I_q^\beta u(\xi_2)$, where $0 \leqslant t \leqslant 1$, $1 < \alpha \leqslant 2$, $q \in (0, 1)$, $\beta \in (0, 2]$, $\xi_1, \xi_2 \in (0, 1)$ with $\xi_1 < \xi_2$, $\lambda_1, \lambda_2, \mu_1, \mu_2, \eta_1, \eta_2 \in \mathbb{R}$ and

$f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous map ([9]). Recently, it has been published some works about approximate solutions of some fractional differential equations ([11] and [12]). By using and mixing idea of the works, we study the existence of approximate solutions for the fractional q-difference equation

$$({}^cD_q^\alpha u)(t) = f(t, u(t), I_q^\beta u(t)) \tag{1}$$

with the q-integral boundary value conditions

$$u(0) = u(1) = 0, \tag{2}$$

where ${}^cD_q^\alpha$ denotes the fractional q-derivative of the Caputo type of order α, $0 \leqslant t \leqslant 1$, $1 < \alpha\ leq2$, $q \in (0,1)$, $\beta \in (0,2]$and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous map.

Now, we provide some basic needed notions. Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ a map, F a selfmap on X and $\varepsilon > 0$. We say that F is α-admissible whenever $\alpha(x, y) \geqslant 1$ implies $\alpha(Fx, Fy) \geqslant 1$ ([12]). An element $x_0 \in X$ is called ε-fixed point of F whenever $d(Fx_0, x_0) \leqslant \varepsilon$. We say that F has the approximate fixed point property whenever F has an ε-fixed point for all $\varepsilon > 0$ ([12]). Some mappings have approximate fixed points while have no fixed points ([12]). Denote by $\Re$ the set of all continuous mappings $g : [0, \infty)^5 \to [0, \infty)$ satisfying $g(1, 1, 1, 2, 0) = g(1, 1, 1, 0, 2) := h \in (0, 1)$, $g(\mu x_1, \mu x_2, \mu x_3, \mu x_4, \mu x_5) \leqslant \mu g(x_1, x_2, x_3, x_4, x_5)$, $g(x_1, x_2, x_3, 0, x_4) \leqslant g(y_1, y_2, y_3, 0, y_4)$ and

$$g(x_1, x_2, x_3, x_4, 0) \leqslant g(y_1, y_2, y_3, y_4, 0)$$

for all $\mu \geqslant 0$ and $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5 \in [0, \infty)$ with $x_i \leqslant y_i$ for $i = 1, 2, 3, 4$ ([20]). Finally, we say that F is a generalized α-contractive mapping whenever there exists $g \in \Re$ such that

$$\alpha(x, y)d(Fx, Fy) \leqslant g(d(x, y), d(x, Fx), d(y, Fy), d(x, Fy), d(y, Fx))$$

for all $x, y \in X$([20]). We need next fixed point theorem for our main result.

Theorem 1.1. *[20] Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ a map and F a generalized α-contractive and α-admissible selfmap on X. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Fx_0) \geqslant 1$. Then F has an approximate fixed point.*

Lemma 1.2. *The function u_0 is a solution for the problem (1) with the boundary value conditions (2) if and only if u_0 is a solution for the fractional q-integral equation*

$$u_0(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u_0(s), I_q^\beta u_0(s)) d_qs - t \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u_0(s) I_q^\beta u_0(s)) d_qs.$$

Proof. Let u_0 be a solution for the fractional q-difference equation (1) with the q-integral boundary value conditions. Choose $c_0, c_1 \in \mathbb{R}$ such that $u_0(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u_0(s), I_q^\beta u_0(s))d_qs + c_0 + c_1 t$ (see [13]). Since $u_0(0) = 0$ and $u_0(1) = 0$, we get $c_0 = 0$ and $c_1 = -\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s))d_qs$.
Thus, we conclude that
$u_0(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u_0(s), I_q^\beta u_0(s))d_qs - t\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u_0(s), I_q^\beta u_0(s))d_qs$.
By using some easy calculation, one can get that the converse part is obvious. This completes the proof. □

2. Main Results

Consider the space

$$X = \{u : u, I_q^\beta u \in C_{\mathbb{R}}([0,1]), {}^c D_q^\alpha u(t) = f(t, u(t), I_q^\beta u(t)), u(0) = u(1) = 0\}$$

via the metric $d(u, v) = \|u - v\|$, where $\|u\| = \sup_{t\in[0,1]} |I_q^\beta(u(t))| + sup_{t\in[0,1]}|u(t)|$. As we know, (X, d) is not a Banach space. By considering Lemma 1.2, define the operator $F : X \to X$ by

$$(Fu)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s))d_qs$$

$$-t\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s))d_qs. \quad (*)$$

It is easy to check that u_0 is aa approximate solution for the problem if and only if u_0 is an approximate fixed point of F.

Theorem 2.1. *Suppose that $f : [0,1] \times X^2 \to X$ is a continuous function and there exists a q-integrable function $L : [0,1] \to \mathbb{R}$ such that $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leqslant L(t)\,[|u_1 - u_2| + |v_2 - v_2|]$ for all $t \in [0,1]$ and $u_1, u_2, v_1, v_2 \in X$. If $h < 1$, then the problem (1) has an approximate solution. Here, $h = 2(I_q^\alpha)L(1)$.*

Proof. Choose $r \geqslant \dfrac{KN}{1-h}$, where $K = \sup_{t\in[0,1]} |f(t,0,0)|$ and $N = \frac{2}{\Gamma_q(\alpha+1)}$. Put $B_r = \{u \in X : \ \|u\| \leqslant r\}$. Consider the operator $F : X \to X$ defined by $(*)$. We show that $FB_r \subset B_r$. Let $u \in B_r$ and $t \in [0,1]$. Then,

$$|(Fu)(t)| \leqslant \Big|\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s))d_qs\Big|$$

$$+t\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),I_q^\beta u(s))|d_qs$$

$$\leqslant \sup_{t\in[0,1]}(\int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),I_q^\beta u(s))-f(s,0,0)+f(s,0,0)|d_qs$$

$$+\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),I_q^\beta u(s))-f(s,0,0)+f(s,0,0)|d_qs)$$

$$\leqslant \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big[L(s)(|I_q^\beta u(s)|+K)\big]d_qs$$

$$+\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big[L(s)(|I_q^\beta u(s)|+K)\big]d_qs$$

$$\leqslant K(2\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs)+r(2\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L(s)d_qs)\leqslant r.$$

Also, we have

$$|(I_q^\beta Fu)(t)|\leqslant \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|f(s,u(s),I_q^\beta u(s))|d_qs$$

$$+\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),I_q^\beta u(s))|d_qs$$

$$\leqslant \sup_{t\in[0,1]}\int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}|f(s,u(s),I_q^\beta u(s))-f(s,0,0)+f(s,0,0)|d_qs$$

$$+\sup_{t\in[0,1]}\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}|f(s,u(s),I_q^\beta u(s))-f(s,0,0)+f(s,0,0)|d_qs$$

$$\leqslant \int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}\big[L(s)r+K\big]d_qs+\frac{1}{\Gamma_q(\beta+2)}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}\big[L(s)r+K\big]d_qs$$

$$=K(\int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}d_qs+\frac{1}{\Gamma_q(\beta+2)}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}d_qs)$$

$$+r(\int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)}L(s)d_qs+\frac{1}{\Gamma_q(\beta+2)}\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}L(s)d_qs)$$

$$=K(\frac{1}{\Gamma_q(\alpha+\beta+1)}+\frac{1}{\Gamma_q(\beta+2)}\times\frac{1}{\Gamma_q(\alpha+1)})+r((I_q^{\alpha+\beta}L)(1)+\frac{1}{\Gamma_q(\beta+2)}(I_q^\alpha L)(1))$$

$$\leqslant K(\frac{2}{\Gamma_q(\alpha+1)})+r(2(I_q^\alpha L)(1))\leqslant r.$$

This implies that $FB_r \subset B_r$. Let $u, v \in X$ be given. Then, we have

$$|Fu - Fv| \leqslant \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_qs$$

$$+\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_qs$$

$$\leqslant \|u-v\| \sup_{t\in[0,1]} \Big\{ \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs + \frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs \Big\}$$

$$\leqslant \|u-v\| \Big\{ \int_0^1 \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs + \frac{1}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs \Big\}$$

$$\leqslant h\|u-v\|.$$

Also, we have

$$|(I_q^\beta F)(u) - (I_q^\beta F)(v)|$$

$$\leqslant \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+1)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_qs$$

$$+\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_qs$$

$$\leqslant \|u-v\| \Big\{ \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+1)} l(s) d_qs + \frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs \Big\}$$

$$\leqslant \|u-v\| \Big\{ \int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+1)} l(s) d_qs + \frac{1}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} l(s) d_qs \Big\}$$

$$\leqslant h\|u-v\|.$$

Now, consider the maps $g : [0,\infty)^5 \to [0,\infty)$ and $\alpha : X \times X \to [0,\infty)$ defined by $g(t_1,t_2,t_3,t_4,t_5) = ht_1$ and $\alpha(x,y) = 1$ for all $x, y \in X$. One can easily check that $g \in \Re$ and F is a generalized α-contraction. By using Theorem 1.1, F has an approximate fixed point which is approximate solution for the problem. □

Theorem 2.2. *Let $M > 0$ be given and $f : [0,1] \times X^2 \to X$ a continuous function such that*

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leqslant M(|u_1-u_2| + |v_2-v_2|)$$

for all $t \in [0,1]$ and $u_1, u_2, v_1, v_2 \in X$. Then the problem (1) has an approximate solution whenever $\Lambda = M\Lambda_1 < 1$, where $\Lambda_1 = 2(I_q^\alpha 1)(1)$.

Proof. Define map $F : X \to X$ by

$$(Fu)(t) = \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s)) d_q s$$

$$-t \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s)) d_q s.$$

Put $M_0 = \sup_{t\in[0,1]} |f(t,0,0)|$ and $r \geqslant \dfrac{M_0 \Lambda_1}{1-\delta}$, where δ is such that $\Lambda \leqslant \delta < 1$. We show that $FB_r \subset B_r$, where $B_r = \{u \in X : \ \|u\| \leqslant r\}$. Let $u \in B_r$ and $t \in [0,1]$. Then, we have

$$|(Fu)(t)| \leqslant \Big| \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} f(s, u(s), I_q^\beta u(s)) d_q s \Big|$$

$$+t \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s), I_q^\beta u(s))| d_q s$$

$$\leqslant \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s), I_q^\beta u(s)) - f(s,0,0) + f(s,0,0)| d_q s$$

$$+t \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s), I_q^\beta u(s)) - f(s,0,0) + f(s,0,0)| d_q s$$

$$\leqslant \int_0^t \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [M(|u(s)| + |I_q^\beta u(s)|) + M_0] d_q s$$

$$+t \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [M(|u(s)| + |I_q^\beta u(s)|) + M_0] d_q s$$

$$\leqslant (M\|u\| + M_0)(2 \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s) \leqslant (Mr + M_0)\Lambda_1 \leqslant \Lambda r + r(1-\delta) \leqslant r.$$

Also, we have

$$|(I_q^\beta Fu)(t)| \leqslant \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |f(s, u(s), I_q^\beta u(s))| d_q s$$

$$+\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s, u(s), I_q^\beta u(s))| d_q s$$

$$\leqslant (Mr + M_0) \int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} d_q s + \frac{1}{\Gamma_q(\beta+2)} \int_0^t \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s \leqslant r.$$

This implies that $FB_r \subset B_r$. Let $u, v \in X$ be given. Then, we obtain

$$|Fu - Fv| \leqslant \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_q s$$

$$+t\int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_q s$$

$$\leqslant \sup_{t\in[0,1]} \int_0^t \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} \big[M(|u(s)-v(s)| + |I_q^\beta u(s) - I_q^\beta v(s)|)\big] d_q s$$

$$+ \sup_{t\in[0,1]} \int_0^1 \frac{(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [M(|u(s)-v(s)| + |I_q^\beta u(s) - I_q^\beta v(s)|)] d_q s$$

$$\leqslant M\|u-v\|\Lambda_1 = \Lambda\|u-v\|.$$

Also, we have

$$|(I_q^\beta F)(u) - (I_q^\beta F)(v)|$$

$$\leqslant \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_q s$$

$$+\frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} |f(s,u(s),I_q^\beta u(s)) - f(s,v(s),I_q^\beta v(s))| d_q s$$

$$\sup_{t\in[0,1]} \int_0^t \frac{(t-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} [M(|u(s)-v(s)| + |I_q^\beta u(s) - I_q^\beta v(s)|)] d_q s$$

$$+ \sup_{t\in[0,1]} \frac{t^{(\beta+1)}}{\Gamma_q(\beta+2)} \int_0^t \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} [M(|u(s)-v(s)| + |I_q^\beta u(s) - I_q^\beta v(s)|)] d_q s$$

$$\leqslant M\|u-v\|\big\{\int_0^1 \frac{(1-qs)^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} d_q s + \frac{1}{\Gamma_q(\beta+2)} \int_0^1 \frac{(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} d_q s\big\}$$

$$\leqslant M\|u-v\|\Lambda_1 = \Lambda\|u-v\|.$$

Now, consider the maps $g : [0,\infty)^5 \to [0,\infty)$ and $\alpha : X \times X \to [0,\infty)$ defined by $g(t_1,t_2,t_3,t_4,t_5) = \Lambda t_1$ and $\alpha(x,y) = 1$ for all $x, y \in X$. One can easily check that $g \in \Re$ and F is a generalized α -contraction. By using Theorem 1.1, F has an approximate fixed point which is approximate solution for the problem. $\Box$

Example 2.3. Consider the fractional q-difference equation

$$({}^c D_{\frac{1}{2}}^2 u)(t) = f(t,u(t),I_{\frac{1}{2}}^{\frac{1}{3}} u(t))$$

with the q-integral boundary value conditions $u(0) = u(1) = 0$. Put $q = \dfrac{1}{2}$, $\alpha = 2$, $\beta = \dfrac{1}{3}$ and

$$f(t, u(t), I_{\frac{1}{2}}^{\frac{1}{3}} u(t)) = t^2(1 + t^2 + \sin u + I_{\frac{1}{2}}^{\frac{1}{3}} u(t))$$

for all $t \in [0, 1]$. Then, $|f(t, u(t), I_{\frac{1}{2}}^{\frac{1}{3}} u(t) - f(t, v(t), I_{\frac{1}{2}}^{\frac{1}{3}} v(t))| \leqslant t^2(|u - v| + |I_{\frac{1}{2}}^{\frac{1}{3}} u(t) - I_{\frac{1}{2}}^{\frac{1}{3}} v(t)|)$. Define $L(t) = t^2$ for all t. Then, an easy calculations shows that $h = \frac{64}{105} < 1$. Now by using Theorem 1.1, the problem has an approximate solution.

Example 2.4. Consider the fractional q-difference equation

$$({}^cD_{\frac{1}{2}}^2 u)(t) = M(t^2 + \cos t + 1 + \tan^{-1} u(t) + I_{\frac{1}{2}}^{\frac{1}{2}} u(t))$$

with the q-integral boundary value conditions $u(0) = u(1) = 0$. Put $q = \frac{1}{2}$, $\beta = \frac{1}{2}$ and $\alpha = 2$. Then, $\Lambda_1 = (2I_{\frac{1}{2}}^{\frac{1}{2}})(1) = \frac{4}{3}$. Choose $M < \frac{1}{\Lambda_1} = \frac{3}{4}$. Consider the function

$$f(t, u(t), I_{\frac{1}{2}}^{\frac{1}{2}} u(t)) = M(t^2 + \cos t + 1 + \tan^{-1} u(t) + I_{\frac{1}{2}}^{\frac{1}{2}} u(t))$$

for all $t \in [0, 1]$. Then, $|f(t, u(t), I_{\frac{1}{2}}^{\frac{1}{2}} u(t)) - f(t, v(t), I_{\frac{1}{2}}^{\frac{1}{2}} v(t))| \leqslant M(|u - v| + |I_{\frac{1}{2}}^{\frac{1}{2}} u(t) - I_{\frac{1}{2}}^{\frac{1}{2}} v(t)|)$. Now by using Theorem **??**, the problem has an approximate solution.

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