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Multiplication Operators with Adjoint in a Cowen-Douglas Class Operator

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Abstract. In this paper, we will consider multiplication operators on Hilbert spaces of analytic functions on a domain $\Omega \subset \mathbb{C}$. Also, we determine the commutants of certain multiplication operators with adjoints in a Cowen-Douglas class operator.

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1. Introduction

In this section we include some preparatory material which will be needed later.

For a positive integer n and a domain $U \subset \mathbb{C}$, the Cowen-Douglas class $B_n(U)$ consists of bounded linear operators T on any fixed separable infinite dimensional Hilbert space H with the following properties:

(a) U is a subset of the spectrum of T.

(b) $ran(\lambda - T) = H$ for every $\lambda \in U$.

(c) $Span\{ker(\lambda - T) : \lambda \in U\} = H$.

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(d) $dim[ker(\lambda - T)] = n$ for every $\lambda \in U$.

Here *Span* denotes the closed linear span of a collection of sets in H. Conditions (a) and (b) insure that U is contained in the point spectrum of T and $T - \lambda$ is right invertible for $\lambda \in U$. Clearly, (d) implies (a), Also, note that condition (d) implies (a), and since (a) and (b) imply that $ker(\lambda - T)$ is constant, condition (d) imposes only that it is finite dimensional. Recall that if T is semi-Fredholm, then $ran(T)$ is closed and at least one of $dim\ ker(T)$ and $dim\ ker(T^*)$ is finite. Now since $ind(T - \lambda)$ is continuous and $T - \lambda$ is right invertible, we can see that $ind(T - \lambda) = dim\ ker(T)$ is constant.

The classes $B_n(U)$ were introduced by Cowen and Douglas ([5]), and each element of $B_n(U)$ is called a Cowen-Douglas class operator. By B_n we mean $B_n(U)$ for some complex domain U. For the study of the Cowen-Douglas classes B_n, we mention [1, 5, 6, 16, 18, 22, 23].

Also, if X is a Banach space of functions analytic on a plane domain Ω, a complex-valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a multiplier of X and the multiplier φ on X determines a multiplication operator M_φ on X by $M_\varphi f = \varphi f,\ f \in X$. The set of all multipliers of X is denoted by $M(X)$. Clearly $M(X) \subset H^\infty(\Omega)$ where $H^\infty(\Omega)$ is the space of all bounded analytic functions on Ω. In fact $||\varphi||_\infty \leqslant ||M_\varphi||$ ([17]).

If X is a Banach space of functions analytic on a domain $\Omega \subset \mathbf{C}$ and X holds the axioms:

Axiom (1). Every point $w \in \Omega$ is a nonzero bounded linear functional on X,

Axiom (2). Every function $\varphi \in H^\infty(\Omega)$ is a multiplier of X,

Axiom (3). If $f \in X$ and $f(\lambda) = 0$, then there is a function $g \in X$ such that $(z - \lambda)g = f$,

then X is called a *Banach space of analytic functions on* Ω. Also, if X is a Hilbert space, it is called a *Hilbert space of analytic functions on* Ω ([13, 15, 19, 21, 23]). The Hardy and Bergman spaces are examples for Hilbert spaces of analytic functions on the open unit disk.

In this paper, we suppose that $\mathcal{H}$ is a Hilbert space of functions analytic on a domain $\Omega \subset \mathbf{C}$. Here, we want to investigate the intertwining multiplication operators on B_n. For some other sources on these topics one can see [2, 3, 7, 8, 9, 10, 11, 12, 14, 17, 20].

2. Intertwining Multiplication Operators

By Propositions 3.1 and 5.2 in [23], K. Zhu gives sufficient conditions for the adjoint of multiplication operators on Hilbert spaces of analytic functions belong to the Cowen-Douglas classes B_n for a positive integer n. Then in [18], B. Yousefi and S. Foroutan investigate the converse of Zhu's results. Also, in [18, 23], the commutant of special multiplication operators with adjoints in a Cowen-Douglas class operator, has been considered. Here, under Axioms (1), (2), (3), we want to determine the commutants of certain multiplication operators with adjoints in a Cowen-Douglas class operator.

Regarding the given axioms on $\mathcal{H}$, we note that a few comments are in order: Since by Axiom (2) every function $\varphi \in H^\infty(\Omega)$ is a pointwise multiplication of $\mathcal{H}$, so by the closed graph theorem, the operator of multiplication by φ, M_φ, is a bounded linear operator on $\mathcal{H}$. Also, Axiom (3) says that if $f \in \mathcal{H}$ and $f(\lambda) = 0$, then $f/(z-\lambda)$ is in $\mathcal{H}$. Thus, this condition implies that $\ker(M_z - \lambda)^* = \mathbb{C}e_\lambda$ for every λ in Ω ([13]). Now, we give an example satisfying Axioms (1), (2), (3):

Example 2.1. Consider the Hilbert Bergman space $L^2_a(\mathbb{D})$ where $\mathbb{D}$ is the open unit disc in the complex domain. Then $L^2_a(\mathbb{D})$ holds in the Axioms (1), (2), (3) ([4, Theorem 8.5, page 67]).

The following characterization of the commutant $\{T\}'$ of T is given in Theorem 3.7 of [6], which is stated for the convenience of the reader. In the following K is the reproducing kernel for a coanalytic functional Hilbert space $\mathcal{K}$ defined in [6].

Theorem 2.2. *If S is in $B_n(\Omega)$ and the operator X commutes with S, then there exists an analytic function $\Phi : \Omega \to \mathcal{B}(\mathbb{C}^{\mathbf{n}})$ such that $XK(\lambda, .) = K(\lambda, .)\Phi(\lambda)$ (all $\lambda \in \Omega$) and for every $f \in \mathcal{K}$, $X^* f(.) = (\Phi(.))^* f(.)$.*

In the following let Ω be such that if $\lambda \in \Omega$ then $-\lambda \in \Omega$. Also, we assume that the composition operator $C_{-z} : \mathcal{H} \to \mathcal{H}$ defined by $C_{-z} f = f(-z)$ is bounded.

Theorem 2.3. *Suppose that $\varphi \in H^\infty(\Omega)$ is odd and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(w)$ is a singleton for every $w \in V$. If for some integer $n \geqslant 3$, $SM_{\varphi^n} = -M_{\varphi^n} S$, $M_{\varphi^i} S M_\varphi = -M_\varphi S M_{\varphi^i}$ for $2 \leqslant i < n$, and $SM_\varphi + M_\varphi S$ is compact, then $S = M_h C_{-z}$ for some $h \in H^\infty(\Omega)$.*

Proof. First, note that by Proposition 2.2 in [23], the adjoint of the operator $M_\varphi : \mathcal{H} \to \mathcal{H}$ belongs to the Cowen-Douglas class $B_1(U)$, where

$$U = \{\bar{z} : z \in V\}.$$

Clearly we can get $T_1 M_\varphi = -M_\varphi T_1$, where

$$T_1 = SM_{\varphi^{n-1}} + M_{\varphi^{n-1}} S.$$

Thus by Proposition 3 in [18], $T_1 = M_{h_1} C_{-z}$ for some h_1 in $H^\infty(\Omega)$. Note that $SM_\varphi + M_\varphi S$ is compact, so the operators

$$M_\varphi(SM_\varphi + M_\varphi S)$$

and

$$(SM_\varphi + M_\varphi S)M_\varphi$$

are also compact. By subtracting them, we conclude that $SM_{\varphi^2} - M_{\varphi^2} S$ is compact. This implies that the operators

$$M_\varphi(SM_{\varphi^2} - M_{\varphi^2} S)$$

and

$$(SM_{\varphi^2} - M_{\varphi^2} S)M_\varphi$$

are also compact. Again by subtracting them and using the fact that

$$M_{\varphi^2} S M_\varphi = -M_\varphi S M_{\varphi^2},$$

we obtain that the operator $SM_{\varphi^3} + M_{\varphi^3} S$ is compact. By repeating this method, we can see that $SM_{\varphi^i} + M_{\varphi^i} S$ is compact for $3 \leqslant i < n$. Now, if $i = n-1$, then T_1 is compact. But $T_1 = M_{h_1} C_{-z}$, thus

$$T_1 \circ C_{-z} = M_{h_1} \circ C_{-z} \circ C_{-z} = M_{h_1}.$$

Hence M_{h_1} is also compact. By the Fredholm Alternative Theorem, we show that $h_1 = 0$. For this suppose that λ is an arbitrary nonzero element of $\mathbf{C}$. Then by the Fredholm Alternative Theorem, $ran(M_{h_1-\lambda})$ is closed and

$$dimker(M_{h_1-\lambda}) = dimker(M_{h_1-\lambda})^* < \infty.$$

Clearly, $M_{h_1-\lambda}$ is injective. This implies that $(M_{h_1-\lambda})^*$ is also injective and so $ran(M_{h_1})$ is dense in $\mathcal{H}$. But $ran(M_{h_1})$ is closed, thus $M_{h_1} - \lambda$ is surjective. Therefore, $M_{h_1-\lambda}$ is invertible for all $\lambda \neq 0$. Hence, $h_1 - \lambda$ is nonvanishing on Ω for all $\lambda \neq 0$, and so $h_1(z) \neq \lambda$ for all $z \in \Omega$ and all $\lambda \in \mathbf{C} \backslash \{\mathbf{0}\}$. Now, clearly it should be $h_1 = 0$ on Ω from which we conclude that $T_1 = 0$. Thus

$$SM_{\varphi^{n-1}} = -M_{\varphi^{n-1}} S.$$

By continuing this way, we conclude that $SM_{\varphi^3} = -M_{\varphi^3} S$ which implies that

$$(SM_{\varphi^2} - M_{\varphi^2} S)M_\varphi = M_\varphi(SM_{\varphi^2} - M_{\varphi^2} S).$$

Now by Proposition 4.1 in [23], we can write $SM_{\varphi^2} - M_{\varphi^2}S = M_g$ for some g in $H^\infty(\Omega)$. Now, by the same method used earlier, by applying the Fredholm Alternative Theorem, we see that $g = 0$. Thus, $SM_{\varphi^2} = M_{\varphi^2}S$ and so $T_2M_\varphi = M_\varphi T_2$ where

$$T_2 = SM_\varphi + M_\varphi S.$$

Hence $T_2 = M_{h_2}$ for some $h_1 \in H^\infty(\Omega)$. But by the hypothesis T_2 is compact, so M_{h_2} is also compact. Therefore, $h_2 = 0$ and so $T_2 = 0$. This implies that $SM_\varphi = -M_\varphi S$ and now by Proposition 3 in [18], we get $S = M_hC_{-z}$ for some $h \in H^\infty(\Omega)$. Thus the proof is complete. □

Theorem 2.4. *Suppose that $\varphi \in H^\infty(\Omega)$ is odd and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(w)$ is a singleton for every $w \in V$. Let for some integer $n \geqslant 3$, $SM_{\varphi^n} = -M_{\varphi^n}S$ and $M_{\varphi^i}SM_\varphi = -M_\varphi SM_{\varphi^i}$ for $2 \leqslant i < n$. If $SM_\varphi - M_\varphi S$ is compact, then $S = M_h$ for some $h \in H^\infty(\Omega)$.*

Proof. First note that since

$$(SM_{\varphi^{n-1}} + M_{\varphi^{n-1}}S)M_\varphi = -M_\varphi(SM_{\varphi^{n-1}} + M_{\varphi^{n-1}}S),$$

thus there exists $h_1 \in H^\infty(\Omega)$ such that $SM_{\varphi^{n-1}} + M_{\varphi^{n-1}}S = M_{h_1}C_{-z}$. Now compactness of $SM_\varphi - M_\varphi S$ implies that the operator

$$SM_{\varphi^2} - M_{\varphi^2} = M_\varphi(SM_\varphi - M_\varphi S) + (SM_\varphi - M_\varphi S)M_\varphi$$

is compact. Hence

$$SM_{\varphi^3} + M_{\varphi^3} = (SM_{\varphi^2} - M_{\varphi^2}S)M_\varphi - M_\varphi(SM_{\varphi^2} - M_{\varphi^2}S)M_\varphi$$

is also compact. Finally by continuing this method, we can see that $SM_{\varphi^{n-1}} + M_{\varphi^{n-1}}S$ and so $M_{h_1}C_{-z}$ is compact. Thus it should be $h_1 = 0$ on Ω which implies that $SM_{\varphi^{n-1}} = -M_{\varphi^n-1}S$. Now, by a similar method used in the proof of Theorem 2.3, we have $SM_{\varphi^2} = M_{\varphi^2}S$. Put $W = SM_\varphi - M_\varphi S$. Clearly, $WM_\varphi = M_\varphi W$ and so $W = M_g$ for some $g \in H^\infty(\Omega)$. By compactness of W, we get $g = 0$ on Ω. Hence $SM_\varphi = M_\varphi S$ and by Proposition 4.1 in [23], there exists $h \in H^\infty(\Omega)$ such that $S = M_h$. □

Theorem 2.5. *Suppose that $\varphi \in H^\infty(\Omega)$ is odd and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(w)$ is a singleton for every $w \in V$. If for some integer $n \geqslant 3$, $SM_{\varphi^n} = M_{\varphi^n}S$, $M_{\varphi^i}SM_\varphi = M_\varphi SM_{\varphi^i}$ for $2 \leqslant i < n$, and $SM_\varphi + M_\varphi S$ is compact, then $S = M_hC_{-z}$ for some $h \in H^\infty(\Omega)$.*

Proof. By a the method used in the proof of Theorem 2.3, we can see that $SM_{\varphi^2} = M_{\varphi^2}S$. Again by the proof of Theorem 2.3, if $T = SM_\varphi + M_\varphi S$, then

$TM_\varphi = M_\varphi T$. Hence $T = M_h$ for some $h \in H^\infty(\Omega)$. Compactness of T implies that $h = 0$. Thus $T = 0$ and so $SM_\varphi = -M_\varphi S$. Now by Proposition 3 in [18], there exists $h \in H^\infty(\Omega)$ satisfying $S = M_h C_{-z}$. This completes the proof. □

Theorem 2.6. *Suppose that $\varphi \in H^\infty(\Omega)$ is odd and there exists a domain $V \subset \varphi(\Omega)$ such that $\Omega \cap \varphi^{-1}(w)$ is a singleton for every $w \in V$. If for some integer $n \geqslant 3$, $SM_{\varphi^n} = M_{\varphi^n} S$, $M_{\varphi^i} SM_\varphi = M_\varphi SM_{\varphi^i}$ for $2 \leqslant i < n$, and $SM_\varphi - M_\varphi S$ is compact, then $S = M_h$ for some $h \in H^\infty(\Omega)$.*

Proof. Clearly, we can see that $T_1 M_\varphi = -M_\varphi T_1$ where

$$T_1 = SM_{\varphi^{n-1}} - M_{\varphi^{n-1}} S.$$

So by Proposition 3 in [17], $T_1 = M_{h_1} C_{-z}$ for some h_1 in $H^\infty(\Omega)$. Now we show that M_{h_1} is compact. Note that we can write

$$T_1 = (SM_\varphi - M_\varphi S)M_{\varphi^{n-2}} + M_{\varphi^{n-2}}(SM_\varphi - M_\varphi S).$$

Therefore, T_1 and so

$$M_{h_1} = M_{h_1} C_{-z} \circ C_{-z} = T_1 \circ C_{-z}$$

is compact. By using the Fredholm Alternative Theorem, we get $h_1 = 0$. Hence $SM_{\varphi^{n-1}} = M_{\varphi^{n-1}} S$. By continuing this manner, we conclude that $SM_\varphi = M_\varphi S$. Now, by Proposition 4.1 in [23], $S = M_h$ for some $h \in H^\infty(\Omega)$ and so the proof is complete. □

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