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Some Fixed Point Theorems for Nonexpansive Self-Mappings and Multi-Valued Mappings in b-Metric Spaces

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Abstract. In this paper, motivated by [F. Vetro, Filomat, 29:9 (2015), 2011-2020] we present some fixed point results for a class of nonexpansive self-mappings and multi-valued mappings in the framework of b-metric spaces. Our results generalize and improve the consequences of [Khojasteh et al. Abstract and Applied Analysis, vol. 2014, Article ID 325840, 5 pages, 2014.] and [F. Vetro, Filomat, 29:9 (2015), 2011-2020]. Some examples are provided to illustrate our results.

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1. Introduction

Assume that (X, d) be a metric space and $f : X \to X$ be a single-valued mapping. Then, f is called a k-Lipscitz mapping if $d(fx, fy) \leqslant kd(x, y)$

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for all $x, y \in X$, where $k \geqslant 0$. If $k \in [0,1)$, then it is called a contractive mapping and if $k = 1$, f is called a nonexpansive mapping. In [2] Banach proved a very important fixed point result for a contractive self-mapping. Obviously, any contractive mapping is nonexpansive, but the reverse is not true in general. Several researchers investigated fixed point theory for nonexpansive mappings such as [7, 8]. In [9] Khojasteh et al. investigated fixed point results for a new type of self-mappings and multivalued mappings. Then, Vetro in [17] extended their results for nonexpansive mappings using a binary relation on X, called f-invariant. Also, he established fixed point results for nonexpansive multi-valued mappings with a new type of contraction. On the other hand, fixed point theory for mappings on b-metric spaces as a generalization of metric spaces has attracted many researchers for many years (see [1, 3, 4, 5, 6, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22]). In this paper, motivated by Vetro [17] we give some fixed point results for a class of nonexpansive self-mappings and multi-valued mappings on b-metric spaces. Our results generalize and improve the results of Khojasteh et al. [9] and Vetro [17]. Some examples are given to illustrate our results.

2. Preliminaries

In this section we give some notions and results that will be needed in the sequel.

Definition 2.1. *[1] Let X be a nonempty set and $s \geqslant 1$ be a constant real number. The function $d : X \times X \to [0, \infty)$ is called a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *iff* $x = y$ *for all* $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leqslant s[d(x, z) + d(z, y)]$ *for all* $x, y, z \in X$.

In this case, (X, d) is called a b-metric space with parameter s.

Denote by $CB(X)$ the set of all nonempty closed bounded subsets of X.

Assume that H be the Pompeiu-Hausdorff metric on $CB(X)$ defined by

$$H(A,B)=\max\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,A)\},$$

for all $A,B\in CB(X)$, where $d(x,B)=\inf_{y\in B} d(x,y)$. An element $x\in X$ is said to be a fixed point of a multi-valued mapping $T:X\to CB(X)$ whenever $x\in Tx$. It is said that $T:X\to CB(X)$ is contractive whenever $H(Tx,Ty)\leqslant kd(x,y)$ for all $x,y\in X$, where $k\in[0,1)$. If $k=1$, then T is called a nonexpansive multivalued mapping. Nadler [11] proved the existence of fixed point for contractive multivalued mappings.

Lemma 2.2. *[17] If $\{a_n\}$ be a nonincreasing sequence of nonnegative real numbers, then the sequence $\{\frac{a_n+a_{n+1}}{a_n+a_{n+1}+1}\}$ is nonincreasing too.*

Corollary 2.3. *Let (X,d) be a b-metric space and $f:X\to X$ be a nonexpansive mapping. If $x_0\in X$ and $\{x_n\}$ be a Picard sequence starting with x_0, that is, $x_n=fx_{n-1}$ for all $n\in\mathbb{N}$, then, the sequence*

$$\{\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_{n-1},x_n)+d(x_n,x_{n+1})+1}\}$$

is nonincreasing too.

3. Fixed Point Results for Nonexpansive Self-Mappings

We prove some results for single-valued mappings defined on a b-metric space endowed with an arbitrary binary relation. Let X be a nonempty set, $f:X\to X$ be a mapping and $\mathcal{R}$ be a binary relation on X, that is, $\mathcal{R}$ is a subset of $X\times X$. Then, $\mathcal{R}$ is Banach f-invariant if $(fx,f^2x)\in\mathcal{R}$, whenever $(x,fx)\in\mathcal{R}$. Also, a subset Y of X is well ordered with respect to $\mathcal{R}$ if for all $x,y\in Y$ we have $(x,y)\in\mathcal{R}$ or $(y,x)\in\mathcal{R}$. Let $Fix(f)=\{x\in X: x=fx\}$ denotes the set of all fixed points of f on X.

Theorem 3.1. *Let (X,d) be a complete b-metric space with parameter $s>1$ endowed with a binary relation $\mathcal{R}$ on X and $f:X\to X$ be a* *nonexpansive mapping such that*

$$d(fx,fy)\leqslant (\frac{d(x,fy)+d(y,fx)}{s[d(x,fx)+d(y,fy)+1]}+k)d(x,y) \tag{1}$$

for all $(x,y)\in\mathcal{R}$*, where* $k\in[0,1)$*. Also assume that*

(a) $\mathcal{R}$ *is Banach* f *-invariant,*

(b) if $\{x_n\}$ *is a sequence in* X *such that* $(x_{n-1},x_n)\in\mathcal{R}$ *for all* $n\in\mathbb{N}$ *and* $x_n\to z\in X$ *as* $n\to\infty$*, then* $(x_{n-1},z)\in\mathcal{R}$*, for all* $n\in\mathbb{N}$*;*

(c) $Fix(f)$ *is well ordered with respect to* $\mathcal{R}$*.*

Let there exists $x_0\in X$ *such that* $(x_0,fx_0)\in\mathcal{R}$ *and* $\lambda s<1$*, where*

$$\lambda=\frac{d(x_0,fx_0)+d(fx_0;f^2x_0)}{d(x_0,fx_0)+d(fx_0;f^2x_0)+1}+k.$$

Then,

(i) f *has at least one fixed point* $z\in X$*,*

(ii) the Picard sequence of initial point $x_0\in X$ *converges to a fixed point of* f*,*

(iii) if $z,w\in X$ *are two distinct fixed points of* f*, then* $d(z,w)\geqslant\frac{s-k}{2}$*.*

Proof. Let $x_0\in X$ be such that $(x_0,fx_0)\in\mathcal{R}$, $\lambda s<1$ and let $\{x_n\}$ be a Picard sequence with initial point x_0. If $x_{n-1}=x_n$ for some $n\in\mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1}\neq x_n$ for all $n\in\mathbb{N}$. From $(x_0,x_1)=(x_0,fx_0)\in\mathcal{R}$, since $\mathcal{R}$ is Banach f-invariant, we deduce that $(x_1,x_2)=(fx_0,f^2x_0)\in\mathcal{R}$. This implies that $(x_{n-1},x_n)=(f^{n-1}x_0,f^nx_0)\in\mathcal{R}$ for all $n\in\mathbb{N}$. Using the contractive condition (1) with $x=x_{n-1}$ and $y=x_n$, we get

$$\begin{aligned} d(x_n,x_{n+1})=d(fx_{n-1},fx_n) &\leqslant (\tfrac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{s[d(x_{n-1},fx_{n-1})+d(x_n,fx_n)+1]}+k)d(x_{n-1},x_n)\\ &= (\tfrac{d(x_{n-1},x_{n+1})}{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})+1]}+k)d(x_{n-1},x_n)\\ &\leqslant (\tfrac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+1]}+k)d(x_{n-1},x_n) \quad (2)\end{aligned}$$

for all $n\in\mathbb{N}$. From (2), by Corollary 2.3, we get

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leqslant \left(\tfrac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{[d(x_{n-1},x_n)+d(x_n,x_{n+1})+1]} + k\right) d(x_{n-1}, x_n) \\
&\leqslant \left(\tfrac{d(x_0,x_1)+d(x_1,x_2)}{[d(x_0,x_1)+d(x_1,x_2)+1]} + k\right) d(x_{n-1}, x_n) \\
&= \lambda d(x_{n-1}, x_n),
\end{aligned} \tag{3}
$$

for all $n \in \mathbb{N}$. Thus, for any $m, n \in \mathbb{N}$ with $m > n$, we have

$$
\begin{aligned}
d(x_n, x_m) &\leqslant s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\
&\leqslant sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + ... + s^{m-n-1} d(x_{m-1}, x_m) \\
&= \textstyle\sum_{i=1}^{m-n-1} s^i d(x_{n+i-1}, x_{n+i}) \leqslant \sum_{i=1}^{m-n-1} s^i \lambda^{i+n-1} d(x_0, x_1) \\
&= \lambda^{n-1} d(x_0, x_1) \textstyle\sum_{i=1}^{m-n-1} (\lambda s)^i \\
&= \lambda^{n-1} d(x_0, x_1)(\lambda s)^{1-n} \textstyle\sum_{i=n}^{m-1} (\lambda s)^i \\
&= d(x_0, x_1) s^{1-n} \textstyle\sum_{i=n}^{m-1} (\lambda s)^i.
\end{aligned} \tag{4}
$$

As $\lambda s < 1$ and $s > 1$, the last term in the above tends to zero, as $m, n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. Now we show that z is a fixed point of f. By assumption (b), we deduce that $(x_n, z) \in R$. So, by (1), we have

$$
\begin{aligned}
d(x_{n+1}, fz) = d(fx_n, fz) &\leqslant \left(\tfrac{d(x_n,fz)+d(z,fx_n)}{s[d(x_n,fx_n)+d(z,fz)+1]} + k\right) d(x_n, z) \\
&= \left(\tfrac{d(x_n,fz)+d(z,x_{n+1})}{s[d(x_n,x_{n+1})+d(z,fz)+1]} + k\right) d(x_n, z).
\end{aligned} \tag{5}
$$

Taking limit as $n \to \infty$, in the above inequality, we get $d(z, fz) \leqslant 0$. Thus, $d(z, fz) = 0$, that is $z = fz$. Thus (i) and (ii) hold. Now, let z, w are two distinct fixed points of f. Then, we have

$$
d(z, w) = d(fz, fw) \leqslant \left(\frac{d(z, fw) + d(w, fz) + k}{s}\right) d(z, w),
$$

which implies that $d(z, w) \geqslant \frac{s-k}{2}$. $\square$

Also, we can prove the following result with a weaker contractive condition.

Theorem 3.2. *Let (X, d) be a complete b-metric space with parameter $s > 1$ endowed with a binary relation $\mathcal{R}$ on X and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leqslant (\frac{d(x, fy) + d(y, fx)}{s[d(x, fx) + d(y, fy) + 1]} + k)d(x, y) + Ld(y, fx) \quad (6)$$

for all $(x, y) \in \mathcal{R}$, where $k \in [0, 1)$ and L is a nonnegative real number. Also, assume that

(a) $\mathcal{R}$ is Banach f-invariant,

(b) if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in \mathcal{R}$, for all $n \in \mathbb{N}$;

(c) $Fix(f)$ is well ordered with respect to $\mathcal{R}$.

Let there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{R}$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, fx_0) + d(fx_0; f^2x_0)}{d(x_0, fx_0) + d(fx_0; f^2x_0) + 1} + k.$$

Then,

(i) f has at least one fixed point $z \in X$,

(ii) the Picard sequence with initial point $x_0 \in X$ converges to a fixed point of f,

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geqslant \max\{\frac{s(1-L)-k}{2}, 0\}$.

Proof. Let $x_0 \in X$ be such that $(x_0, fx_0) \in \mathcal{R}$, $\lambda s < 1$ and let $\{x_n\}$ be a Picard sequence with initial point x_0. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From $(x_0, x_1) = (x_0, fx_0) \in \mathcal{R}$, since $\mathcal{R}$ is Banach f-invariant, we deduce $(x_1, x_2) = (fx_0, f^2x_0) \in \mathcal{R}$. This implies that $(x_{n-1}, x_n) = (f^{n-1}x_0, f^nx_0) \in \mathcal{R}$

for all $n \in \mathbb{N}$. Using the contractive condition (6) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) &\leqslant (\tfrac{d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})}{s[d(x_{n-1}, fx_{n-1}) + d(x_n, fx_n) + 1]} + k) d(x_{n-1}, x_n) + Ld(x_n, x_n) \\ &= (\tfrac{d(x_{n-1}, x_{n+1})}{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1]} + k) d(x_{n-1}, x_n) \\ &\leqslant (\tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1]} + k) d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. As in the proof of Theorem 3.1, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. Now we show that z is a fixed point of f. By assumption (b), we deduce that $(x_n, z) \in \mathcal{R}$. So, by (6), we have

$$\begin{aligned} d(x_{n+1}, fz) = d(fx_n, fz) &\leqslant (\tfrac{d(x_n, fz) + d(z, fx_n)}{s[d(x_n, fx_n) + d(z, fz) + 1]} + k) d(x_n, z) + Ld(x_n, z) \\ &= (\tfrac{d(x_n, fz) + d(z, x_{n+1})}{s[d(x_n, x_{n+1}) + d(z, fz) + 1]} + k) d(x_n, z) + Ld(x_n, z). \end{aligned} \tag{7}$$

Taking limit as $n \to \infty$ in the above inequality, we get $d(z, fz) \leqslant 0$. Thus, $d(z, fz) = 0$, that is $z = fz$. Thus, (i) and (ii) hold. Now, let z, w are two distinct fixed points of f. Then, we have

$$d(z, w) = d(fz, fw) \leqslant (\frac{d(z, fw) + d(w, fz) + k}{s}) d(z, w) + Ld(z, w),$$

which implies that $d(z, w) \geqslant \frac{s(1-L)-k}{2}$. Thus, (iii) holds. $\square$

Putting $\mathcal{R} = X \times X$ in Theorems 3.1 and 3.2, we obtain the following results in b-metric spaces:

Theorem 3.3. *Let (X, d) be a complete b-metric space with parameter $s > 1$ and let $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leqslant (\frac{d(x, fy) + d(y, fx)}{s[d(x, fx) + d(y, fy) + 1]} + k) d(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$. Assume that there exists $x_0 \in X$ such that $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, fx_0) + d(fx_0; f^2x_0)}{d(x_0, fx_0) + d(fx_0; f^2x_0) + 1} + k.$$

Then,

(i) f *has at least one fixed point* $z \in X$,

(ii) *the Picard sequence with initial point* $x_0 \in X$ *converges to a fixed point of* f,

(iii) *if* $z, w \in X$ *are two distinct fixed points of* f, *then* $d(z,w) \geqslant \frac{s-k}{2}$.

Theorem 3.4. *Let* (X,d) *be a complete* b*-metric space with parameter* $s > 1$ *and let* $f : X \to X$ *be a nonexpansive mapping such that*

$$d(fx,fy) \leqslant \left(\frac{d(x,fy)+d(y,fx)}{s[d(x,fx)+d(y,fy)+1]} + k\right)d(x,y) + Ld(y,fx)$$

for all $(x,y) \in X$, *where* $k \in [0,1)$ *and* L *is a nonnegative real number. Let there exists* $x_0 \in X$ *such that* $\lambda s < 1$, *where*

$$\lambda = \frac{d(x_0,fx_0)+d(fx_0,f^2x_0)}{d(x_0,fx_0)+d(fx_0,f^2x_0)+1} + k.$$

Then,

(i) f *has at least one fixed point* $z \in X$,

(ii) *the Picard sequence with initial point* $x_0 \in X$ *converges to a fixed point of* f,

(iii) *if* $z, w \in X$ *are two distinct fixed points of* f, *then* $d(z,w) \geqslant \max\{\frac{s(1-L)-k}{2}, 0\}$.

Example 3.5. Let $X = [0,1] \cup [\frac{5}{2}, \infty)$ and $d : X \times X :\to [0,\infty)$ be defined by $d(x,y) = (x-y)^2$. Define $f : X \to X$ by $fx = \begin{cases} \frac{1}{2} + \frac{1}{2}x & if\ x \in [0,1], \\ \frac{5}{4} + \frac{1}{2}x & if\ x \in [\frac{5}{2}, \infty). \end{cases}$

It is clear that (X,d) is a complete b-metric space with parameter $s = 2$ and f is nonexpansive. Also, if $x, y \in [0,1]$ or $x, y \in [\frac{5}{2}, \infty)$, then

$$d(fx,fy) = \frac{1}{4}(x-y)^2 \leqslant \frac{1}{2}(x-y)^2 = \frac{1}{2}d(x,y).$$

If $x \in [0,1]$ and $y \in [\frac{5}{2}, \infty)$, then

$$\frac{d(x,fy)+d(y,fx)}{s[d(x,fx)+d(y,fy)+1]} \geqslant \frac{3}{4}. \tag{8}$$

In fact,

$$\begin{aligned}
&\tfrac{d(x,fy)+d(y,fx)}{s[d(x,fx)+d(y,fy)+1]} \geqslant \tfrac{3}{4}\\
&\Longleftrightarrow \tfrac{(x-\frac{5}{4}-\frac{1}{2}y)^2+(y-\frac{1}{2}-\frac{1}{2}x)^2}{[(x-\frac{1}{2}-\frac{1}{2}x)^2+(y-\frac{5}{4}-\frac{1}{2}y)^2+1]} \geqslant \tfrac{3}{2}\\
&\Longleftrightarrow 2x^2+2(\tfrac{5}{4}+\tfrac{1}{2}y)^2-4x(\tfrac{5}{4}+\tfrac{1}{2}y)+2y^2+2(\tfrac{1}{2}+\tfrac{1}{2}x)^2-4y(\tfrac{1}{2}+\tfrac{1}{2}x)\\
&\geqslant 3x^2+3(\tfrac{1}{2}+\tfrac{1}{2}x)^2-6x(\tfrac{1}{2}+\tfrac{1}{2}x)+3y^2+3(\tfrac{5}{4}+\tfrac{1}{2}y)^2-6y(\tfrac{5}{4}+\tfrac{1}{2}y)+3\\
&\Longleftrightarrow 7x^2+7y^2-16xy-10x+17y \geqslant \tfrac{77}{4}\\
&\Longleftrightarrow 7(y-x)^2+17y \geqslant \tfrac{63}{4}+\tfrac{14}{4}+2x(y+5).
\end{aligned} \tag{9}$$

Since $(y-x) \geqslant \frac{3}{2}$, thus $7(y-x)^2 \geqslant \frac{63}{4}$. It is sufficient to show that $17y \geqslant \frac{14}{4}+2x(y+5)$. Now, since $x \leqslant 1$, it is sufficient to show $17y \geqslant \frac{14}{4}+2(y+5)$ or equally $y \geqslant \frac{54}{60}$ which is desired. Thus, (7) holds. Now

$$\begin{aligned}
d(fx,fy) &= (\tfrac{5}{4}+\tfrac{1}{2}y-\tfrac{1}{2}-\tfrac{1}{2}x)^2\\
&= (\tfrac{3}{4}+\tfrac{1}{2}(y-x))^2 \leqslant (\tfrac{3}{4}+\tfrac{1}{4})(y-x)^2\\
&\leqslant (\tfrac{d(x,fy)+d(y,fx)}{s[d(x,fx)+d(y,fy)+1]}+\tfrac{1}{4})d(x,y).
\end{aligned} \tag{10}$$

Also, for $k=\frac{1}{4}$ and $x_0=\frac{1}{8}$, we have

$$\lambda = \frac{d(x_0,fx_0)+d(fx_0,f^2x_0)}{d(x_0,fx_0)+d(fx_0,f^2x_0)+1}+k = \frac{245}{1269}+\frac{1}{4}.$$

Therefore, $\lambda s = (\frac{245}{1269}+\frac{1}{4})2 = \frac{490}{1269}+\frac{1}{2} < 1$. Thus, all of the conditions of Theorem 3.3 are satisfied and so T has a fixed point. Here, $z=1$ and $w=\frac{5}{2}$ are two fixed points of f. Also,

$$d(z,w) = (1-\frac{5}{2})^2 = \frac{9}{4} \geqslant \frac{7}{8} = \frac{2-\frac{1}{4}}{2} = \frac{s-k}{2}.$$

Note that f is not a contraction. In fact, $d(f1, f\frac{5}{2}) = d(1, \frac{5}{2})$.

Let (X, d) be a b-metric space and $\preceq$ be an order on X. Then, the triple $(X, d, \preceq)$ is called a partial ordered b-metric space. Then, two elements $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$. Also, $(X, d, \preceq)$ is called regular if for any nondecreasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$ as $n \to \infty$, then $x_n \preceq z$, for all $n \in \mathbb{N}$. Note that $\mathcal{R} = \{(x, y) : x \preceq y\}$ is a binary relation on X. Also, if $f : X \to X$ be nondecreasing, then $\mathcal{R}$ is Banach f-invariant.

Theorem 3.6. *Let $(X, d, \preceq)$ be a complete ordered b-metric space with parameter $s > 1$ and $f : X \to X$ be a nonexpansive nondecreasing mapping such that*

$$d(fx, fy) \leqslant \left(\frac{d(x, fy) + d(y, fx)}{s[d(x, fx) + d(y, fy) + 1]} + k\right)d(x, y)$$

for all comparable elements $x, y \in X$, where $k \in [0, 1)$. Also assume that

(a) $(X, d, \preceq)$ is regular,

(b) $Fix(f)$ is well ordered with respect to $\preceq$.

Let there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, fx_0) + d(fx_0; f^2x_0)}{d(x_0, fx_0) + d(fx_0; f^2x_0) + 1} + k.$$

Then,

(i) f has at least one fixed point $z \in X$,

(ii) the Picard sequence with initial point $x_0 \in X$ converges to a fixed point of f,

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geqslant \frac{s-k}{2}$.

Theorem 3.7. *Let $(X, d, \preceq)$ be a complete ordered b-metric space with parameter $s > 1$ and $f : X \to X$ be a nonexpansive nondecreasing mapping such that*

$$d(fx, fy) \leqslant \left(\frac{d(x, fy) + d(y, fx)}{s[d(x, fx) + d(y, fy) + 1]} + k\right)d(x, y) + Ld(x, y)$$

for all comparable elements $x, y \in X$, where $k \in [0,1)$ and L is a non-negative real number. Also assume that

(a) $(X, d, \preceq)$ is regular,

(b) $Fix(f)$ is well ordered with respect to $\preceq$.

Let there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, fx_0) + d(fx_0; f^2x_0)}{d(x_0, fx_0) + d(fx_0; f^2x_0) + 1} + k.$$

Then,

(i) f has at least one fixed point $z \in X$,

(ii) the Picard sequence with initial point $x_0 \in X$ converges to a fixed point of f,

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geqslant \frac{s(1-L)-k}{2}$.

4. Fixed Point Results for Nonexpansive Multi-Valued Mappings

In this section, we give some fixed point results for nonexpansive multi-valued mappings in b-metric spaces. Let (X, d) be a b-metric space and $K(X)$ be the set of all nonempty compact subsets of X.

Definition 4.1. *Let $\mathcal{R}$ be a binary relation on X. Then, $\mathcal{R}$ is called Banach T-invariant if for any $x \in X$ and $y \in Tx$ with $(x, y) \in \mathcal{R}$, then we have $(y, z) \in \mathcal{R}$ for all $z \in Ty$.*

Note that if T be a single-valued mapping, then Definition 4.1 reduces to the definition of f-invariant for single-valued mappings.

Theorem 4.2. *Let (X, d) be a complete b-metric space with parameter $s > 1$ endowed with a binary relation $\mathcal{R}$ on X and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leqslant (\frac{d(x, Ty) + d(y, Tx)}{s[d(x, Tx) + d(y, Ty) + 1]} + k)d(x, y) \tag{11}$$

for all $(x, y) \in \mathcal{R}$, where $k \in [0,1)$. Also, assume that *(a)* *$\mathcal{R}$ is Banach T-invariant,*

(b) *if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in \mathcal{R}$, for all $n \in \mathbb{N}$;*

Let there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$, $d(x_0, x_1) = d(x_0, Tx_0)$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T has at least one fixed point $z \in X$.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ be such that $(x_0, x_1) \in \mathcal{R}$, $d(x_0, x_1) = d(x_0, Tx_0)$ and $\lambda s < 1$. Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$.

From $(x_0, x_1) \in \mathcal{R}$, since $\mathcal{R}$ is Banach T-invariant, we get $(x_1, x_2) \in \mathcal{R}$. Continuing this process, we have a sequence $\{x_n\}$ in X, such that $(x_{n-1}, x_n) \in \mathcal{R}$, $x_n \in Tx_{n-1}$ and $d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1})$ for all $n \in \mathbb{N}$. Since T is nonexpansine, we have $d(x_n, x_{n+1}) = d(x_n, Tx_n) \leqslant H(Tx_{n-1}, Tx_n) \leqslant d(x_{n-1}, x_n)$. Thus, by Corollary 2.3,

$$\{\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1}\}$$

is nonincreasing. Using the contractive condition (11) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) \leqslant H(Tx_{n-1}, Tx_n) \\ &\leqslant (\tfrac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{s[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + 1]} + k) d(x_{n-1}, x_n) \\ &= (\tfrac{d(x_{n-1}, x_{n+1})}{s[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1]} + k) d(x_{n-1}, x_n) \\ &\leqslant (\tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1]} + k) d(x_{n-1}, x_n) \end{aligned} \tag{12}$$

for all $n \in \mathbb{N}$. Thus, we get

$$\begin{aligned} d(x_n, x_{n+1}) &\leqslant (\tfrac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1]} + k) d(x_{n-1}, x_n) \\ &\leqslant (\tfrac{d(x_0, x_1) + d(x_1, x_2)}{[d(x_0, x_1) + d(x_1, x_2) + 1]} + k) d(x_{n-1}, x_n) \\ &= \lambda d(x_{n-1}, x_n), \end{aligned} \tag{13}$$

for all $n \in \mathbb{N}$. Thus, as in Theorem 3.1, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $z \in X$ such that $x_n \to z$. Now, we show that z is a fixed point of T. By assumption (b), we deduce that $(x_n, z) \in \mathcal{R}$. So, by (11), we have

$$\begin{aligned} d(z, Tz) &\leqslant s[d(z, x_{n+1}) + d(x_{n+1}, Tz)] \\ &\leqslant sd(z, x_{n+1}) + sH(Tx_n, Tz) \\ &\leqslant sd(z, x_{n+1}) + (\tfrac{d(x_n, Tz) + d(z, Tx_n)}{[d(x_n, Tx_n) + d(z, Tz) + 1]} + k)d(x_n, z) \\ &= sd(z, x_{n+1} + (\tfrac{d(x_n, Tz) + d(z, x_{n+1})}{[d(x_n, x_{n+1}) + d(z, Tz) + 1]} + k)d(x_n, z). \end{aligned} \tag{14}$$

Taking limit as $n \to \infty$, in the above inequality, we get $d(z, Tz) \leqslant 0$. Thus, $d(z, Tz) = 0$, that is, $z \in Tz$. $\square$

Also, we can prove the following result for nonexpansive multi-valued mapping with a weaker contractive condition.

Theorem 4.3. *Let (X, d) be a complete b-metric space with parameter $s > 1$ endowed with a binary relation $\mathcal{R}$ on X and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leqslant (\frac{d(x, Ty) + d(y, Tx)}{s[d(x, Tx) + d(y, Ty) + 1]} + k)d(x, y) + Ld(x, y) \tag{15}$$

for all $(x, y) \in \mathcal{R}$, where $k \in [0, 1)$ and L is a nonnegative real number. Also, assume that

(a) $\mathcal{R}$ is Banach T-invariant,

(b) if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to \infty$, then $(x_{n-1}, z) \in \mathcal{R}$, for all $n \in \mathbb{N}$;

Let there exists $x_0 \in X$, $x_1 \in Tx_0$ such that $(x_0, x_1) \in \mathcal{R}$, $d(x_0, x_1) = d(x_0, Tx_0)$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T has at least one fixed point $z \in X$.

Putting $\mathcal{R} = X \times X$ in Theorems 4.2 and 4.3, we obtain the following results in b-metric spaces:

Theorem 4.4. *Let (X, d) be a complete b-metric space with parameter $s > 1$ and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leqslant (\frac{d(x, Ty) + d(y, Tx)}{s[d(x, Tx) + d(y, Ty) + 1]} + k)d(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$. Assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $d(x_0, x_1) = d(x_0, Tx_0)$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T has at least one fixed point $z \in X$.

Theorem 4.5. *Let (X, d) be a complete b-metric space with parameter $s > 1$ and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leqslant (\frac{d(x, Ty) + d(y, Tx)}{s[d(x, Tx) + d(y, Ty) + 1]} + k)d(x, y) + Ld(x, y)$$

for all $x, y \in X$, where $k \in [0, 1)$ and L is a nonnegative real number. Assume that there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $d(x_0, x_1) = d(x_0, Tx_0)$ and $\lambda s < 1$, where

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T has at least one fixed point $z \in X$.

Example 4.6. Let $X = [0, 1] \cup [\frac{5}{2}, \infty)$ and let $d : X \times X :\to [0, \infty)$ be defined by $d(x, y) = (x - y)^2$. Define $T : X \to K(X)$ by

$$Tx = \begin{cases} \left[\frac{1}{2} + \frac{1}{2}x, 1\right], \text{if } x \in [0, 1], \\ \left[\frac{5}{2}, \frac{5}{4} + \frac{1}{2}x\right], \text{if } x \in [\frac{5}{2}, \infty). \end{cases}$$

It is clear that (X, d) is a complete b-metric space with parameter $s = 2$ and T is nonexpansive. Also, if $x, y \in [0, 1]$ or $x, y \in [\frac{5}{2}, \infty)$, then

$$H(Tx, Ty) = \frac{1}{4}(x - y)^2 \leqslant \frac{1}{2}(x - y)^2 = \frac{1}{2}d(x, y).$$

If $x \in [0,1]$ and $y \in [\frac{5}{2}, \infty)$, then

$$\frac{d(x,Ty)+d(y,Tx)}{s[d(x,Tx)+d(y,Ty)+1]} \geqslant \frac{3}{4}.$$

Now,

$$\begin{aligned} H(Tx,Ty) &= \max\{\tfrac{5}{2}-\tfrac{1}{2}-\tfrac{1}{2}x, 1+\tfrac{1}{2}y-1\})^2 \\ &= \max\{2-\tfrac{1}{2}x, \tfrac{1}{2}y\})^2 \leqslant (\tfrac{3}{4}+\tfrac{1}{4})(y-x)^2 \\ &\leqslant (\tfrac{d(x,Ty)+d(y,Tx)}{s[d(x,Tx)+d(y,Ty)+1]}+\tfrac{1}{4})d(x,y). \end{aligned}$$

Also, for $k = \frac{1}{4}$ and $x_0 = \frac{1}{8}$, $d(x_0, Tx_0) = d(\frac{1}{8}, \frac{3}{16})$. Thus, with $x_1 = \frac{3}{16}$ we have

$$\lambda = \frac{d(x_0,x_1)+d(x_1,Tx_1)}{d(x_0,x_1)+d(x_1,Tx_1)+1} + k = \frac{245}{1269} + \frac{1}{4}.$$

Therefore, $\lambda s = (\frac{245}{1269}+\frac{1}{4})2 = \frac{490}{1269}+\frac{1}{2} < 1$. Thus, all of the conditions of Theorem 4.4 are satisfied and so T has a fixed point. Here, $z = 1$ and $w = \frac{5}{2}$ are two fixed points of T. Also,

$$d(z,w) = (1-\frac{5}{2})^2 = \frac{9}{4} \geqslant \frac{7}{8} = \frac{2-\frac{1}{4}}{2} = \frac{s-k}{2}.$$

Note that T is not a contraction. In fact, $H(T1, T\frac{5}{2}) = d(1, \frac{5}{2})$.

Theorem 4.7. *Let $(X, d, \preceq)$ be a complete ordered b-metric space with parameter $s > 1$ and $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx,Ty) \leqslant (\frac{d(x,Ty)+d(y,Tx)}{s[d(x,Tx)+d(y,Ty)+1]} + k)d(x,y)$$

for all comparable elements $x, y \in X$, where $k \in [0,1)$. Also, assume that

(a) for any $x \in X$ and $y \in Tx$ with $x \preceq y$, then we have $y \preceq z$ for all $z \in Ty$.

(b) $(X, d, \preceq)$ is regular,

Let there exists $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $d(x_0, Tx_0) = d(x_0, x_1)$, $x_0 \preceq x_1$ *and* $\lambda s < 1$, *where*

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T *has at least one fixed point* $z \in X$.

Theorem 4.8. *Let* $(X, d, \preceq)$ *be a complete ordered* b*-metric space with parameter* $s > 1$ *and* $T : X \to K(X)$ *be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leqslant (\frac{d(x, Ty) + d(y, Tx)}{s[d(x, Tx) + d(y, Ty) + 1]} + k)d(x, y) + Ld(x, y)$$

for all comparable elements $x, y \in X$, *where* $k \in [0, 1)$ *and* L *is a non-negative real number. Also, assume that*

(a) *for any* $x \in X$ *and* $y \in Tx$ *with* $x \preceq y$, *then we have* $y \preceq z$ *for all* $z \in Ty$,

(b) $(X, d, \preceq)$ *is regular.*

Let there exists $x_0 \in X$ *and* $x_1 \in Tx_0$ *such that* $d(x_0, Tx_0) = d(x_0, x_1)$, $x_0 \preceq x_1$ *and* $\lambda s < 1$, *where*

$$\lambda = \frac{d(x_0, x_1) + d(x_1, Tx_1)}{d(x_0, x_1) + d(x_1, Tx_1) + 1} + k.$$

Then, T *has at least one fixed point* $z \in X$.

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