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Generalized Partial Metric Spaces With A Fixed Point Theorem

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Abstract. In this paper, we introduce the notion of extended partial metric space and we present some fixed point theorems in generalized partial metric spaces involving linear and nonlinear contractions.

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1. Introduction and Preliminaries

Very recently, Aydi and Czerwik [2] proposed a new notion, generalized b-metric space and investigated the existence and uniqueness of a fixed point of certain mappings on this new space. In this paper, we introduce the generalized partial metric space inspired of the notion of a partial metric space was introduced by Matthews [18] in 1994 as a part to study the denotational semantics of dataflow networks which play an important role in constructing models in the theory of computation (see also e.g. ([1, 2, 5, 14, 19]).

Definition 1.1. *(cf. [18]) A generalized partial metric on a nonempty set X is a function $p : X \times X \to [0, \infty]$ such that for all $x, y, z \in X$*

(PM1) $p(x, x) = p(x, y) = p(y, y)$, *then* $x = y$;

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(PM2) $p(x,x) \leqslant p(x,y)$;

(PM3) $p(x,y) = p(y,x)$;

(PM4) $p(x,z) + p(y,y) \leqslant p(x,y) + p(y,z)$.

The pair (X,p) *is then called a generalized partial metric space (gpms).*

As usual, by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}_+$ we denote the set of all natural numbers, the set of all nonnegative integers or the set of all nonnegative real numbers, respectively. If $f\colon X \to X$, by f^n we denote the n-th iterate of f:

$$f^0(x) = x, \quad x \in X; \quad f^{n+1} = f \circ f^n.$$

Here the symbol $\varphi \circ f$ denotes the function $\varphi[f(x)]$ for $x \in X$.

As in [18], we may state the following definitions and remarks. If p is a generalized partial metric on X, then the function $d_p : X \times X \to [0,\infty]$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

for all $x, y \in X$, is a generalized metric on X (defined in [**?**] with $s = 1$). More precisely, for a nonempty set X, a function $d_p : X \times X \to [0,\infty]$ is called a generalized metric space if and only if for $x, y, z \in X$ the conditions are satisfied:

(d_1) $d_p(x,y) = 0$ if and only if $x = y$, (self-distance)

(d_2) $d_p(x,y) = d_p(y,x)$, (symmetry)

(d_3) $d_p(x,y) \leqslant d_p(x,z) + d_p(z,y)$ (triangle inequality).

Note that if a sequence converges in a generalized partial metric space (X,p) with respect to the topology of d_p, then it converges with respect to the topology of p.

Also, a sequence $\{x_n\}$ is Cauchy in a generalized partial metric space (X,p) if and only if it is Cauchy in the generalized metric space (X,d_p). Consequently, a generalized partial metric space (X,p) is complete if and only if the generalized metric space (X,d_p) is complete. Moreover, if $\{x_n\}$ is a sequence in a generalized partial metric space (X,p) and $x \in X$, one has that

$$\lim_{n\to\infty} d_p(x_n,x) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 1.2. *Let* (X,p) *be a generalized partial metric space. We say that* $T : X \to X$ *is (sequentially) continuous if* $p(x_n,x) \to p(x,x)$, *then* $p(Tx_n,Tx) \to p(Tx,Tx)$ *as* $n \to \infty$.

Lemma 1.3. *Let* (X,p) *be a generalized partial metric space. Then*
(1) *if* $p(x,y) = 0$, *we have* $x = y$,
(2) *if* $x \neq y$, *we have* $p(x,y) > 0$.

2. Linear Quasi-Contractions

We start with the following theorem

Theorem 2.1. *Let (X, d) be a complete generalized partial metric space. Assume that $T\colon X \to X$ is continuous on (X, d_p). If there exists an $\alpha \in [0, 1)$ such that*

$$p(T(x), T^2(x)) \leqslant \alpha p(x, T(x)), \tag{1}$$

for $x \in X$ with$p(x, T(x)) < \infty$, then, for an arbitrary fixed $x \in X$, one of the following alternative holds : either

(A) for every nonnegative integer $n \in \mathbb{N}_0$,

$$p(T^n(x), T^{n+1}(x)) = \infty,$$

or

(B) there exists an $k \in \mathbb{N}_0$ such that

$$p(T^k(x), T^{k+1}(x)) < \infty.$$

If (B) holds, then, we also conclude the followings:

(i) the sequence $\{T^m(x)\}$ is a Cauchy sequence in (X, p);

(ii) there exists a point $u \in X$ such that

$$\lim_{m\to\infty} d_p(T^m(x), u) = 0 \quad \textit{and} \quad T(u) = u.$$

Proof. From (1) we get (in case (B))

$$p(T^{k+1}(x), T^{k+2}(x)) \leqslant \alpha p(T^k(x), T^{k+1}(x)) < \infty$$

and by induction

$$p(T^{k+n}(x), T^{k+n+1}(x)) \leqslant \alpha^n p(T^k(x), T^{k+1}(x)), \quad n = 0, 1, 2, \dots\ . \tag{2}$$

Consequently, for $n, v \in \mathbb{N}_0$, by (2) we obtain

$$\begin{aligned} p(T^{k+n}(x), T^{k+n+v}(x)) &\leqslant p(T^{k+n}(x), T^{k+n+1}(x)) + \ldots + p(T^{k+n+v-2}(x), T^{k+n+v-1}(x)) \\ &+ p(T^{k+n+v-1}(x), T^{k+n+v}(x)) \\ &\leqslant \alpha^n p(T^k(x), T^{k+1}(x)) + \ldots + \alpha^{n+v-2} p(T^k(x), T^{k+1}(x)) \\ &+ \alpha^{n+v-1} p(T^k(x), T^{k+1}(x)) \\ &\leqslant \alpha^n [1 + s\alpha + \ldots + (\alpha)^{v-1}] p(T^k(x), T^{k+1}(x)) \\ &\leqslant \alpha^n \sum_{m=0}^{\infty} (\alpha)^m p(T^k(x), T^{k+1}(x)) \\ &\leqslant \frac{\alpha^n}{1-\alpha} p(T^k(x), T^{k+1}(x)). \end{aligned}$$

Finally, we derive that

$$p(T^{k+n}(x), T^{k+n+v}(x)) \leqslant \frac{\alpha^n}{1-\alpha} p(T^k(x), T^{k+1}(x)) \tag{3}$$

for $n, v \in \mathbb{N}_0$. By (3) it follows that $\{T^n(x)\}$ is a Cauchy sequence in (X, p), which is complete, so there exists $u \in X$ such that

$$\lim_{n\to\infty} p(T^n(x), u) = p(u, u) = \lim_{n,m\to\infty} p(T^n(x), T^m(x)) = 0.$$

We have $\lim_{n\to\infty} d_p(T^n(x), u) = 0$. Since T is continuous on (X, d_p), we have

$$\lim_{n\to\infty} d_p(T^{n+1}(x), Tu) = \lim_{n\to\infty} d_p(T(T^n(x)), Tu) = 0.$$

Moreover, $\lim_{n\to\infty} d_p(T^{n+1}(x), Tu) = d_p(u, Tu)$. By uniqueness of limit, we get $T(u) = u$. and u is a fixed point of T, which ends the proof. □

Remark 2.2. *Theorem 2.1 extends the results of Aydi and Czerwik ([2] with $s = 1$), Diaz and Margolis [4], Luxemburg [15, 16] and Banach ([3] to generalized partial metric spaces.*

3. Nonlinear Contractions

In this section, we present the following result.

Theorem 3.1. *Assume that (X, p) is a complete generalized partial space. Suppose that $T\colon X \to X$ satisfies the condition*

$$p(T(x), T(y)) \leqslant \varphi[p(x, y)] \tag{4}$$

for $x, y \in X$, $p(x, y) < \infty$, *where* $\varphi\colon [0, \infty) \to [0, \infty)$ *is nondecreasing and*

$$\lim_{n\to\infty} \varphi^n(z) = 0 \quad for\ z > 0. \tag{5}$$

Let $x \in X$ *be arbitrarily fixed. Then the following alternative holds: either*

(C) *for every nonnegative integer* $n \in \mathbb{N}_0$

$$p(T^n(x), T^{n+1}(x)) = \infty,$$

or

(D) *there exists an* $k \in \mathbb{N}_0$ *such that*

$$p(T^k(x), T^{k+1}(x)) < \infty.$$

In (D), T *has a unique fixed point in* $A := \{t \in X : d_p(T^k(x), t) < \infty\}$.

Proof. First, take $x \in X$ and $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that

$$\varphi^n(\varepsilon) < \frac{\varepsilon}{2}.$$

Put $\alpha = \varphi^n$ and $x_m = T^{m+n}(x)$ for $m \in \mathbb{N}$. Then for all $x, y \in X$ such that $p(x, y) < \infty$, one gets

$$p(T^n(x), T^n(y)) \leqslant \varphi^n[p(x, y)] = \alpha[p(x, y)]. \tag{6}$$

Consider the following set

$$B := \{t \in X : p(T^k(x), t) < \infty\}.$$

Clearly, $B \subset A$ and $T^k(x),\ T^{k+1}(x) \in B$.

Now we observe that $T\colon B \to B$. Indeed, if $t \in B$, i.e., $p(T^k(x), t) < \infty$, then

$$\begin{aligned} p(T^k(x), T(t)) &\leqslant p(T^k(x), T^{k+1}(x)) + p(T^{k+1}(x), T(t))] \\ &\leqslant \varepsilon_1 + \varphi[p(T^k(x), t)] \\ &\leqslant \varepsilon_1 + \varepsilon_2 < \infty, \end{aligned}$$

where ε_1 and ε_2 are some positive numbers. Consequently, $T^n\colon B \to B$. Put $T^n = F$. We have $F\colon B \to B$. We rewrite (6) as

$$p(F(x), F(y)) \leqslant \varphi^n[p(x, y)] = \alpha[p(x, y)]. \tag{7}$$

For $t \in B$, we have $\{F^m(t)\} \subset B$, for all $m \in \mathbb{N}_0$. We verify that $\{F^m(t)\}$ is a Cauchy sequence. In fact, putting $y_m = F^m(t)$, $m \in \mathbb{N}_0$, we get

$$p(F(t), F^2(t)) = p(T^n(t), T^{n+1}(t)) \leqslant \alpha[p(t, T^n(t))].$$

By induction, we get

$$p(F^m(t), F^{m+1}(t)) \leqslant \alpha^m[p(t, F(t))],$$

that is equivalent to

$$p(y_m, y_{m+1}) \leqslant \alpha^m[p(t, F(t))].$$

Consequently, $p(y_m, y_{m+1}) \to 0$ as $m \to \infty$. Let m be such that

$$p(y_m, y_{m+1}) < \frac{\varepsilon}{2}.$$

Then for every $z \in K(y_m, \varepsilon) := \{y \in X : p(y_m, y) \leqslant \varepsilon\}$, we obtain

$$p(F(z), F(y_m)) \leqslant \alpha[p(z, y_m)] \leqslant \alpha(\varepsilon) = \varphi^n(\varepsilon) < \frac{\varepsilon}{2}.$$

Also, we know that

$$p(F(y_m), y_m) = p(y_{m+1}, y_m) < \frac{\varepsilon}{2}.$$

Thus we have

$$p(T^n(z), y_m) = p(F(z), y_m) \leqslant p(F(z), F(y_m)) + p(F(y_m), y_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $F = T^n$ maps $K(y_m, \varepsilon)$ into itself. Therefore

$$p(y_r, y_l) \leqslant 2\varepsilon \quad for \quad r, l \geqslant m,$$

so $\{y_r\} = \{F^r(t)\}_r$ is a Cauchy sequence in B. Since $B \subset A$, $\{y_r\} = \{F^r(t)\}_r$ is a Cauchy sequence in A. Since (X, p) is complete, (X, d_p) is also complete. Clearly, (A, d_p) is closed, so it is complete. Hence there exists $u \in A \subset X$ such that

$$\lim_{r\to\infty} d_p(y_r, u) = 0.$$

We deduce that

$$p(u, u) = \lim_{r\to\infty} p(y_r, u) = \lim_{r,j\to\infty} p(y_r, y_j) = 0. \tag{8}$$

Thus, for a large r,

$$p(y_r, u) < \infty. \tag{9}$$

Also, we have

$$\lim_{r\to\infty} d_p(y_{r+1}, Fu) = d_p(u, F(u)). \tag{10}$$

Moreover, by (7) and (9),

$$p(y_{r+1}, F(u)) = p(F(y_r), F(u)) \leqslant \alpha[p(y_r, u)] \tag{11}$$

letting $r \to \infty$ in (11), due to (8), we get

$$\lim_{r\to\infty} p(y_{r+1}, F(u)) = 0. \tag{12}$$

Consequently, we find

$$\lim_{r\to\infty} d_p(y_{r+1}, F(u)) = 0. \tag{13}$$

Comparing (10) to (13) yields that $d_p(u, F(u)) = 0$, i.e., $u = F(u)$, that is, u is a fixed point of F. Suppose there are two different fixed points u and v of F in A. Then

$$d_p(u, v) \leqslant d_p(u, T^n(x)) + d_p(T^n(x), v) < \infty.$$

Now, applying (4),

$$p(u, v) = p(F(u), F(v)) \leqslant \alpha[p(u, v)].$$

Taking into consideration that $\alpha(t) = \varphi^n(t) < t$ for any $t > 0$, we get a contradiction. Thus, F has exactly one fixed point in A. Now, we shall show that u is also a fixed point of T. Applying (4) and (9),

$$p(T(u), T(y_r)) \leqslant \varphi(p(y_r, u).$$

In view of (8),

$$\lim_{r\to\infty} p(T(u), T(y_r)) = 0. \tag{14}$$

On the other hand,

$$p(T(u), Ty_r) = p(T(u), T(F^r(t))) = p(T(u), F^r(T(t))) \to p(T(u), u). \quad \square$$

By comparison, we deduce that $p(u, T(u)) = 0$, so $u = T(u)$, hence u is a fixed point of T. Again, obviously by (4) such point is the unique fixed point of T in A.

If X is a partial metric space, then $B = A = X$ and we have from Theorem 3.1.

Corollary 3.2. *Let (X, d) be a complete partial space. Suppose that $T\colon X \to X$ satisfies*

$$p(T(x), T(y)) \leqslant \varphi[p(x, y)], \quad x, y \in X,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. Then T has exactly one fixed point $u \in X$.

Remark 3.3. *Corollary 3.2 corresponds to Corollary 1 of Romaguera [19], which a Matkowski type result [17]. Theorem 2.1 extended the main result of Aydi and Czerwik [2] to generalized partial matric spaces.*

Competing interests
The authors declare that they have no competing interests.

Authors contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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