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# Smarandache Strong Hoop-algebras

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**Abstract.** In this paper, we define the $Q$-Smarandache hoop-algebras and $Q$-Smarandache filters, we obtain some related results. After that, by considering the notions of these filters we determine relationships between filters in hoop-algebras and $Q$-Smarandache filters in hoop-algebras. Finally, we introduce the concept of Smarandache 2-structure and Smarandache 2-filter on a hoop-algebras.

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## 1. Introduction

Hoop-algebras are naturally ordered commutative residuated integral monoids, introduced by Bosbach in [8, 9] and then studied by Buchi and Owens in [10], a paper never published. Subsequently, hoops theory was enriched with deep structure theorems [1, 8, 9]. Many of these results have a strong impact with fuzzy logic. Particularly, from the structure theorem of finite basic hoops one obtains an elegant short proof of the completeness theorem for propositional basic logic, introduced by Hajek [1, 12]. The algebraic structures corresponding to Hajek's propositional (fuzzy) basic logic, BL-algebras, are particular cases of hoops. The main example of BL-algebras is the $[0, 1]$ endowed with

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the structure induced by a t-norm. MV-algebras are the best known classes of BL-algebras. Recent investigations are concerned with non-commutative generalizations for these structures.

The Smarandache algebraic structures theory was introduced in 1998 by Padilla [15]. Kandasamy studied Smarandache groupoids, subgroupoids, ideal of groupoids, seminormal sub groupoids, Smarandache Bol groupoids, and strong Bol groupoids and obtained many interesting results about them [14]. Smarandache semigroups are very important for the study of congruences, and they were studied by Padilla [15]. In [13] Jun discussed the Smarandache structure in BCI-algebras. He introduced the notion of Smarandache (positive implicative, commutative, implicative) BCI-algebras, Smarandache subalgebras and Smarandache ideals and investigated some related properties. Smarandache BL-algebras and Smarandache n-structure on CI-algebras were devised by Borumand Saeid et al. [5, 6], and they deal with Smarandache ideal structures in Smarandache BL-algebras.

The aim of this paper is to study the Smarandache $Q$-hoop-algebras and $Q$-Smarandache filters. Also by considering the notions of these filters we determine relationships between filters in hoop-algebras and $Q$-Smarandache filters in hoop-algebras.

## 2. Preliminaries

In this section, we recollect some definitions and results which will be used in the following and we shall not cite them every time they are used.

**Definition 2.1.** *[1]. A hoop-algebra or hoop is an algebra* $(A, \odot, \longrightarrow, 1)$ *of type* $(2, 2, 0)$ *such that:*
*(HP1)* $(A, \odot, 1)$ *is an abelian monoid,*
*(HP2)* $x \to x = 1$,
*(HP3)* $(x \odot y) \to z = x \to (y \to z)$,
*(HP4)* $x \odot (x \to y) = y \odot (y \to x)$,
*for all* $x, y, z \in A$.

In every hoop-algebra $(A, \odot, \longrightarrow, 1)$, we define $x \leqslant y$ if and only if $x \to y = 1$. It is easy to see that $\leqslant$ is a partial order relation on $A$. A hoop $A$ is bounded if there is an element $0 \in A$ such that $0 \leqslant x$, for all $x \in A$.
We let $x^0 = 1$, $x^n = x^{n-1} \odot x$, for any $n \in N$. Let $A$ be a bounded hoop. We define a negation " $-$ " on $A$ by, $x^- = x \to 0$, for all $x \in A$. If $(x^-)^- = x$, for all $x \in A$, then the bounded hoop $A$ is said to have the double negation property, or (DNP), for short.

**Theorem 2.2.** *[8, 9]. In any* $(A, \odot, \to, 1)$ *the following properties are valid:*
*(HP5)* $(A, \leqslant)$ *is a meet-semilattice with* $x \wedge y = x \odot (x \to y)$,
*(HP6)* $x \odot z \leqslant y$ *if and only if* $z \leqslant x \to y$,
*(HP7)* $x \odot y \leqslant x, y$ *and* $x^n \leqslant x$, *for any* $n \in N$,
*(HP8)* $1 \to x = x$,
*(HP9)* $x \leqslant y \to x$,
*(HP10 )* $x \odot (x \to y) \leqslant y$,
*(HP11)* $x \leqslant (x \to y) \to y$,
*(HP12)* $x \to y \leqslant (y \to z) \to (x \to z)$,
*(HP13)* $(x \to y) \odot (y \to z) \leqslant (x \to z)$,
*(HP14)* $x \leqslant y$ *implies* $x \odot z \leqslant y \odot z$,
*(HP15)* $x \leqslant y$ *implies* $z \to x \leqslant z \to y$,
*(HP16)* $x \leqslant y$ *implies* $y \to z \leqslant x \to z$,
*(HP17)* $x \to (y \to z) = y \to (x \to z)$,
*for all* $x, y, z, \in A$.

**Proposition 2.3.** *[11]. Let* $A$ *be a hoop and for any* $x, y \in A$, *we define,*

$$x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

*Then the following conditions are equivalent:*
*(i)* $\sqcup$ *is associative,*
*(ii)* $x \leqslant y$ *implies* $x \sqcup z \leqslant y \sqcup z$, *for all* $x, y, z \in A$,
*(iii)* $x \sqcup (y \wedge z) \leqslant (x \sqcup y) \wedge (x \sqcup z)$, *for all* $x, y, z \in A$,
*(iv)* $\sqcup$ *is the join operation on* $A$.

**Definition 2.4.** *[11]. A hoop* $A$ *is called a* $\sqcup$*-hoop, if* $\sqcup$ *is a join operation on* $A$.

**Definition 2.5.** *[11]. Let* $A$ *be a hoop-algebra. A nonempty subset* $F$ *of* $A$ *is called a filter of* $A$, *if* $F$ *satisfies the following conditions:*
$(F1)$ *if* $x \in F$, $x \leqslant y$ *and* $y \in A$, *then* $y \in F$,
$(F2)$ $x \odot y \in F$ *for every* $x, y \in F$, *that is,* $F$ *is a subsemigroup of* $A$.

Clearly, $1 \in F$, for all filter of $A$ such as, $F$. A filter $F$ of $A$ is called a proper filter if $F \neq A$. It can be easily seen that, if $A$ is a bounded hoop, then a filter is proper if and only if it is not containing 0.

Denote by $\mathcal{F}(A)$ the set of all filters of a hoop-algebra $A$.

**Definition 2.6.** *[12]. A residuated lattice is an algebra* $(A, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *such that:*
$(RL1)$ $(A, \vee, \wedge, 0, 1)$ *is a bounded lattice,*
$(RL2)$ $(A, \odot, 1)$ *is an abelian monoid,*
$(RL3)$ $x \odot z \leqslant y$ *if and only if* $z \leqslant x \to y$, *for all* $x, y, z \in A$.

**Definition 2.7.** *[12]. (i) A divisible residuated lattice, is a residuated lattice $A$ such that for any $x, y \in A$,*
$(BL4)$ $x \odot (x \to y) = x \wedge y$.

*(ii) An $MTL$-algebra, is a residuated lattice $A$ such that for all $x, y \in A$,*
$(BL5)$ $(x \to y) \vee (y \to x) = 1$.

*(iii) A $BL$-algebra, is a divisible residuated lattice $A$, such that for all $x, y \in A$, $(BL5)$ is satisfied.*

*(iv) A $BL$-algebra $A$ is called an $MV$-algebra, if for all $x \in A$, $x^{**} = x$, where $x^* = x \to 0$.*

*(v) An $MV$-algebra $A$ is a boolean algebra iff the operation $\oplus$ is idempotent, i.e., the equation $x \oplus x = x$ is satisfied by $A$.*

**Theorem 2.8.** *[7]. $A$ is a divisible residuated lattice if and only if $A$ is a bounded $\sqcup$-hoop.*

**Theorem 2.9.** *[7]. $A$ is a $BL$-algebra if and only if $A$ is a bounded $\sqcup$-hoop with condition $(BL5)$.*

**Theorem 2.10.** *[7]. $A$ is a bounded $\sqcup$-hoop with $(DNP)$ and condition $(BL5)$ if and only if $A$ is an $MV$-algebra.*

**Definition 2.11.** *[5]. A Smarandache $BL$-algebra is defined to be a $BL$-algebra $A$ in which there exists a proper subset $Q$ of $A$ such that:*

*$(S1)$ $0, 1 \in Q$ and $\mid Q \mid > 2$,*

*$(S2)$ $Q$ is an $MV$-algebra under the operations of $A$.*

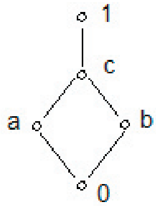
Note: From now on $(A, \odot, \longrightarrow, 1)$ is a hoop-algebra and $Q = (Q, \vee, \wedge, \odot, \longrightarrow, 0, 1)$ is a $BL$-algebras unless otherwise specified.

# 3. $Q$-Smarandache Hoop-Algebras and $Q$-Smarandache Filters

Note that every $BL$-algebras is a hoop-algebra, but the converse is not true. A hoop-algebra which is not a $BL$-algebra is called a proper hoop-algebra.

**Definition 3.1.** *A $Q$-Smarandache hoop-algebra defined to be a hoop-algebra $A$ in which there exists a proper subset $Q$ of $A$ such that:*
*$(S1)$ $0, 1 \in Q$ and $\mid Q \mid > 2$,*
*$(S2)$ $Q$ is a $BL$-algebra under the operations of $A$.*

**Example 3.2.** $(a)$ Suppose $0 < a, b < c < 1$ and let $A = \{0, a, b, c, 1\}$. For all $x, y \in A$, we define, $\odot$ and $\rightarrow$ as follows:

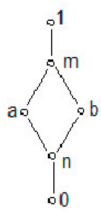



| $\odot$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $a$ | $a$ |
| $b$ | $0$ | $0$ | $b$ | $b$ | $b$ |
| $c$ | $0$ | $a$ | $b$ | $c$ | $c$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

| $\rightarrow$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $b$ | $1$ | $b$ | $1$ | $1$ |
| $b$ | $a$ | $a$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

Then $(A, \odot, \rightarrow, 1)$ is a hoop-algebra but is not a $BL$-algebra, because $(a \rightarrow b) \vee (b \rightarrow a) = c \neq 1$. If $Q = \{0, a, b, 1\}$, then $Q$ is a $BL$-algebra. So $A$ is a $Q$-Smarandache hoop-algebra.

$(b)$ Let $A = \{0, n, a, b, m, 1\}$. For all $x, y \in A$, we define, $\odot$ and $\rightarrow$ as follows:

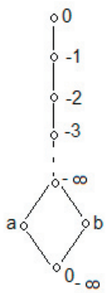



| $\rightarrow$ | $0$ | $n$ | $a$ | $b$ | $m$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $n$ | $0$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $0$ | $b$ | $1$ | $b$ | $1$ | $1$ |
| $b$ | $0$ | $a$ | $a$ | $1$ | $1$ | $1$ |
| $m$ | $0$ | $n$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $n$ | $a$ | $b$ | $m$ | $1$ |

| $\odot$ | $0$ | $n$ | $a$ | $b$ | $m$ | $1$ |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $n$ | $0$ | $n$ | $n$ | $n$ | $n$ | $n$ |
| $a$ | $0$ | $n$ | $a$ | $n$ | $a$ | $a$ |
| $b$ | $0$ | $n$ | $n$ | $b$ | $b$ | $b$ |
| $m$ | $0$ | $n$ | $a$ | $b$ | $m$ | $m$ |
| $1$ | $0$ | $n$ | $a$ | $b$ | $m$ | $1$ |

Then $(A, \odot, \to, 1)$ is a hoop-algebra but is not a $BL$-algebra, because $(a \to b) \vee (b \to a) = m \neq 1$. If $Q_1 = \{0, n, a, b, 1\}$, $Q_2 = \{0, n, 1\}$ and $Q_3 = \{0, m, 1\}$, then $Q_1$, $Q_2$ and $Q_3$ are $BL$-algebras. So $A$ is a $Q_1$-Smarandache, $Q_2$-Smarandache and $Q_3$-Smarandache hoop-algebra.

$(c)$ Let $A = \{0_{-\infty}, a, b, -\infty, ..., -3, -2, -1, 0\}$. For all $x, y \in A$, we define, $\odot$ and $\to$ as follows:




| $\odot$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|---|---|---|---|
| $0_{-\infty}$ | $0_{-\infty}$ | $0_{-\infty}$ | $0_{-\infty}$ | $0_{-\infty}$ | ... | $0_{-\infty}$ | $0_{-\infty}$ | $0_{-\infty}$ | $0_{-\infty}$ |
| $a$ | $0_{-\infty}$ | $a$ | $0_{-\infty}$ | $a$ | ... | $a$ | $a$ | $a$ | $a$ |
| $b$ | $0_{-\infty}$ | $0_{-\infty}$ | $b$ | $b$ | ... | $b$ | $b$ | $b$ | $b$ |
| $-\infty$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-\infty$ | $-\infty$ | $-\infty$ | $-\infty$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $-3$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-6$ | $-5$ | $-4$ | $-3$ |
| $-2$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-5$ | $-4$ | $-3$ | $-2$ |
| $-1$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-4$ | $-3$ | $-2$ | $-1$ |
| $0$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-3$ | $-2$ | $-1$ | $0$ |

| $\to$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|---|---|---|---|
| $0_{-\infty}$ | $0$ | $0$ | $0$ | $0$ | ... | $0$ | $0$ | $0$ | $0$ |
| $a$ | $b$ | $0$ | $b$ | $0$ | ... | $0$ | $0$ | $0$ | $0$ |
| $b$ | $a$ | $a$ | $0$ | $0$ | ... | $0$ | $0$ | $0$ | $0$ |
| $-\infty$ | $0_{-\infty}$ | $a$ | $b$ | $0$ | ... | $0$ | $0$ | $0$ | $0$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $-3$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $0$ | $0$ | $0$ | $0$ |
| $-2$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-1$ | $0$ | $0$ | $0$ |
| $-1$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-2$ | $-1$ | $0$ | $0$ |
| $0$ | $0_{-\infty}$ | $a$ | $b$ | $-\infty$ | ... | $-3$ | $-2$ | $-1$ | $0$ |

Then $(A, \odot, \rightarrow, 0)$ is a hoop-algebra but is not a $BL$-algebra, because $(a \rightarrow b) \vee (b \rightarrow a) = -\infty \neq 0$. If $Q = \{0_{-\infty}, -\infty, ..., -3, -2, -1, 0\}$ then $Q$ is a $BL$-algebra. So $A$ is a infinit $Q$-Smarandache hoop-algebra.

**Definition 3.3.** *A nonempty subset $F$ of $A$ is called Smarandache filter of $A$ related to $Q$ (or briefly $Q$-Smarandache filter of $A$) if:*
$(SF1)$ $1 \in F$,
$(SF2)$ $x \in F$, $y \in Q$ *and* $x \rightarrow y \in F$, *then* $y \in F$.

**Example 3.4.** $(a)$ In Example 3.2 (a), $\{1\}$, $\{c, 1\}$, $\{a, c, 1\}$, $\{b, c, 1\}$ and $A$ are $Q$-Smarandache filters of $A$.

$(b)$ In Example 3.2 (b), for $Q_3$, $\{1\}$, $\{m, 1\}$, $\{a, 1\}$, $\{b, 1\}$, $\{n, a, b, 1\}$, $\{n, a, b, m, 1\}$ and $A$ are $Q_3$-Smarandache filters of $A$.

Note: If $F$ is a filter of $A$ related to every $BL$-algebra contained in $A$, we simply say that $F$ is a Smarandache filter of $A$.

**Example 3.5.** In Example 3.2 (b), $F = \{n, a, b, m, 1\}$ is a Smarandache filter of $A$.

**Theorem 3.6.** *If $\{F_\lambda : \lambda \in \Lambda\}$ is an indexed set of $Q$-Smarandache filters of $A$, where $\Lambda \neq \phi$, then $F = \cap\{F_\lambda : \lambda \in \Lambda\}$ is a $Q$-Smarandache filter of $A$.*

**Proof.** For all $\lambda \in \Lambda, 1 \in F_\lambda$, so $1 \in \cap\{F_\lambda : \lambda \in \Lambda\} = F$. Let $x \in F,\ y \in Q$ and $x \rightarrow y \in F$. Thus for all $\lambda \in \Lambda,\ x \in F_\lambda,\ x \rightarrow y \in F_\lambda$ and $y \in Q$. Then $y \in F_\lambda$, for all $\lambda \in \Lambda$. Therefore $y \in F$. $\square$

**Proposition 3.7.** *Any filter $F$ of a $Q$-Smarandache hoop-algebra $A$ is a $Q$-Smarandache filter of $A$.*

**Proof.** By Definitions of 2.5 and 3.3, its trivial. $\square$

Note: By the following example, we show that the converse of Proposition 3.7 is not correct in general.

**Example 3.8.** In Example 3.2 (b), for $Q_2$, $\{a, 1\}$, $\{b, 1\}$, $\{n, a, b, 1\}$ are $Q_2$-smarandache filters of $A$, but are not filters of $A$.

**Theorem 3.9.** *Let $F$ be a $Q$-Smarandache filter of $A$. $F$ is a filter of $A$ if and only if for every $x \in F, y \in A \setminus Q$ and $x \leqslant y$, then $y \in F$.*

**Proof.** Let $F$ be a filter of $A$. If $x \in F,\ y \in A$ and $x \leqslant y$, then $y \in F$. So, if $y \in A \setminus Q$ and $x \leqslant y$, then $y \in F$. Conversly, let $x \in F,\ x \leqslant y$ and $y \in A \setminus Q$, also since $F$ is a $Q$-Smarandache filter of $A$, for $y \in Q$, we get $y \in F$. Its $F$ is a filter of $A$. $\square$

**Theorem 3.10.** *Let $F$ be a $Q$-Smarandache filter of $A$. Then:*
*(i)* $F \neq \emptyset$,
*(ii) if* $x \in F, x \leqslant y$ *and* $y \in Q$, *then* $y \in F$,
*(iii) if* $x, y \in F$, *then* $x \odot y \in F$.

**Proof.** $(i)$ $1 \in F$, then $F \neq \emptyset$.
$(ii)$ Let $x \in F,\ x \leqslant y$ and $y \in Q$, then $x \to y \in F$, therefore by $(SF2)$, we get that $y \in F$.
$(iii)$ Let $x, y \in F$. Since $x \to [y \to (x \odot y)] = 1 \in F$, we have $y \to (x \odot y) \in F$, therefore $x \odot y \in F$. $\square$

**Theorem 3.11.** *If $F$ is a $Q$-Smarandache filter of hoop-algebra $A$, then*

$$(\forall x, y \in F),\ (\forall z \in Q),\ (x \to (y \to z) = 1 \text{ then } z \in F).$$

**Proof.** Assume that $F$ is a $Q$-Smarandache filter of hoop-algebra $A$. Suppose that $x \to (y \to z) = 1$, for all $x, y \in F$ and $z \in Q$, then $x \leqslant y \to z \Rightarrow y \to z \in F$. So since $y \in F$, $z \in Q$ and $y \to z \in F$, thus $z \in F$. $\square$

**Theorem 3.12.** *Let $Q_1$ and $Q_2$ be two $BL$-algebras which are properly contained in hoop-algebra $A$ and $Q_1 \subseteq Q_2$. Then every $Q_2$-Smarandache filter is a $Q_1$-Smarandache filter.*

**Proof.** Let $F$ be a $Q_2$-Smarandache filter of $A$. So $1 \in F$. Now let $x \in F$ $y \in Q_1$ and $x \to y \in F$. Since $F$ is a $Q_2$-Smarandache filter and $y \in Q_1 \subseteq Q_2$, so $y \in F$. Therefore $F$ is $Q_1$-Smarandache filter of $A$. $\square$

The following example shows that the converse of Theorem 3.12, is not true in general.

**Example 3.13.** In Example 3.2 (b), we have $Q_1 = \{0, n, a, b, 1\}$ and $Q_2 = \{0, n, 1\}$ are two $BL$-algebras which are properly contained in hoop algebra $A$ and $Q_2 \subseteq Q_1$. Then $F = \{n, 1\}$ is $Q_2$-Smarandache filter but is not a $Q_1$-Smarandache filter.

**Theorem 3.14.** *The relation $\sim_Q$ on a $Q$-Smarandache hoop-algebra $A$ which is defined by*

$$x \sim_Q y \Leftrightarrow (x \to y \in Q, y \to x \in Q)$$

*is a congruence relation.*

**Proof.** (1) $x \to x = 1 \in Q$, then $x \sim_Q x$.
(2) Let $x \sim_Q y$. Then $x \to y \in Q,\ y \to x \in Q$, therefore $y \sim_Q x$.

(3) $x \sim_Q y,\ y \sim_Q z$ if and only if $(x \to y \in Q,\ y \to x \in Q)$ and $(y \to z \in Q,\ z \to y \in Q)$. In BL-algebra $Q$, we have

$$(x \to y) \odot (y \to z) \leqslant x \to z \ \ \textit{and} \ \ (z \to y) \odot (y \to x) \leqslant z \to x.$$

Since $Q$ is a $Q$-Smarandache filter and $Q$ is a $BL$-algebra, then $x \to z \in Q$ and $z \to x \in Q$, so $x \sim_Q z$. Clearly $\sim_Q$ is a congruence relation. □

**Definition 3.15.** *Let $A$ be a $Q$-Smarandache hoop-algebra. Then $\frac{A}{Q} = \{[x] | x \in A\}$ is a quotient algebra via the congruence relative $\sim_Q$ and $[x] = \{y \in A | x \sim_Q y\}$. We define on $\frac{A}{Q}$:*

$$[x] \to [y] = [x \to y],\ [x] \odot [y] = [x \odot y],\ [1] = \frac{1}{Q}.$$

**Theorem 3.16.** *The $(\frac{A}{Q}, \to, \odot, \frac{1}{Q})$ which defined in Definition 3.15 is a hoop-algebra.*

## 4. Smarandache $n$-Structure on Hoop-Algebra

**Example 4.1.** Suppose $0 < a < b < 1$ and let $A = \{0, a, b, 1\}$. We define, $\odot$ and $\to$ as follows:


| $\odot$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $a$ | $a$ |
| $b$ | $0$ | $a$ | $b$ | $b$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

| $\to$ | $0$ | $a$ | $b$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $a$ | $1$ | $1$ | $1$ |
| $b$ | $0$ | $a$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $1$ |

then $(A, \vee, \wedge, \odot, \to, 0, 1)$ is a $BL$-algebra. Consider $Q = \{0, a, 1\}$, with the following tables:

| $\oplus$ | $0$ | $a$ | $1$ |
|---|---|---|---|
| $0$ | $0$ | $a$ | $1$ |
| $a$ | $a$ | $1$ | $1$ |
| $1$ | $1$ | $1$ | $1$ |

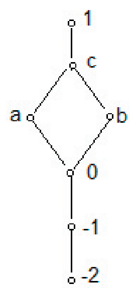
| $*$ | $0$ | $a$ | $1$ |
|---|---|---|---|
| | $1$ | $a$ | $0$ |

Then $Q$ is a $MV$-algebra which is properly contained in $A$. So $A$ is a $Q$-Smarandache $BL$-algebra.

**Remark 4.2.** *A Smarandache strong $n$-structure on a set $S$ refers to a structure $\{W_0\}$ on a set $S$ such that there exists a chain of proper subsets $P_{n-1} < P_{n-2} < ... < P_2 < P_1 < S$ where $<$ means "included in" whose corresponding structures verify the inverse chain $W_{n-1} > W_{n-2} > ... > W_2 > W_1 > W_0$ where $>$ signifies strictly stronger (i.e structure satisfying more axioms) and $<$ signifies 'strictly weaker' (i.e. structure satisfying less axioms). And by structure on $S$ we mean a structure $\{w\}$ on $S$ under the given operation(s).*

**Definition 4.3.** *A Smarandache strong 2-structure on hoop-algebra $A$ is a chain $A = A_1 > A_2 > A_3$ where $A_1$ is a hoop-algebra, $A_2$ is a $BL$-algebra and $A_3$ is a $MV$-algebra.*

**Example 4.4.** Let $A = \{-2, -1, 0, a, b, c, 1\}$. For all $x, y \in A$, we define, $\odot$ and $\rightarrow$ as follows:




| $\odot$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|---|---|
| $-2$ | $-2$ | $-2$ | $-2$ | $-2$ | $-2$ | $-2$ | $-2$ |
| $-1$ | $-2$ | $-2$ | $-1$ | $-1$ | $-1$ | $-1$ | $-1$ |
| $0$ | $-2$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $-2$ | $-1$ | $0$ | $a$ | $0$ | $a$ | $a$ |
| $b$ | $-2$ | $-1$ | $0$ | $0$ | $b$ | $b$ | $b$ |
| $c$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $c$ | $c$ |
| $1$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

| $\rightarrow$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|---|---|
| $-2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $-1$ | $-1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $0$ | $-2$ | $-1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $-2$ | $-1$ | $b$ | $1$ | $b$ | $1$ | $1$ |
| $b$ | $-2$ | $-1$ | $a$ | $a$ | $1$ | $1$ | $1$ |
| $c$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $-2$ | $-1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

Then $A = A_1 = \{-2, -1, 0, a, b, c, 1\}$ is a hoop-algebra and is not a $BL$-algebra. $A_2 = \{-2, -1, 0, a, b, 1\}$ is a BL-algebra and is not a $MV$-algebra. $A_3 = \{-2, -1, 1\}$ is a $MV$-algebra. So $A$ is a Smarandache strong 2-structure hoop-algebra.

**Definition 4.5.** *Let $A$ be a Smarandache strong 2-structure of hoop-algebra $A$, with chain $A = A_1 > A_2 > A_3$ where $A_1$ is a hoop-algebra, $A_2$ is a $BL$-algebra and $A_3$ is a $MV$-algebra. A subset $F$ of $A$ is called a Smarandache 2-filter with chain $F = F_1 > F_2$ if $F_1$ is a $A_2$-Smarandache filter and $F_2$ is a $A_3$-Smarandache filter.*

**Example 4.6.** a) In Example 4.4, $F = F_1 = \{0, a, b, 1\}$ with chain $F_1 > F_2 = \{0, 1\}$ is a Smarandache 2-filter of $A$.

b) In Example 4.4, $F = \{c, 1\}$ is a $A_1$-Smarandache filter, $A_2$-Smarandache filter and $A_3$-Smarandache filter but is not a Smarandache 2-filter of $A$.

# 5. Conclusion and Future Research

Hoops are a particular class of algebraic structures which were introduced in an unpublished manuscript by Buchi and Owens in the mid-1970s. In fact, hoops are partially ordered commutative residuated integral monoids satisfying a further divisibility condition.

Smarandache structurs are a weak structure in any structure. In the present paper, by using this notion we have introduced the concept of $Q$-Smarandache hoop-algebras and investigated some of their useful properties. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras.

It is our hope that this work would other foundations for further study of the theory of hoop-algebra. Our obtained results can be perhaps applied in engineering, soft computing or even in medical diagnosis.

In our future study of Smarandache structure of hoop-algebras, may be the following topics should be considered:

(1) To get more results in $Q$-Smarandache hoop-algebras and application;

(2) To get more connection to divisible residuated lattice and hoop-algebra;

(3) To define another Smarandache structure, if put divisible residuated lattice instead of $BL$-algebra;

(4) To define fuzzy structure of Smarandache hoop-algebras.

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