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Total $[1, k]$-Sets in the Lexicographic Product of Graphs

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Abstract. A subset $S \subseteq V$ in a graph $G = (V, E)$ is called a $[1, k]$-set, if for every vertex $v \in V \setminus S$, $1 \leqslant |N_G(v) \cap S| \leqslant k$. The $[1, k]$-domination number of G, denoted by $\gamma_{[1,k]}(G)$ is the size of the smallest $[1, k]$-sets of G. A set $S' \subseteq V(G)$ is called a total $[1, k]$-set, if for every vertex $v \in V$, $1 \leqslant |N_G(v) \cap S| \leqslant k$. If a graph G has at least one total $[1, k]$-set then the cardinality of the smallest such set is denoted by $\gamma_{t[1,k]}(G)$. In this paper, we investigate the existence of $[1, k]$-sets in lexicographic products $G \circ H$. Furthermore, we completely characterize graphs whose lexicographic product has at least one total $[1, k]$-set. Finally, we show that finding smallest total $[1, k]$-set is an NP-complete problem.

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1. Introduction and Terminology

The concept of dominating set and domination number is a well studied topic in graph theory and has many extensions and applications [8, 9]. Many variants of domination numbers have been proposed and surveyed in the literature such as total domination number [10], efficient and open efficient domination numbers

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[1], k-tuple domination number [2] and others like [8]. Most of these problems are shown to be NP-hard. Recently, Chellali et al. introduced the notion of a $[j,k]$-dominating set for a graph and studied some problems in this respect [4]. They have also pointed out a number of open problems on $[1,2]$-dominating sets in [4]. Some of those problems are solved by X. Yang et al. [12] and AK. Goharshady et al. [5].

In [3], Chellali et al. investigated independent $[1,k]$-sets for graphs and gave a constructive characterization of the trees having an independent $[1,k]$-set. Also they proved that the corona of two graphs G and H has an independent $[1,k]$-set if and only if each component of G is an isolated vertex or $i(H) \leqslant k$, where $i(H) = i_{[1,k]}(G)$ is the minimum cardinality of an independent $[1,k]$-set of G.

All graphs in this paper are assumed to be a simple ones, i.e., finite, undirected, loopless graphs without multiple edges. For notation and terminology that are not defined here, we refer the reader to [11]. For given simple graph G with vertex set $V(G)$ and edge set $E(G)$, the degree of vertex $v \in V(G)$ is denoted by $d_G(v)$, or simply $d(v)$. We denote the minimum and maximum degrees of vertices in G by $\delta(G)$ and $\Delta(G)$, respectively. The open neighborhood $N_G(v)$ of a vertex $v \in V(G)$ equals $\{u : \{u,v\} \in E(G)\}$ and its closed neighborhood $N_G[v]$ is defined $N_G(v) \cup \{v\}$. The open (resp. closed) neighborhood of $S \subseteq V$ is defined to be the union of open (resp. closed) neighborhoods of vertices in S and is denoted by $N(S)$ (resp. $N[S]$). A set $D \subseteq V$ is called a dominating set of G if for every $v \in V \setminus D$, there exists some vertex $u \in D$ such that $v \in N(u)$. The domination number of G is the minimum number among cardinalities of all dominating sets of G and is denoted by $\gamma(G)$. A set $D \subseteq V$ is called a total dominating set of G if for every $v \in V$, there exists some vertex $u \in D$ such that $v \in N(u)$. The total domination number is the minimum number among cardinalities of all total dominating sets of G and is denoted by $\gamma_t(G)$. For two given integers j and k such that $j \leqslant k$, a subset $D \subseteq V$ is called a $[j,k]$-set (resp. total $[j,k]$-set) if for every vertex $v \in V \setminus D$ (resp. $v \in V$), $j \leqslant |N(v) \cap D| \leqslant k$. Note that total $[j,k]$-sets might not exist for an arbitrary graph. The sets of all graphs like G which have at least one total $[j,k]$-set is denoted by $\mathcal{D}^t_{[j,k]}$. Other types of dominating sets, that we are used in this work are summarized in the Table 1.

Table 1: Some types of domination studied in this paper where $S \subseteq V$

Name	$v \in V \setminus S$	$v \in S$
$[1,k]$-set	$\|N(v) \cap S\| \in [1,k]$	-
Independent $[1,k]$-set	$\|N(v) \cap S\| \in [1,k]$	$\|N(v) \cap S\| = 0$
j-dependent $[1,k]$-set	$\|N(v) \cap S\| \in [1,k]$	$\|N(v) \cap S\| \in [0,j]$
Total $[1,k]$-set	$\|N(v) \cap S\| \in [1,k]$	$\|N(v) \cap S\| \in [1,k]$
j-dependent total $[1,k]$-set	$\|N(v) \cap S\| \in [1,k]$	$\|N(v) \cap S\| \in [1,j]$
Efficient dominating	$\|N(v) \cap S\| = 1$	$\|N(v) \cap S\| = 0$

The rest of the paper is organized as follows: In Section 2, we study total $[1, k]$-sets of lexicographic product of graphs and then, we completely characterize graphs which their lexicographic product has at least one total $[1, k]$-set. Then, we determine the structure of all total $[1, k]$-sets for these graphs. In Section 3, we prove that finding a total $[1, 2]$-set with minimum cardinality for a graph is NP-complete.

2. Total $[1, 2]$-Sets of Lexicographic Products of Graphs

The lexicographic product of graphs G and H, denoted by $G \circ H$ is a graph with the vertex set $V(G \circ H) = V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent in $G \circ H$ if and only if either $\{g, g'\} \in E(G)$ or $g = g'$ and $\{h, h'\} \in E(H)$.

Note that if G is not connected, then $G \circ H$ is not connected, too. So in this section, we always assume that G is a connected graph.

In this section, we investigate properties of graphs G and H such that $G \circ H$ has a total $[1, 2]$-set. Then we extend these results to total $[1, k]$-set. Note that, it is possible that $G \in \mathcal{D}^t_{[1,2]}$, whereas $G \circ H \notin \mathcal{D}^t_{[1,2]}$, or vice versa.

Definition 2.1. *Let H and G be two graphs. Let $g_0 \in V(H)$ and $h_0 \in V(H)$. The sets $G^{h_0} = \{(g, h_0) \in V(G \circ H) : g \in V(G)\}$ and $H^{g_0} = \{(g_0, h) \in V(G \circ H) : h \in V(H)\}$ are called a G_Layer and a H_Layer respectively.*

Lemma 2.2. *Let v and v' be two adjacent vertices of G and $u, u' \in V(H)$. Then*

$$\begin{aligned} N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u)) &= N_{G\circ H}((v, u')) \cup N_{G\circ H}((v', u')) \\ &= N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u')). \end{aligned}$$

Proof. We know that

$$N_{G\circ H}((v, u)) = \bigcup_{v_i \in N_G(v)} V(H^{v_i}) \cup \{(v, u_j) : u_j \in N_H(u)\},$$

so

$$N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u')) = D_1 \cup D_2 \tag{1}$$

where $D_1 = (\bigcup_{v_i \in N_G(v)} V(H^{v_i})) \cup \{(v, u_j) : u_j \in N_H(u)\}$ and $D_2 = (\bigcup_{v_i \in N_G(v')} V(H^{v_i})) \cup \{(v', u_j) : u_j \in N_H(u')\}$.

It is easy to see that

$$\{(v, u_j) : u_j \in N_H(u)\} \subseteq V(H^v). \tag{2}$$

and

$$\{(v', u_j) : u_j \in N_H(u')\} \subseteq V(H^{v'}). \tag{3}$$

By hypotheses $\{v, v'\} \in E(G)$, we have

$$\begin{aligned} V(H^v) &\subseteq N_{G\circ H}((v', u')),\\ V(H^{v'}) &\subseteq N_{G\circ H}((v, u)). \end{aligned} \tag{4}$$

So by Relations 1, 2, 3 and 4, it is implied that

$$N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u')) = \bigcup_{v_i \in N_G(\{v, v'\})} V(H^{v_i}).$$

The equation above shows that the union of neighbors of the vertices (v, u) and (v', u') is independent from u and u'. Therefore, we have

$$\begin{aligned} N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u)) &= N_{G\circ H}((v, u')) \cup N_{G\circ H}((v', u'))\\ &= N_{G\circ H}((v, u)) \cup N_{G\circ H}((v', u')). \quad \square \end{aligned}$$

Lemma 2.3. *Let D be a total $[1, 2]$-set for $G \circ H \in \mathcal{D}^t_{[1,2]}$ which contains more than two vertices of a H_Layer H^v. Then $G = K_1$ and $H \in \mathcal{D}^t_{[1,2]}$.*

Proof. Let D be a total $[1, 2]$-set of $G\circ H$ that contains vertices $(x, v), (y, v)$ and (z, v) where $v \in V(G)$ and $x, y, z \in V(H)$. If there exists a vertex $v' \in V(G)$ such that $\{v, v'\} \in E(G)$, then all vertices of $H^{v'}$ are dominated by three vertices $(x, v), (y, v)$ and (z, v). This is a contradiction. So there is not any vertex adjacent to v. Since G is a connected graph, $G = K_1 = (\{v\}, \emptyset)$ and $S = \{u : (v, u) \in D\}$ is a total $[1, 2]$-set for H and hence $H \in \mathcal{D}^t_{[1,2]}$. $\square$

Corollary 2.4. *Let G be a nontrivial connected graph and $G\circ H \in \mathcal{D}^t_{[1,2]}$. Then, every total $[1, 2]$-set of $G \circ H$ has at most two vertices of each H_Layer. For a total $[1, 2]$-set D, we define A^D_1 as $\{(v, u) : |V(H^v) \cap D| = 1\}$ and A^D_2 as $\{(v, u) : |V(H^v) \cap D| = 2\}$. The set D satisfies in one of the following conditions:*

1) $A^D_1 = \emptyset$ and $A^D_2 \neq \emptyset$,
2) $A^D_1 \neq \emptyset$ and $A^D_2 \neq \emptyset$,
3) $A^D_2 = \emptyset$ and $A^D_1 \neq \emptyset$.

Lemma 2.5. *Let D be a total $[1, 2]$-set of $G \circ H \in \mathcal{D}^t_{[1,2]}$ such that $A^D_2 = \emptyset$. Then, $S = \{u : (u, v) \in D\}$ is a total $[1, 2]$-set for G. In addition, if there is a vertex $u \in S$ such that $|N(u) \cap S| = 2$; then H contains an isolated vertex.*

Proof. The proof is by contradiction. Assume D is a total $[1,2]$-set of $G \circ H$ with $A_2^D = \emptyset$ and $S = \{u : (u,v) \in D\}$ is not a total set of G. Then, we have three cases to consider.

1. There exists a vertex like $u \in S$ such that $|N(u) \cap S| = 0$. It means that there is no vertex $u' \in N_G(u)$ such that $u' \in S$. The set D is a total $[1,2]$-set and $u \in S$, so there exists a vertex $v \in V(H)$ such that $(u,v) \in D$. Similarly there exists a vertex $v' \in V(H)$ such that $(u,v') \in D$. This is a contradiction against $A_2^D = \emptyset$.

2. There exists a vertex like $w \in V(G) \setminus S$ such that $|N_G(w) \cap S| = 0$. Then, there is no vertex like $v \in V(H)$ such that $(u,v) \in D$. Moreover, there is no vertex $w' \in N_G(w)$ such that $w' \in S$. Therefore vertices of H^w can not be dominated by any vertex in D, which is a contradiction.

3. There exists a vertex like $w \in V(G) \setminus S$ such that $|N(w) \cap S| > 2$. Then, there are at least three distinct vertices $w', w'', w''' \in N_G(w) \cap S$. By the definition of S, there are vertices $v', v'', v''' \in V(H)$ such that $(w',v'), (w'',v''), (w''',v''') \in D$. These vertices dominate all vertices of H^w, which is a contradiction. $\square$

Lemma 2.6. Let $G \circ H \in \mathcal{D}^t_{[1,2]}$ and H does not contain any isolated vertex. Then, there exists either a 1-dependent total $[1,2]$-set for G or for each total $[1,2]$-set D of G, $A_1^D = \{(v,u) : |V(H^v) \cap D| = 1\} \neq \emptyset$ and $A_2^D = \{(v,u) : |V(H^v) \cap D| = 2\} \neq \emptyset$.

Proof. Let D be a total $[1,2]$-set of $G \circ H$ which contains at most one vertex from each H_Layer. Since H does not contain any isolated vertex then by Lemma 2.5 there is a 1-dependent total $[1,2]$-set like S for G such that $S = \{v : (v,u) \in D\}$ and $A_2^D = \emptyset$. $\square$

For a given graph $G \circ H \in \mathcal{D}^t_{[1,2]}$ and a total $[1,2]$-set D of $G \circ H$ where $A_2^D \neq \emptyset$, we define the set B^D as $B^D = \{\{u', u''\} : (v,u'), (v,u'') \in A_2^D\}$.

Lemma 2.7. *Let $G \circ H \in \mathcal{D}^t_{[1,2]}$ where H does not contain any isolated vertex and for any total $[1,2]$-set D of $G \circ H$, $A_1^D \neq \emptyset$ and $A_2^D \neq \emptyset$. Then, the following conditions hold:*

1) *Every element of B^D is a total $[1,2]$-set for H.*

2) *The set $S' = \{v : (v,u) \in D\}$ is a 1-dependent $[1,2]$-set for G.*

3) *If there is a vertex $v \in S'$ such that $|N(v) \cap S'| = 0$ then $dist_G(v,v') \geqslant 3$ for every $v' \in S' \setminus \{v\}$.*

Proof. Let D be a total $[1,2]$-set of $G \circ H \in \mathcal{D}^t_{[1,2]}$; there are three cases to consider.

1) Suppose that $S = \{u^\star, u^\bullet\} \in B$ is not a total $[1,2]$-set for H. Then two cases occur and in each case, we can establish a contradiction with D is a total $[1,2]$-set.

   - Let $\{u^\star, u^\bullet\} \notin E(H)$ and there is a $(v', u') \in D$ such that $\{(v, u^\star), (v', u')\} \in E(G \circ H)$. Since H dose not contain any isolated vertex, so any vertex $u'' \in N_H(u')$ is dominated by $(v', u'), (v, u^\star)$ and $(v, u^\bullet)$.
   - Let $\{u^\star, u^\bullet\}$ does not dominate all vertices of $V(H)$. So, there is a vertex $(v', u') \in D$ such that $\{v, v'\} \in E(G)$ and (v', u') dominates all vertices of H^v. Then any vertex $u'' \in N_H(u')$ is dominated by $(v', u'), (v, u^\star)$ and $(v, u^\bullet)$.

2) Suppose that $S' = \{v : (v, u) \in D\}$ is not a 1-dependent $[1,2]$-set for G. Then, three cases occur and in each case, we have a contradiction with D being a total $[1,2]$-set.

   - There is a vertex $v \in S'$ that is dominated by at least two vertices $v', v'' \in S'$. So there are vertices $u, u', u'' \in V(H)$ such that $(v, u), (v', u'), (v'', u'') \in D$. Since H dose not contain any isolated vertex, there is a vertex $u''' \in V(H)$ such that $\{u, u'''\} \in E(H)$. Then, (v, u''') is dominated by (v, u),(v', u') and (v'', u'').
   - There is a vertex $v \in V(G) \setminus S'$ such that $|N_G(v) \cap S'| = 0$. So no vertex of H^v is dominated by D.
   - There is a vertex $v \in V(G) \setminus S'$ such that $|N_G(x) \cap S'| > 2$. Then there are at least three vertices distinct $v', v'', v''' \in S'$ to dominate v. By definition of S', there are vertices $u', u'', u''' \in V(H)$ such that $(v', u'), (v'', u''), (v''', u''') \in D$. These vertices dominate all vertices of H^v.

3) Let $v \in S'$ such that $|N(v) \cap S'| = 0$ and there is a vertex $v' \in S'$ such that $dist_G(v, v') = 2$.

   By $|N(v) \cap S'| = 0$, there exist vertices $u', u'' \in V(H)$ such that $(v, u'), (v, u'') \in D$ and $\{u', u''\} \in E(H)$. Suppose there is a vertex $v' \in S'$ such that $dist_G(v, v') = 2$. So, there is a vertex $v'' \in V(G)$ such that $\{v, v''\}, \{v', v''\} \in E(G)$. The vertices (v, u'), (v, u'') and (v', u') dominate all vertices of $H^{v''}$. It is contradictory with D being a total $[1,2]$-set. So we have $dist_G(v, v') \geqslant 3$. $\square$

Lemma 2.8. *Let D be a total $[1,2]$-set of $G \circ H \in \mathcal{D}^t_{[1,2]}$ such that $A_1^D = \emptyset$. Then $S' = \{v : (v, u) \in D\}$ is an efficient dominating set of G.*

Proof. Since D be a total $[1,2]$-set of $G \circ H$, then there is a vertex $v \in S'$ such that the set D contains $(v, u'), (v, u'')$ for some vertex $u', u'' \in V(H)$. By Lemma 2.7, $\{u', u''\}$ is a total $[1,2]$-set for H. So for any vertex $v' \in N_G(v)$, none of vertices in $H^{v'}$ cannot be contained in D. Thus $dist_G(v, v') \geqslant 3$ and S is an efficient dominating set of G. $\square$

In the sequel $\mathcal{SD}^k_{[i,j]}(G)$ is used to denote the set of all k-dependent $[i, j]$-set S of G such that S satisfies in the following condition

$$(\forall v \in S \ \ |N(v) \cap S| = 0) \to (\forall v' \in S \setminus \{v\} \ \ d(v, v') \geqslant 3).$$

Corollary 2.9. *Let G be a connected nontrivial graph and D be a total $[1,2]$-set of $G \circ H \in \mathcal{D}^t_{[1,2]}$, one of the following cases holds:*

- *If $A_1^D = \{(u, v) : |V(H^v) \cap D| = 1\} = \emptyset$, then there is a total $[1,2]$-set $S = \{u^\star, u^\bullet\}$ in H and an efficient dominating set S' in G such that $D' = S' \times S$ is a total $[1,2]$-set for $G \circ H$ and $|D| = |D'| = 2|S'|$.*
- *If $A_2^D = \{(u, v) : |V(H^v) \cap D| = 2\} = \emptyset$ and H contains an isolated vertex v. Then there is a total $[1,2]$-set S in G where $D' = S \times \{v\}$ and D' is a total $[1,2]$-set for $G \circ H$. Moreover, we have $|D| = |D'| = |S|$.*
- *If $A_2^D = \{(u, v) : |V(H^v) \cap D| = 2\} = \emptyset$ and H does not contain any isolated vertex, then for every vertex $v \in V(H)$ there is a 1-dependent total $[1,2]$-set S in G such that $D' = S \times \{v\}$ and D' is a total $[1,2]$-set for $G \circ H$. Clearly, $|D| = |D'| = |S|$.*
- *If $A_1^D \neq \emptyset$ and $A_2^D \neq \emptyset$, then there is a total $[1,2]$-set $S = \{u^\star, u^\bullet\}$ in H and a 1-dependent total $[1,2]$-set S' in G such that for any vertex $v \in S$ and $u \in X$ where $X = \{x : |N_G(x) \cap S'| = 0\}$, $dist(v, u) \geqslant 3$. Moreover $D' = ((X \times S) \cup (S' \setminus X) \times \{u^\star\})$ is a total $[1,2]$-set of size $|D|$ in $G \circ H$ and $|D| = |D'| = |S'| + |X|$.*

Proof. This corollary is a direct result of Lemma 2.2, 2.5, 2.7 and 2.8. $\square$

Theorem 2.10. *Let G and H be two graphs. Then, $G \circ H \in \mathcal{D}^t_{[1,2]}$ if and only if one of the following conditions holds:*

1. *$G = K_1$ and $H \in \mathcal{D}^t_{[1,2]}$;*
2. *G has a total $[1,2]$-set S such that if S has a vertex v where $|N(v) \cap S| = 2$ then H has an isolated vertex;*
3. *G is an efficient domination graph and $\gamma_{t[1,2]}(H) = 2$;*

4. $\mathcal{SD}^1_{[1,2]}(G) \neq \emptyset$ *and* $\gamma_{t[1,2]}(H) = 2$.

Proof. Suppose that D be a total $[1,2]$-set of $G \circ H \in \mathcal{D}^t_{[1,2]}$. If D contains more than two vertices of a $H_$Layer, then by Lemma 2.3, $G = K_1$ and $H \in \mathcal{D}^t_{[1,2]}$. If D contains at most two vertices of each $H_$Layer, then there is a total $[1,2]$-set D' for $G \circ H$ such that $|D'| = |D|$ and vertices of D' have been choosen from two $G_$Layers as $G^{u^\star}$ and $G^{u^\bullet}$. Without lose of generality we consider that $S = \{v : (v,u) \in D'\}$ and $S' = \{u^\star, u^\bullet\}$. Then, the set D' satisfies one of the following conditions:

a) By Lemma 2.5, $D = \{(v,u^\star) : v \in S\}$, so S is a total $[1,2]$-set for G and if there exists a vertex $v \in D$ such that $|N(v) \cap S| = 2$, then H has an isolated vertex.

b) $D' = \{(v,u^\star) : v \in S \text{ and } u \in S'\}$, by Corollary 2.9, S is an efficient dominating set of G and S' is a total $[1,2]$-set for H.

c) There is a vertex $w \in S$ such that $(w,u^\star) \in D'$ but $(w,u^\bullet) \notin D'$. By Lemma 2.7, we have $S \in \mathcal{SD}^1_{[1,2]}(G)$ and S' is a total $[1,2]$-set for H.

Now, we show the other side as follows:

1. If $G = K_1$ and H has a total $[1,2]$-set S', then it is easy to see that $G \circ H = H$ and S' is a total $[1,2]$-set of $G \circ H$.
2. Assume that S is a total $[1,2]$-set of G and $u^\star \in V(H)$. We define D as $S \times \{u^\star\}$. Since every vertex of $G^{u^\star}$ is dominated by at least one of vertices of D, then every vertex of other $G_$Layers is dominated by D. So, for any vertex $(v',u') \in G \circ H$, we have $|N((v',u')) \cap D| \geqslant 1$. Now, it is sufficient to show that $|N((v',u')) \cap D| \leqslant 2$. To this end, we consider two cases:

   a) For every vertex $v \in S$, $|N(v) \cap S| = 1$: So, it is clear that for any vertex $(v',u^\star)$ of $G^{u^\star}$, $|N((v',u^\star)) \cap D| \leqslant 2$. If $u' \neq u^\star$, we need to show that $|N((v',u')) \cap D| \leqslant 2$. Then following cases can happen:

   a1) $(v',u^\star) \in D$ and $\{u',u^\star\} \in E(H)$; for every $v'' \in S$ adjacent to v', (v',u') is dominated by $(v',u^\star)$ and $(v'',u^\star)$. Since $(v',u^\star) \in D$ and $v' \in S$, so $|N(v') \cap S| = 2$ and $|N((v',u')) \cap D| = |N(v') \cap S| + 1 = 2$.

   a2) $(v',u^\star) \in D$ and $\{u',u^\star\} \notin E(H)$; if $v'' \in S$ and $\{v',v''\} \in E(G)$ then (v',u') is dominated by $(v'',u^\star)$. So $|N((v',u')) \cap D| = |N(v') \cap S| = 1$.

a3) $(v',u^\star) \notin D$; for every $v'' \in S$ and $\{v',v''\} \in E(G)$, (v',u') is dominated by $(v'',u^\star)$. Since $(v',u^\star) \notin D$, $v' \notin S$. We have$|N((v',u')) \cap D| = |N(v') \cap S| \leqslant 2$.

b) There is a vertex $v \in S$ such that $|N(v) \cap S| = 2$ and $u^\star$ is an isolated vertex in H. For every vertex $v'' \in S$ and $\{v',v''\} \in E(G)$, (v',u') is dominated by $(v'',u^\star)$. So it is the case that $|N((v',u')) \cap D| = |N(v') \cap S| \leqslant 2$.

3. Let S be an efficient dominating set of G, $S' = \{u^\star, u^\bullet\}$ is a total $[1,2]$-set for H and $D = \{(v,u) : v \in S \textit{ and } u \in S'\}$. It is easy to see that D is a total dominating set of $G \circ H$.

   If $v' \in S$, then every $(v',u') \in V(H^{v'})$ are dominated by either $(v',u^\star)$ or $(v',u^\bullet)$. Since S is an efficient dominating set of G, then $N_G(v') \cap S = \emptyset$ and (v',u') is not dominated by any other vertices. If $v' \notin S$, then there is exactly one vertex $v'' \in S$ such that $\{v',v''\} \in E(G)$ and every $(v',u') \in V(H^{v'})$ are dominated by either $(v'',u^\star)$ and $(v'',u^\bullet)$. So, D is a total $[1,2]$-set for $G \circ H$.

4. Suppose that $S \in \mathcal{SD}^1_{[1,2]}$, $S' = \{u^\star, u^\bullet\}$ is a total $[1,2]$-set for H and

$$\begin{aligned} D = \{(v,u^\star),(v,u^\bullet) &: v \in S \text{ and } |N(v) \cap S| = 0\} \\ \cup\{(v,u^\star) &: v \in S \text{ and } |N(v) \cap S| = 1\}. \end{aligned}$$

   By definition of D, It is easy to see that for any vertex $(v,u) \in D$, there is a vertex $(v',u') \in D$ such that $\{(v,u),(v',u')\} \in E(G \circ H)$. So, D is a total set of $G \circ H$. Now, we must show that D dominates all vertices of $G \circ H$ at least one and at most two times. It is clear $S = \{v : (v,u^\star) \in D\} \in \mathcal{SD}^1_{[1,2]}$. We consider three kind of vertices and we will show vertices of each $H_$Layer are dominated by at least one and two vertices of D.

a) $v \in S$ and $|N(v) \cap S| = 0$: Since $S' = \{u^\star, u^\bullet\}$ is a total $[1,2]$-set for $G \circ H$, $(v,u^\star) \in D$ and $(v,u^\bullet) \in D$. Then, all of the vertices of H^v are dominated by $(v,u^\star)$ and $(v,u^\bullet)$. Since $|N(v) \cap S| = 0$. So, any other vertex cannot dominate vertices of H^v. Therefore $1 \leqslant |N(v,u) \cap D| \leqslant 2$.

b) $v \in S$ and $|N(v) \cap S| = 1$: So, there is a vertex $v' \in S$ such that $\{v,v'\} \in E(G)$, $(v',u^\star)$ dominates all of the vertices of H^v and these vertices can also be dominated by $(v,u^\star)$. Since S is a 1-dependent $[1,2]$-set for G, then there is not any other vertex in neighborhood of v in S, so $1 \leqslant |N(v,u) \cap D| \leqslant 2$.

c) $v \notin S$: Since S is a 1-dependent $[1,2]$-set for G, it is easy to see that there is a vertex $v' \in S$ such that $\{v, v'\} \in E(G)$. So, all of the vertices of H^v are dominated by $(v', u^\star)$. If $|N(v') \cap S| = 0$, then $(v', u^\bullet)$ dominates vertices of H^v and any other vertices can not dominate them. If there exist a $v'' \in S$ such that $\{v, v''\} \in E(G)$ and it is contradict to $dist_G(v', v'') \geqslant 3$. If $|N(v') \cap S| = 0$, there maybe exists a vertex $(v'', u^\star) \in D$ such that $|N(v') \cap S| \neq 0$ and there is no vertex in $H^{v''}$ and other $H_$Layers dominate vertices of H^v. $\square$

In the sequel, we express necessary and sufficient conditions for the given graphs G and H such that $G \circ H$ has a total $[1,k]$-set. The Lemma 2.3, 2.5, 2.7 and Corollary 2.9 are generalized to total $[1,k]$-set. Since proofs in this section can be similarly obtained from the case on total $[1,2]$-sets, we omit them.

Theorem 2.11. *Let D be a total $[1,k]$-set for $G \circ H$.*

a) *If D contains more than k vertices of a $H_$Layer, then $G = K_1$ and $H \in \mathcal{D}^t_{[1,k]}$.*

b) *If D contains at most one vertex of every $H_$Layers, then $S = \{v \in V(G) : (v,u) \in D\}$ is a $(k-1)$-dependent total $[1,k]$-set of G. Moreover if there is a vertex $v \in S$ such that $|N(v) \cap S| = k$, then H contains an isolated vertex.*

c) *If H does not contain any isolated vertex and $S = \{v \in V(G) : (v,u) \in D\}$ is not a total set of G, then D contains at most k vertices of each H^v and satisfies the following conditions:*

   c1) *The set $S' = \{u \in V(H) : (v,u) \in D\}$ is a total $[1,k]$-set of H with cardinality to at most k and there is a vertex $x \in S$ such that $1 < |D \cap V(H^x)| \leqslant |S'|$;*

   c2) *S is a $(k-1)$-dependent $[1,k]$-set for G;*

   c3) *If there exist a vertex $v \in S$ such that $|N(v) \cap S| = 0$, then $1 < |D \cap V(H^v)| \leqslant \lfloor k/2 \rfloor$ or for any vertex $v' \in S - \{v\}$, we have $dist_G(v, v') \geqslant 3$.*

Theorem 2.12. *Let G and H be two graphs. $G \circ H \in \mathcal{D}^t_{[1,k]}$ if and only if G and H satisfy one of the following conditions*

1. *$G = K_1$ and $H \in \mathcal{D}^t_{[1,k]}$;*
2. *G has a total $[1,k]$-set S and if S has a vertex v such that $|N(v) \cap S| = k$ then H has an isolated vertex;*

3. *G is an efficient domination graph and $\gamma_{t[1,k]}(H) \leqslant k$;*

4. *G has a $(k-1)$-dependent $[1,k]$-set S and if $S \in \mathcal{SD}^{k-1}_{[1,k]}(G)$ then $\gamma_{t[1,k]}(H) \leqslant k$ and otherwise $\gamma_{t[1,k]}(H) \leqslant k/2$.*

3. Complexity

In this section, we will show that the decision problem for total $[1,2]$-set is NP-complete. We will do this by reduction the NP-complete problem, Exact 3-Cover, to Total $[1,2]$-Set.

Exact 3-cover problem: Input of this problem is a finite set $X = \{x_1, x_2, ..., x_{3q}\}$ with $|X| = 3q$ and a collection C of 3-element subsets of X such as $C_i = \{x_{i_1}, x_{i_2}, x_{i_3}\}$. our goal is to understand is there a $C' \subseteq C$ such that every element of X appears in exactly one element of C'?

Total $[1,2]$-set problem: Input of this problem is a graph $G = (V,E)$ and a positive integer $k \leqslant |V|$. We want to investigate is there any total $[1,2]$-set of cardinality at most k for G.

Theorem 3.1. Total [1, 2]-SET is NP-complete for bipartite graphs.

Proof. Let $D \subseteq V$ is given, we verify D is a total $[1,2]$-set. For any vertex $v \in D$, we check neighborhood of each vertex and compute span number of any vertex $v \in V$. If there is a vertex v with span number more than 2, this set isn't a total $[1,2]$-set for G. It is obvious this algorithm is done in polynomial time and total $[1,2]$-set is a NP problem. Now for a set X, and a collection C of 3-element subsets of X, we build a graph and transform EXACT 3-COVER into a total $[1,2]$-set problem. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, ..., C_t\}$. For each $C_i \in C$, we build a cycle C_4 with a vertex u_i and add new vertices $\{v_{1_1}, v_{1_2}, v_{1_3}, v_{2_1}, v_{2_2}, v_{2_3}, \cdots, v_{t_1}, v_{t_2}, v_{t_3}\}$. We and connect all vertices v_{i1}, v_{i2}, v_{i3} to u_i. Then add some other vertices $\{x_1, x_2, ..., x_{3q}\}$ and edges $x_i v_{j_1}$, $x_i v_{j_2}$ and $x_i v_{j_3}$, if $x_i \in C_j$.G is a bipartite graph.

Let $k = 2t + q$. Suppose that C' is a solution for set X and collection C of EXACT 3-COVER. We build a set D of vertices of G contain every u_i, $1 \leqslant i \leqslant t$, and another vertex of C_4 adjacent to u_i and one of the v_{j_1}, v_{j_2} or v_{j_3} for each $C_j \in C'$. If C' exists, then it's cardinality is precisely q, and so $|D| = 2t + q = k$. We can check easily that D is a [1, 2]-total set of G.

Conversely, suppose that G has a total $[1,2]$-set D with $|D| \leqslant 2t + q = k$. Then D must contain two vertices of every C_4, in the best case we select u_i and one of the vertices in that adjacency in C_4. We select $2t$ vertices that dominate all vertices of cycles and all vertices of form v_{i_1}, v_{i_2} or v_{i_3} for $1 \leqslant i \leqslant t$. Since each v_{i_j} dominates only three vertices of $\{x_1, x_2, ..., x_{3q}\}$ We have to select exactly q vertices of them, i.e. we select q 3-element subsets of form $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ and one element of each of them. Each of this v_{i_j} correspond to a C_i and union of them is a exact cover for C. $\square$

Example 3.2. Let $C = \{C_1, C_2, C_3, C_4\}$ where $C_1 = \{x_1, x_2, x_4\}$,$C_2 = \{x_3, x_5, x_7\}$,$C_3 = \{x_4, x_5, x_6, x_7\}$ and $C_4 = \{x_6, x_8, x_9\}$, Corresponding graph was shown in Figure 1.

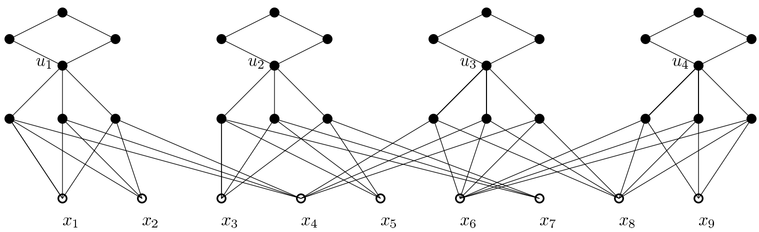

Figure 1. NP-completeness for bipartite graph

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