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Some Fixed Point Theorems Via $\widetilde{G}$-Rational Contractive Mappings in Ordered Modified G-Metric Spaces

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Abstract. In this paper, recalling the structure of modified G-metric spaces (as a generalization of both G-metric and G_b-metric spaces), we present the notions of $\widetilde{G}$-rational contractive mappings and investigate the existence of fixed point for such mappings. We also provide examples and an application to illustrate the results presented herein.

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1. Introduction

principle via using different form of contractive conditions in generalized metric spaces. Some of such generalizations are obtained via contractive conditions expressed by rational terms (see, [31], [19], [4], [5], [16], [24] and [32]).

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Ran and Reurings initiated the study of fixed point results on partially ordered sets in [30]. Also, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order. For more details we refer the reader to [25, 26].

Parvaneh and Ghoncheh in [28] introduced the concept of an extended b-metric space (p-metric space).

Definition 1.1. *[28] Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to R^+$ is a p-metric iff there exists a strictly increasing continuous function $\Omega : [0, \infty) \to [0, \infty)$ with $\Omega^{-1}(x) \leqslant x \leqslant \Omega(x)$ and $\Omega^{-1}(0) \leqslant 0 \leqslant \Omega(0)$ such that for all $x, y, z \in X$, the following conditions hold:*

($\widetilde{d}_1$) $\widetilde{d}(x, y) = 0$ *iff* $x = y$,

($\widetilde{d}_2$) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$,

($\widetilde{d}_3$) $\widetilde{d}(x, z) \leqslant \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z))$.

In this case, the pair $(X, \widetilde{d})$ is called a p-metric space, or, an extended b-metric space.

A b-metric [6] is a p-metric, when $\Omega(x) = sx$ while a metric is a p-metric, when $\Omega(x) = x$.

We have the following proposition.

Proposition 1.2. *[28] Let (X, d) be a metric space and let $\widetilde{d}(x, y) = \xi(d(x, y))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $x \leqslant \xi(x)$ and $0 = \xi(0)$. In this case, $\widetilde{d}$ is a p-metric with $\Omega(t) = \xi(t)$.*

The above proposition constructs the following example:

Example 1.3. Let (X, d) be a metric space and let $\widetilde{d}(x, y) = e^{d(x,y)} \sec^{-1}(e^{d(x,y)})$. Then $\widetilde{d}$ is a p-metric with $\Omega(t) = e^t \sec^{-1}(e^t)$.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims. For more details in this field the reader can refer to [15, 12, 13]

Definition 1.4. *[23] Let X be a nonempty set and G: $X \times X \times X \to R^+$ be a function satisfying the following properties:*

(G1) $G(x, y, z) = 0$ *iff* $x = y = z$;

(G2) $0 < G(x, x, y)$, *for all* $x, y \in X$ *with* $x \neq y$;

(G3) $G(x, x, y) \leqslant G(x, y, z)$, *for all* $x, y, z \in X$ *with* $y \neq z$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$, *(symmetry in all three variables);*

(G5) $G(x,y,z) \leqslant G(x,a,a)+G(a,y,z)$, *for all* $x,y,z,a \in X$ *(rectangle inequality).*

Then, the function G *is called a* G*-metric on* X *and the pair* (X,G) *is called a* G*-metric space.*

Aghajani *et al.* in [2] motivated by the concept of b-metric [6] introduced the concept of generalized b-metric spaces (G_{b}-metric spaces) and then they presented some basic properties of G_{b}-metric spaces.

The following is the definition of modified G-metric spaces which is a proper generalization of the notions of G-metric spaces and G_b-metric spaces.

Definition 1.5. *[29] Let* X *be a nonempty set and* $\Omega : [0,\infty) \to [0,\infty)$ *be a strictly increasing continuous function such that* $\Omega^{-1}(x) \leqslant x \leqslant \Omega(x)$ *for all* $x > 0$ *and* $\Omega^{-1}(0) = 0 = \Omega(0)$. *Suppose that a mapping* $\widetilde{G} : X \times X \times X \to \mathbb{R}^+$ *satisfies:*

($\widetilde{G}1$*)* $\widetilde{G}(x,y,z) = 0$ *if* $x = y = z$,

($\widetilde{G}2$*)* $0 < \widetilde{G}(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$,

($\widetilde{G}3$*)* $\widetilde{G}(x,x,y) \leqslant \widetilde{G}(x,y,z)$ *for all* $x,y,z \in X$ *with* $y \neq z$,

($\widetilde{G}4$*)* $\widetilde{G}(x,y,z) = \widetilde{G}(p\{x,y,z\})$, *where* p *is a permutation of* x,y,z *(symmetry),*

($\widetilde{G}5$*)* $G(x,y,z) \leqslant \Omega[\widetilde{G}(x,a,a) + \widetilde{G}(a,y,z)]$ *for all* $x,y,z,a \in X$ *(rectangle inequality).*

Then $\widetilde{G}$ *is called a modified* G*-metric and the pair* $(X,\widetilde{G})$ *is called a modified* G*-metric space or a* $\widetilde{G}$*-metric space.*

Each G-metric space is a $\widetilde{G}$-metric space with $\Omega(t) = t$ and every G_b-metric space is a $\widetilde{G}$-metric space with $\Omega(t) = st$.

Proposition 1.6. *[29] Let* (X,G) *be a* G_b*-metric space with coefficient* $s \geqslant 1$ *and let* $\widetilde{G}(x,y,z) = \xi(G(x,y,z))$ *where* $\xi : [0,\infty) \to [0,\infty)$ *is a strictly increasing function with* $x \leqslant \xi(x)$ *for all* $x > 0$ *and* $\xi(0) = 0$. *Then, show that* $\widetilde{G}$ *is a modified* G*-metric with* $\Omega(t) = \xi(st)$.

For each $x,y,z,a \in X$,

$$\begin{aligned}
&\widetilde{G}(x, y, z)\\
&= \xi(G(x, y, z)) \leqslant \xi(sG(x, a, a) + sG(a, y, z))\\
&\leqslant \xi(s\xi(G(x, a, a)) + s\xi(G(a, y, z))\\
&= \xi(s\widetilde{G}(x, a, a) + s\widetilde{G}(a, y, z))\\
&= \Omega(\widetilde{G}(x, a, a) + \widetilde{G}(a, y, z)).
\end{aligned}$$

So, $\widetilde{G}$ is a modified G-metric with $\Omega(t) = \xi(st)$.

The above proposition constructs the following examples:

Example 1.7. [27] Let (X, G) be a G_b-metric space with coefficient $s \geqslant 1$. Then,

1. $\widetilde{G}(x, y, z) = e^{G(x,y,z)} \sec^{-1}(e^{G(x,y,z)})$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st} \sec^{-1}(e^{st})$.
2. $\widetilde{G}(x, y, z) = [G(x, y, z) + 1] \sec^{-1}([G(x, y, z) + 1])$ is a $\widetilde{G}$-metric with $\Omega(t) = [st + 1] \sec^{-1}([st + 1])$.
3. $\widetilde{G}(x, y, z) = e^{G(x,y,z)} \tan^{-1}(e^{G(x,y,z)}-1)$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st} \tan^{-1}(e^{st}-1)$.
4. $\widetilde{G}(x, y, z) = G(x, y, z) \cosh(G(x, y, z))$ is a $\widetilde{G}$-metric with $\Omega(t) = st \cosh(st)$.
5. $\widetilde{G}(x, y, z) = e^{G(x,y,z)} \ln(1+G(x, y, z))$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st} \ln(1+st)$.
6. $\widetilde{G}(x, y, z) = G(x, y, z) + \ln(1 + G(x, y, z))$ is a $\widetilde{G}$-metric with $\Omega(t) = st + \ln(1 + st)$.

Definition 1.8. *A $\widetilde{G}$-metric $\widetilde{G}$ is said to be symmetric if $\widetilde{G}(x, y, y) = \widetilde{G}(y, x, x)$, for all $x, y \in X$.*

Proposition 1.9. *[29] Let X be a $\widetilde{G}$-metric space. Then for each $x, y, z, a \in X$ it follows that:*

(1) if $\widetilde{G}(x, y, z) = 0$ then $x = y = z$,

(2) $\widetilde{G}(x, y, z) \leqslant \Omega(\widetilde{G}(x, x, y) + \widetilde{G}(x, x, z))$,

(3) $\widetilde{G}(x, y, y) \leqslant \Omega[2\widetilde{G}(y, x, x)]$,

(4) $\widetilde{G}(x, y, z) \leqslant \Omega(\widetilde{G}(x, a, z) + \widetilde{G}(a, y, z))$.

Recall that a function f is super-additive if

$$f(x + y) \geqslant f(x) + f(y)$$

for all $x, y \in D(f)$.

Definition 1.10. *Let X be a $\widetilde{G}$-metric space with a super-additive function Ω. We define $\widetilde{d}_{\widetilde{G}}(x,y) = \widetilde{G}(x,y,y) + \widetilde{G}(x,x,y)$, for all $x, y \in X$. It is easy to see that $\widetilde{d}_{\widetilde{G}}$ defines a p-metric $\widetilde{d}$ on X, which we call it the $\widetilde{d}$-metric associated with $\widetilde{G}$.*

Definition 1.11. *Let X be a $\widetilde{G}$-metric space. A sequence $\{x_n\}$ in X is said to be:*

(1) $\widetilde{G}$-Cauchy if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m, n, l \geqslant n_0, \widetilde{G}(x_n, x_m, x_l) < \varepsilon$;

(2) $\widetilde{G}$-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m, n \geqslant n_0, \widetilde{G}(x_n, x_m, x) < \varepsilon$.

(3) A $\widetilde{G}$-metric space X is called $\widetilde{G}$-complete, if every $\widetilde{G}$-Cauchy sequence is $\widetilde{G}$-convergent in X.

Proposition 1.12. *Let X be a $\widetilde{G}$-metric space. Then the following are equivalent:*

(1) the sequence $\{x_n\}$ is $\widetilde{G}$-Cauchy.

(2) for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\widetilde{G}(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geqslant n_0$.

Proposition 1.13. *Let X be a $\widetilde{G}$-metric space. The following are equivalent:*

(1) $\{x_n\}$ is $\widetilde{G}$-convergent to x.

(2) $\widetilde{G}(x_n, x_n, x) \to 0$, as $n \to \infty$.

(3) $\widetilde{G}(x_n, x, x) \to 0$, as $n \to \infty$.

In general, a G_b-metric function $G(x,y,z)$ for $s > 1$ and so a modified G-metric function $\widetilde{G}(x,y,z)$ with nontrivial function Ω is not jointly continuous in all its variables (see [20]).

We will apply the following simple lemma about the $\widetilde{G}$-convergent sequences.

Lemma 1.14. *[29] Let $(X, \widetilde{G})$ be a $\widetilde{G}$-metric space.*

1. Suppose that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are $\widetilde{G}$-convergent to x, y and z, respectively. Then we have

$$(\Omega^{-1})^3[\widetilde{G}(x,y,z)] \leqslant \liminf_{n \longrightarrow \infty} \widetilde{G}(x_n, y_n, z_n) \leqslant \limsup_{n \longrightarrow \infty} \widetilde{G}(x_n, y_n, z_n) \leqslant \Omega^3[\widetilde{G}(x,y,z)].$$

2. Suppose that $\{x_n\}$ and $\{y_n\}$ are $\widetilde{G}$-convergent to x and y, respectively. Then we have

$$(\Omega^{-1})^2[\widetilde{G}(x,y,\alpha)] \leqslant \liminf_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,\alpha) \leqslant \limsup_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,\alpha) \leqslant \Omega^2[\widetilde{G}(x,y,\alpha)].$$

3. If $\{x_n\}$ *be* $\widetilde{G}$*-convergent to* x*, then*

$$(\Omega^{-1})[\widetilde{G}(x,\alpha,\beta)] \leqslant \liminf_{n\longrightarrow\infty} \widetilde{G}(x_n,\alpha,\beta) \leqslant \limsup_{n\longrightarrow\infty} \widetilde{G}(x_n,\alpha,\beta) \leqslant \Omega[\widetilde{G}(x,\alpha,\beta)].$$

In particular, if $x = y = z$*, then we have* $\lim_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,z_n) = 0$.

Proof. 1. Using the rectangle inequality in a $\widetilde{G}$-metric space it is easy to see that,

$$\widetilde{G}(x,y,z) \leqslant \Omega\Big[\widetilde{G}(x,x_n,x_n) + \Omega\big[\widetilde{G}(y,y_n,y_n) + \Omega[\widetilde{G}(z,z_n,z_n) + \widetilde{G}(x_n,y_n,z_n)]\big]\Big]$$

and

$$\widetilde{G}(x_n,y_n,z_n) \leqslant \Omega\Big[\widetilde{G}(x_n,x,x) + \Omega\big[G(y_n,y,y) + \Omega[G(z_n,z,z) + G(x,y,z)]\big]\Big].$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.
2. Using the rectangle inequality we see that,

$$\widetilde{G}(x,y,\alpha) \leqslant \Omega\Big[\widetilde{G}(x,x_n,x_n) + \Omega\big[\widetilde{G}(y,y_n,y_n) + \widetilde{G}(x_n,y_n,\alpha)\big]\Big]$$

and

$$\widetilde{G}(x_n,y_n,\alpha) \leqslant \Omega\Big[\widetilde{G}(x_n,x,x) + \Omega\big[G(y_n,y,y) + G(x,y,\alpha)\big]\Big].$$

3. Similarly,

$$\widetilde{G}(x,\alpha,\beta) \leqslant \Omega\Big[\widetilde{G}(x,x_n,x_n) + \widetilde{G}(x_n,\alpha,\beta)\Big]$$

and

$$\widetilde{G}(x_n,\alpha,\beta) \leqslant \Omega\Big[\widetilde{G}(x_n,x,x) + G(x,\alpha,\beta)\Big]. \quad \square$$

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies that } \; t_n \to 0, \; \textit{as } n \to \infty.$$

In order to generalize the Banach contraction principle, in 1973, Geraghty proved the following.

Theorem 1.15. *[9] Let (X, d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx, fy) \leqslant \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In 2010, Amini-Harandi and Emami [3] characterized the result of Geraghty in the setting of a partially ordered complete metric space.

In [7], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces.

Also, Zabihi and Razani [32] and Shahkoohi and Razani [31] obtained some fixed point results duo to rational Geraghty contractions in b-metric spaces.

Motivated by [19], in this paper we present some fixed point theorems for different rational contractive mappings in partially ordered modified G-metric spaces. Our results extend some existing results in the literature.

2. Main Results

2.1 Fixed point results using $\widetilde{G}$-rational geraghty contractions

Let $(X, \widetilde{G})$ be a $\widetilde{G}$-metric space with function Ω and let $\mathcal{F}_\Omega$ denotes the class of all functions $\beta : [0, \infty) \to [0, \Omega^{-1}(1))$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \Omega^{-1}(1) \text{ implies that } \ t_n \to 0, \ as\ n \to \infty.$$

An example of a function in $\mathcal{F}_\Omega$ may be given by $\beta(t) = (\ln 2)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, \ln 2)$ where $\widetilde{G}(x, y, z) = e^{\max(|x-y|,|y-z|,|z-x|)} - 1$ for all $x, y, z \in \mathbb{R}$.

Another example of a function in $\mathcal{F}_\Omega$ may be given by $\beta(t) = W(1)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, W(1))$ where $\widetilde{G}(x, y, z) = \max(|x - y|, |y - z|, |z - x|)e^{\max(|x-y|,|y-z|,|z-x|)}$ for all $x, y, z \in \mathbb{R}$. Note that W is the Lambert W-function (see, e.g., [8])

Definition 2.1.1. *Let $(X, \widetilde{G})$ be an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational Geraghty contraction if, there exists $\beta \in \mathcal{F}_\Omega$ such that,*

$$\Omega(\widetilde{G}(fx, fy, fz)) \leqslant \beta(M(x, y, z))M(x, y, z) \tag{1}$$

for all comparable elements $x, y, z \in X$, where

$$M(x,y,z) = \max\left\{\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,y,fy)[\widetilde{G}(y,z,fz)]^2}{1+\widetilde{G}(x,fx,f^2x)\widetilde{G}(y,fy,f^2y)}, \frac{\widetilde{G}(x,fx,f^2x)\widetilde{G}(y,fy,f^2y)\widetilde{G}(z,fz,f^2z)}{1+\widetilde{G}(fx,f^2x,f^3x)\widetilde{G}(fy,f^2y,f^3y)}\right\}.$$

Recall that a modified G-metric space $(X, \widetilde{G})$ it said to has the s.l.c. property, if whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$

Theorem 2.1.2. *Let $(X, \preceq, \widetilde{G})$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational Geraghty contraction. If,*

(I) f is continuous, or,

(II) $(X, \preceq, \widetilde{G})$ has the s.l.c. property,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq ... \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots .$$

We will do the proof in the following steps.

Step 1. We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (1) we have

$$G(x_n, x_{n+1}, x_{n+2}) = \widetilde{G}(fx_{n-1}, fx_n, fx_{n+1}) \leqslant \beta(M(x_{n-1}, x_n, x_{n+1}))M(x_{n-1}, x_n, x_{n+1}), \qquad (2)$$

where

$M(x_{n-1}, x_n, x_{n+1})$

$= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \frac{\widetilde{G}(x_{n-1},x_n,fx_n)\widetilde{G}(x_n,x_{n+1},fx_{n+1})^2}{1+\widetilde{G}(x_{n-1},fx_{n-1},f^2x_{n-1})\widetilde{G}(x_n,fx_n,f^2x_n)},$

$\frac{\widetilde{G}(x_{n-1},fx_{n-1},f^2x_{n-1})\widetilde{G}(x_n,fx_n,f^2x_n)\widetilde{G}(x_{n+1},fx_{n+1},f^2x_{n+1})}{1+\widetilde{G}(fx_{n-1},f^2x_{n-1},f^3x_{n-1})\widetilde{G}(fx_n,f^2x_n,f^3x_n)}\}$

$= \max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \frac{\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})^2}{1+\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})},$
$\frac{\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})\widetilde{G}(x_{n+1},x_{n+2},x_{n+3})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+2})\widetilde{G}(x_{n+1},x_{n+2},x_{n+3})}\}$
$\leqslant \max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \widetilde{G}(x_n,x_{n+1},x_{n+2})\}.$

$$\begin{aligned}
&M(x_{n-1},x_n,x_{n+1})\\
&= \max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \frac{\widetilde{G}(x_{n-1},x_n,fx_n)\widetilde{G}(x_n,x_{n+1},fx_{n+1})^2}{1+\widetilde{G}(x_{n-1},fx_{n-1},f^2x_{n-1})\widetilde{G}(x_n,fx_n,f^2x_n)},\\
&\frac{\widetilde{G}(x_{n-1},fx_{n-1},f^2x_{n-1})\widetilde{G}(x_n,fx_n,f^2x_n)\widetilde{G}(x_{n+1},fx_{n+1},f^2x_{n+1})}{1+\widetilde{G}(fx_{n-1},f^2x_{n-1},f^3x_{n-1})\widetilde{G}(fx_n,f^2x_n,f^3x_n)}\}\\
&= \max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \frac{\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})^2}{1+\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})},\\
&\frac{\widetilde{G}(x_{n-1},x_n,x_{n+1})\widetilde{G}(x_n,x_{n+1},x_{n+2})\widetilde{G}(x_{n+1},x_{n+2},x_{n+3})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+2})\widetilde{G}(x_{n+1},x_{n+2},x_{n+3})}\}\\
&\leqslant \max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \widetilde{G}(x_n,x_{n+1},x_{n+2})\}.
\end{aligned}$$

If $\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \widetilde{G}(x_n,x_{n+1},x_{n+2})\} = \widetilde{G}(x_n,x_{n+1},x_{n+2})$, then from (2) we have,

$$\begin{aligned}
\widetilde{G}(x_n,x_{n+1},x_{n+2}) &\leqslant \beta(M(x_{n-1},x_n,x_{n+1}))\widetilde{G}(x_n,x_{n+1},x_{n+2})\\
&< \Omega^{-1}(1)\widetilde{G}(x_n,x_{n+1},x_{n+2})\\
&\leqslant \widetilde{G}(x_n,x_{n+1},x_{n+2}),
\end{aligned} \tag{3}$$

which is a contradiction.

Hence, $\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}), \widetilde{G}(x_n,x_{n+1},x_{n+2})\} = \widetilde{G}(x_{n-1},x_n,x_{n+1})$. So, from (2),

$$\widetilde{G}(x_n,x_{n+1},x_{n+2}) \leqslant \beta(M(x_{n-1},x_n,x_{n+1}))\widetilde{G}(x_{n-1},x_n,x_{n+1}) < \widetilde{G}(x_{n-1},x_n,x_{n+1}). \tag{4}$$

That is, $\{\widetilde{G}(x_n,x_{n+1},x_{n+2})\}$ is a decreasing sequence, then there exists $r \geqslant 0$ such that $\lim_{n\to\infty} \widetilde{G}(x_n,x_{n+1},x_{n+2}) = r$. We will prove that $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$, from (4) we have

$$r \leqslant \lim_{n\to\infty} \beta(M(x_{n-1},x_n,x_{n+1}))r \leqslant \Omega^{-1}(1)r,$$

which implies that $\Omega^{-1}(1) \leqslant 1 \leqslant \lim_{n\to\infty} \beta(M(x_{n-1},x_n,x_{n+1})) \leqslant \Omega^{-1}(1)$. Now, as $\beta \in \mathcal{F}_\Omega$ we conclude that $M(x_{n-1},x_n,x_{n+1}) \to 0$ which yields that $r = 0$, a contradiction. Hence, the assumption that $r > 0$ is false. That is,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) = 0. \tag{5}$$

Consequently,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0. \tag{6}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a $\widetilde{G}$-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \geqslant \varepsilon. \tag{7}$$

This means that

$$\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon. \tag{8}$$

From the rectangular inequality, we get

$$\varepsilon \leqslant \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \leqslant \Omega[\widetilde{G}(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})].$$

Taking the upper limit as $i \to \infty$ and by (6), we get

$$\Omega^{-1}(\varepsilon) \leqslant \limsup_{i\to\infty} \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}). \tag{9}$$

From the definition of $M(x, y, z)$ and the above limits,

$$\begin{aligned}
\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) &= \limsup_{i\to\infty} \max\{\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}), \\
&\quad \frac{\widetilde{G}(x_{m_i}, x_{n_i-1}, fx_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})^2}{1 + \widetilde{G}(x_{m_i}, fx_{m_i}, f^2x_{m_i})\widetilde{G}(x_{n_i-1}, fx_{n_i-1}, f^2x_{n_i-1})}, \\
&\quad \frac{\widetilde{G}(x_{m_i}, fx_{m_i}, f^2x_{m_i})\widetilde{G}(x_{n_i-1}, fx_{n_i-1}, f^2x_{n_i-1})^2}{1 + \widetilde{G}(fx_{m_i}, f^2x_{m_i}, f^3x_{m_i})\widetilde{G}(fx_{n_i-1}, f^2x_{n_i-1}, f^3x_{n_i-1})}\} \\
&\leqslant \varepsilon.
\end{aligned}$$

Now, from (1) and the above inequalities, we have

$$\begin{aligned}
\varepsilon &\leqslant \limsup_{i\to\infty} \Omega(\widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})) \\
&\leqslant \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1})) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \\
&\leqslant \varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1}))
\end{aligned}$$

which implies that $\Omega^{-1}(1) \leqslant \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}_\Omega$ we conclude that
$M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$ which yields that $\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$. Consequently,

$$\widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \leqslant \Omega[\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) + s\widetilde{G}(x_{n_i-1}, x_{n_i}, x_{n_i})] \to 0,$$

a contradiction to (7). Therefore, $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence.
$\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f.
First, let f is continuous, so, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) holds. Using the assumption on X we have $x_n \preceq u$. Now, by Lemma 1.14,

$$\begin{aligned}(\Omega^{-1})^2[\widetilde{G}(u, u, fu)] &\leqslant \limsup_{n\to\infty} \widetilde{G}(x_{n+1}, x_{n+1}, fu)\\ &\leqslant \limsup_{n\to\infty} \beta(M(x_n, x_n, u)) \limsup_{n\to\infty} M(x_n, x_n, u),\end{aligned}$$

where,

$$\begin{aligned}\lim_{n\to\infty} M(x_n, x_n, u) = \lim_{n\to\infty} \max\{&\widetilde{G}(x_n, x_n, u), \frac{\widetilde{G}(x_n, x_n, fx_n)\widetilde{G}(x_n, u, fu)^2}{1 + \widetilde{G}(x_n, fx_n, f^2x_n)^2},\\ &\frac{\widetilde{G}(x_n, fx_n, f^2x_n)^2\widetilde{G}(u, fu, f^2u)}{1 + \widetilde{G}(fx_n, f^2x_n, f^3x_n)^2}\}\\ &= 0.\end{aligned}$$

Therefor, we deduce that $\widetilde{G}(u, u, fu) = 0$, so, $u = fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume on contrary that, u and v are two fixed points of f such that $u \neq v$. Then by (1), we have

$$\begin{aligned}G(u, v, v) = \widetilde{G}(fu, fv, fv) \leqslant \beta(M(u, v, v))M(u, v, v) =\\ \beta(\widetilde{G}(u, v, v))\widetilde{G}(u, v, v) < \Omega^{-1}(1)\widetilde{G}(u, v, v). \qquad (10)\end{aligned}$$

Because

$$M(u, v, v) = \widetilde{G}(u, v, v).$$

So, we get, $G(u, v, v) < \Omega^{-1}(1)G(u, v, v)$, a contradiction. Hence, $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is well ordered. □

2.2 Fixed point results via comparison functions

Let Ψ be the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2.2.1. *If $\psi \in \Psi$, then the following are satisfied.*
(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Definition 2.2.2. *Let $(X, \preceq, \widetilde{G})$ is an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational ψ-contraction if, there exists $\psi \in \Psi$ such that,*

$$\Omega(\widetilde{G}(fx, fy, fz)) \leqslant \psi(M(x, y, z)) \tag{11}$$

for all comparable elements $x, y, z \in X$, where

$$\begin{aligned} &M(x, y, z)\\ &= \max\left\{\widetilde{G}(x, y, z), \frac{\widetilde{G}(x, x, fx)\widetilde{G}(x, x, fy)}{1 + \Omega[\widetilde{G}(x, x, fx) + \widetilde{G}(y, y, fy)]},\right.\\ &\left.\frac{\widetilde{G}(y, y, z)\widetilde{G}(y, y, fz)}{1 + \Omega[\widetilde{G}(y, y, fy) + \widetilde{G}(z, z, fz)]}, \frac{\widetilde{G}(x, x, fx)\widetilde{G}(x, x, z)}{1 + \widetilde{G}(x, x, fy) + \widetilde{G}(y, y, fx)}\right\}. \end{aligned}$$

Theorem 2.2.3. *Let $(X, \preceq, \widetilde{G})$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational ψ-contractive mapping. If*

(I) f is continuous, or,

(II) $(X, \preceq, \widetilde{G})$ has the s.l.c. property,

then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$.

Step I: We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 11 we have

$$\begin{aligned} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) &= \widetilde{G}(fx_{n-1}, fx_n, fx_{n+1})\\ &\leqslant \psi(M(x_{n-1}, x_n, x_{n+1}))\\ &\leqslant \psi(\widetilde{G}(x_{n-1}, x_n, x_{n+1}))\\ &< \widetilde{G}(x_{n-1}, x_n, x_{n+1}), \end{aligned} \tag{12}$$

because

$$\begin{aligned}
&M(x_{n-1},x_n,x_{n+1})\\
&=\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\frac{\widetilde{G}(x_{n-1},x_{n-1},x_n)\widetilde{G}(x_{n-1},x_{n-1},fx_n)}{1+\Omega[\widetilde{G}(x_{n-1},x_{n-1},fx_{n-1})+\widetilde{G}(x_n,x_n,fx_n)]},\\
&\frac{\widetilde{G}(x_n,x_n,x_{n+1})\widetilde{G}(x_n,x_n,fx_{n+1})}{1+\Omega[\widetilde{G}(x_n,x_n,fx_n)+\widetilde{G}(x_{n+1},x_{n+1},fx_{n+1})]},\\
&\frac{\widetilde{G}(x_{n-1},x_{n-1},fx_{n-1})\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})}{1+\widetilde{G}(x_{n-1},x_{n-1},fx_n)+\widetilde{G}(x_n,x_n,fx_{n-1})}\}\\
&\leqslant\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\widetilde{G}(x_{n-1},x_{n-1},x_n),\widetilde{G}(x_n,x_n,x_{n+1})\}\\
&\leqslant\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\widetilde{G}(x_n,x_{n+1},x_{n+2})\},
\end{aligned}$$

and it is easy to see that

$$\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\widetilde{G}(x_n,x_{n+1},x_{n+2})\}=\widetilde{G}(x_{n-1},x_n,x_{n+1}),$$

so from (12), we conclude that $\{\widetilde{G}(x_n,x_{n+1},x_{n+2})\}$ is decreasing.
Then there exists $r\geqslant 0$ such that $\lim_{n\to\infty}\widetilde{G}(x_n,x_{n+1},x_{n+2})=r$.

is easy to see that $r=\lim_{n\to\infty}\widetilde{G}(x_{n-1},x_n,x_n)=0$.

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \text{ and } \widetilde{G}(x_{m_i},x_{n_i},x_{n_i})\geqslant\varepsilon. \tag{13}$$

This means that

$$\widetilde{G}(x_{m_i},x_{n_i-1},x_{n_i-1})<\varepsilon. \tag{14}$$

As in the proof of Theorem 2.1.2, we have,

$$\limsup_{i\to\infty}\widetilde{G}(x_{m_i+1},x_{n_i},x_{n_i}). \tag{15}$$

From the definition of $M(x,y,z)$ and the above limits,

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) = \limsup_{i\to\infty} \max\{&\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}),\\ &\frac{\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})\widetilde{G}(x_{m_i}, x_{m_i}, fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})+\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\ &\frac{\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})+\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\ &\frac{\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})\widetilde{G}(x_{m_i}, x_{m_i}, x_{n_i-1})}{1+\widetilde{G}(x_{m_i}, x_{m_i}, fx_{n_i-1})+\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{m_i})}\}\\ &\leqslant \varepsilon.\end{aligned}$$

Now, from (11) and the above inequalities, we have

$$\begin{aligned}\varepsilon \leqslant \limsup_{i\to\infty} \Omega[\widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})] &\leqslant \limsup_{i\to\infty} \psi(M(x_{m_i}, x_{n_i-1}, x_{n_i-1}))\\ &< \varepsilon\end{aligned}$$

which is a contradiction. Now, we conclude that $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. $\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f. This step is proved as the proof of step 3 of Theorem 2.1.2 with some elementary changes. $\square$

If in the above theorem we take $\psi(t) = \sinh t$ and $\widetilde{G}(x,y,z) = \sinh(G(x,y,z))$ then we have the following corollary in the framework of G_b metric spaces.

Corollary 2.2.4. *Let $(X, G_b, \preceq)$ be an ordered G_b-complete G_b-metric space with coefficient $s > 1$. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$\sinh(s \cdot \sinh(G(fx, fy, fz))) \leqslant \sinh(M(x,y,z)) \tag{16}$$

for all comparable elements $x, y, z \in X$, where

$$\begin{aligned}&M(x,y,z)\\ &= \max\left\{\sinh(G(x,y,z)), \frac{\sinh(G(x,x,fx))\sinh(G(y,y,fy))}{1+\sinh(s\cdot[\sinh(G(x,y,y))+\sinh(G(x,x,fy))])},\right.\\ &\frac{\sinh(G(y,y,fy))\sinh(G(z,z,fz))}{1+\sinh(s\cdot[\sinh(G(y,z,z))+\sinh(G(y,y,fz))])},\\ &\left.\frac{\sinh(G(y,z,z))\sinh(G(y,y,z))}{1+\sinh(G(y,fy,fy))+\sinh(G(z,fz,fz))}\right\}.\end{aligned}$$

If

(I) *f is continuous, or,*

(II) *$(X, G_b, \preceq)$ enjoys the s.l.c. property,*

then f has a fixed point.

2.3 Fixed point results related to JS-contractions

Jleli et al. [17] have introduced the class Θ_0 consists of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is non-decreasing;

(θ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

(θ_3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;

(θ_4) θ is continuous.

They proved the following result:

Theorem 2.3.1. [17, Corollary 2.1] *Let (X, d) be a complete metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \implies \theta\big(d(Tx, Ty)\big) \leqslant \theta\big(d(x, y)\big)^k. \tag{17}$$

Then T has a unique fixed point.

From now on, we denote by Θ the set of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the following conditions:

θ_1. θ is a continuous strictly increasing function;

θ_2. for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

Remark 2.3.2. *[10] It is clear that $f(t) = e^t$ does not belong to Θ_0, but it belongs to Θ. Other examples are $f(t) = \cosh t$, $f(t) = \frac{2\cosh t}{1+\cosh t}$, $f(t) = 1 + \ln(1 + t)$, $f(t) = \frac{2+2\ln(1+t)}{2+\ln(1+t)}$, $f(t) = e^{te^t}$ and $f(t) = \frac{2e^{te^t}}{1+e^{te^t}}$, for all $t > 0$.*

Definition 2.3.3. *Let $(X, \widetilde{G}, \preceq)$ be an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational JS-contraction if*

$$\theta(\Omega[\widetilde{G}(fx, fy, fz)]) \leqslant \theta(M(x, y, z))^k \tag{18}$$

for all comparable elements $x, y, z \in X$, where $\theta \in \Theta$, $k \in [0,1)$ and

$$M(x,y,z) = \max\Big\{\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,x,fx)\widetilde{G}(y,y,fy)}{1+\Omega[\widetilde{G}(x,y,y)+\widetilde{G}(x,x,fy)]}, \frac{\widetilde{G}(y,y,fy)\widetilde{G}(z,z,fz)}{1+\Omega[\widetilde{G}(y,z,z)+\widetilde{G}(y,y,fz)]}, \frac{\widetilde{G}(y,z,z)\widetilde{G}(y,y,z)}{1+\widetilde{G}(y,fy,fy)+\widetilde{G}(z,fz,fz)}\Big\}.$$

Theorem 2.3.4. *Let $(X, \widetilde{G}, \preceq)$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f: X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational JS-contractive mapping. If*
(I) f is continuous, or,
(II) $(X, \widetilde{G}, \preceq)$ enjoys the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$.

Step 1. We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by (18) we have

$$\begin{aligned} \theta(\widetilde{G}(x_n, x_{n+1}, x_{n+2})) &\leqslant \theta(\Omega[\widetilde{G}(x_n, x_{n+1}, x_{n+2})]) \\ &= \theta(\Omega[\widetilde{G}(fx_{n-1}, fx_n, fx_{n+1})]) \\ &\leqslant \theta(M(x_{n-1}, x_n, x_{n+1}))^k \\ &\leqslant \theta(\widetilde{G}(x_{n-1}, x_n, x_{n+1}))^k, \end{aligned} \tag{19}$$

because $M(x_{n-1}, x_n, x_{n+1})$

$$\begin{aligned} &= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \tfrac{\widetilde{G}(x_{n-1},x_{n-1},fx_{n-1})\widetilde{G}(x_n,x_n,fx_n)}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},fx_n)]}, \\ &\tfrac{\widetilde{G}(x_n,x_n,fx_n)\widetilde{G}(x_{n+1},x_{n+1},fx_{n+1})}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,fx_{n+1})]}, \tfrac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,fx_n,fx_n)+\widetilde{G}(x_{n+1},fx_{n+1},fx_{n+1})}\} \\ &= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \tfrac{\widetilde{G}(x_{n-1},x_{n-1},x_n)\widetilde{G}(x_n,x_n,x_{n+1})}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]}, \\ &\tfrac{\widetilde{G}(x_n,x_n,x_{n+1})\widetilde{G}(x_{n+1},x_{n+1},x_{n+2})}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,x_{n+2})]}, \tfrac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_{n+1},x_{n+2},x_{n+2})}\} \\ &\leqslant \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \tfrac{\widetilde{G}(x_{n-1},x_{n-1},x_n)\Omega[\widetilde{G}(x_n,x_n,x_{n-1})+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]}, \\ &\tfrac{\widetilde{G}(x_n,x_n,x_{n+1})\Omega[\widetilde{G}(x_{n+1},x_{n+1},x_n)+\widetilde{G}(x_n,x_n,x_{n+2})]}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,x_{n+2})]}, \tfrac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_{n+1},x_{n+2},x_{n+2})}\} \\ &\leqslant \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_{n-1}, x_{n-1}, x_n), \widetilde{G}(x_n, x_n, x_{n+1})\} \\ &\leqslant \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\}. \end{aligned}$$

From (19) we deduce that,

$$\Theta\big(\widetilde{G}(x_n, x_{n+1}, x_{n+2})\big) \leqslant \Theta\big(\widetilde{G}(x_{n-1}, x_n, x_{n+1})\big)^k.$$

Therefore,

$$1 \leqslant \Theta\big(\widetilde{G}(x_n, x_{n+1}, x_{n+2})\big) \leqslant \Theta\big(\widetilde{G}(x_{n-1}, x_n, x_{n+1})\big)^k \leqslant \ldots \leqslant \Theta(\widetilde{G}(x_0, x_1, x_2))^{k^n}. \quad (20)$$

Taking the limit as $n \to \infty$ in (20) we have,

$$\lim_{n\to\infty} \Theta\big(\widetilde{G}(x_n, x_{n+1}, x_{n+2})\big) = 1$$

and since $\Theta \in \Delta_\Theta$ we obtain,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) = 0. \quad (21)$$

Therefore, we have,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_n, x_{n-1}) = 0. \quad (22)$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, that $\{x_n\}$ is not a $\widetilde{G}$-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \geqslant \varepsilon. \quad (23)$$

This means that

$$\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon. \quad (24)$$

Hence,

$$\widetilde{G}(x_{m_i}, x_{m_i}, x_{n_i-1}) < \Omega(2\varepsilon). \quad (25)$$

From the rectangular inequality, we get

$$\varepsilon \leqslant \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \leqslant \Omega[\widetilde{G}(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})].$$

By taking the upper limit as $i \to \infty$, we get

$$\Omega^{-1}(\varepsilon) \leqslant \limsup_{i\to\infty} \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}). \quad (26)$$

From the definition of $M(x,y,z)$ and the above limits,

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i},x_{n_i-1},x_{n_i-1}) &= \limsup_{i\to\infty}\max\{\widetilde{G}(x_{m_i},x_{n_i-1},x_{n_i-1}),\\ &\frac{\widetilde{G}(x_{m_i},x_{m_i},fx_{m_i})\widetilde{G}(x_{n_i-1},x_{n_i-1},fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{m_i},x_{n_i-1},x_{n_i-1})+\widetilde{G}(x_{n_i-1},x_{n_i-1},fx_{m_i})]},\\ &\frac{\widetilde{G}(x_{n_i-1},x_{n_i-1},fx_{n_i-1})\widetilde{G}(x_{n_i-1},x_{n_i-1},fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{n_i-1},x_{n_i-1},x_{n_i-1})+\widetilde{G}(x_{n_i-1},x_{n_i-1},fx_{n_i-1})]},\\ &\frac{\widetilde{G}(x_{n_i-1},x_{n_i-1},x_{n_i-1})\widetilde{G}(x_{n_i-1},x_{n_i-1},x_{n_i-1})}{1+\widetilde{G}(x_{n_i-1},fx_{n_i-1},fx_{n_i-1})+\widetilde{G}(x_{n_i-1},fx_{n_i-1},fx_{n_i-1})}\}\\ &\leqslant \varepsilon.\end{aligned}$$

Now, from (18) and the above inequalities, we have

$$\begin{aligned}\theta(\Omega[\Omega^{-1}(\varepsilon)]) &\leqslant \limsup_{i\to\infty}\theta(\Omega[\widetilde{G}(x_{m_i+1},x_{n_i},x_{n_i})])\\ &\leqslant \limsup_{i\to\infty}\theta(M(x_{m_i},x_{n_i-1},x_{n_i}))^k\\ &\leqslant \theta(\varepsilon)^k\end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction. So, we conclude that $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. $\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f.

When f is continuous, the proof is straightforward.

Now, let (II) holds. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 1.14,

$$\begin{aligned}\theta((\Omega^{-1})^2[\widetilde{G}(u,u,fu)]) &\leqslant \limsup_{n\to\infty}\theta(\widetilde{G}(x_{n+1},x_{n+1},fu))\\ &\leqslant \limsup_{n\to\infty}\theta(M(x_n,x_n,u))^k,\end{aligned}$$

where,

$$\begin{aligned}&\lim_{n\to\infty} M(x_n,x_n,u)\\ &= \lim_{n\to\infty}\max\left\{\widetilde{G}(x_n,x_n,u),\frac{\widetilde{G}(x_n,x_n,fx_n)\widetilde{G}(x_n,x_n,fx_n)}{1+\Omega[\widetilde{G}(x_n,x_n,x_n)+\widetilde{G}(x_n,x_n,fx_n)]},\right.\\ &\left.\frac{\widetilde{G}(x_n,x_n,fx_n)\widetilde{G}(u,u,fu)}{1+\Omega[\widetilde{G}(x_n,u,u)+\widetilde{G}(x_n,x_n,fu)]},\frac{\widetilde{G}(x_n,u,u)\widetilde{G}(x_n,x_n,u)}{1+\widetilde{G}(x_n,fx_n,fx_n)+\widetilde{G}(u,fu,fu)}\right\} = 0.\end{aligned}$$

Therefor, we deduce that $\widetilde{G}(u,u,fu)=0$, so, $u=fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume on contrary that, u and v are two fixed points of f such that $u\neq v$. Then by (18), we have

$$\theta[\widetilde{G}(u,v,v)]=\theta[\widetilde{G}(fu,fv,fv)]\leqslant\theta(M(u,v,v))^k=\theta(\widetilde{G}(u,v,v))^k. \tag{27}$$

So, we get, $G(u,v,v)=0$, a contradiction. Hence $u=v$, and f has a unique fixed point. $\square$

If in the above theorem we take $\theta(t)=\frac{2e^{te^t}}{1+e^{te^t}}$ and $\widetilde{G}(x,y,z)=e^{G(x,y,z)}-1$ then we have the following corollary in the framework of G_b metric spaces.

Corollary 2.3.5. *Let $(X,G_b,\preceq)$ be an ordered G_b-complete G_b-metric space with coefficient $s>1$. Let $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq f(x_0)$. Suppose that*

$$\frac{2e^{[e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1]e^{e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1}}}{1+e^{[e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1]e^{e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1}}}\leqslant\sqrt{\frac{2e^{M(x,y,z)e^{M(x,y,z)}}}{1+e^{M(x,y,z)e^{M(x,y,z)}}}}$$

for all comparable elements $x,y,z\in X$, where

$$\begin{aligned}&M(x,y,z)\\&=\max\left\{e^{G(x,y,z)}-1,\frac{[e^{G(x,x,fx)}-1][e^{G(y,y,fy)}-1]}{1+e^{s\cdot[e^{G(x,y,y)}-1+e^{G(x,x,fy)}-1]}-1},\right.\\&\left.\frac{[e^{G(y,y,fy)}-1][e^{G(z,z,fz)}-1]}{1+e^{s\cdot[e^{G(y,z,z)}-1+e^{G(y,y,fz)}-1]}-1},\frac{[e^{G(y,z,z)}-1][e^{G(y,y,z)}-1]}{1+e^{G(y,fy,fy)}-1+e^{G(z,fz,fz)}-1}\right\}.\end{aligned}$$

If

(I) *f is continuous, or,*

(II) *$(X,G_b,\preceq)$ enjoys the s.l.c. property,*

then f has a fixed point.

2.4 Examples

Example 2.4.1. Let $X=[0,8]$ be equipped with the $\widetilde{G}$-metric

$$\widetilde{G}(x,y,z)=\sinh(\frac{|x-y|+|y-z|+|z-x|}{3})$$

for all $x,y,z\in X$, where $\Omega(x)=\sinh x$ which $\Omega^{-1}(x)=\sinh^{-1}(x)$.

Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leqslant x$, the function $f : [0, 8] \to [0, 2]$ by

$$fx = \sqrt{2 + \frac{x}{4}}$$

and the function β given by $\beta(t) = \frac{1}{2} < 0.88137358702 = \Omega^{-1}(1)$.
For all comparable elements $x, y \in X$, we have,

$$\begin{aligned}
&\Omega(\widetilde{G}(fx, fy, fz)) \\
&= \sinh(\sinh(\frac{|\sqrt{2+\frac{x}{4}} - \sqrt{2+\frac{y}{4}}| + |\sqrt{2+\frac{y}{4}} - \sqrt{2+\frac{z}{4}}| + |\sqrt{2+\frac{z}{4}} - \sqrt{2+\frac{x}{4}}|}{3})) \\
&\leqslant \sinh(\sinh(\frac{|\frac{x}{4} - \frac{y}{4}| + |\frac{y}{4} - \frac{z}{4}| + |\frac{z}{4} - \frac{x}{4}|}{3})) \\
&\leqslant \sinh(\frac{\widetilde{G}(x, y, z)}{4}) \\
&\leqslant \frac{\widetilde{G}(x, y, z)}{2} = \beta(\widetilde{G}(x, y, z))\widetilde{G}(x, y, z) \leqslant \beta(M(x, y, z))M(x, y, z),
\end{aligned}$$

So, from Theorem 2.1.2 f has a fixed point.

Example 2.4.2. Let $X = [0, \infty]$ be equipped with the

$$\widetilde{G}(x, y, z) = \frac{|x - y| + |y - z| + |z - x|}{3} + \ln(\frac{|x - y| + |y - z| + |z - x|}{3})$$

for all $x, y, z \in X$, where $\Omega(x) = x + \ln x$.
Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leqslant x$, the function $f : X \to X$ by

$$fx = \ln(\frac{x}{5} + 2)$$

and the function ψ given by $\psi(t) = \frac{1}{2}t$. It is obvious that $\psi(t) < t$ for all $t \in X$.
For all comparable elements $x, y \in X$, by mean value theorem, we have,

$$\begin{aligned}
&\Omega[\widetilde{G}(fx, fy, fz)] \\
&= \widetilde{G}(fx, fy, fz) + \ln[1 + \widetilde{G}(fx, fy, fz)] + \ln[1 + \widetilde{G}(fx, fy, fz) + \ln[1 + \widetilde{G}(fx, fy, fz)]] \\
&= \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3} \\
&+ \ln[1 + \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3}] \\
&+ \ln\Big[1 + \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3}
\end{aligned}$$

$$+\ln[1+\frac{\left|\ln 2+\frac{x}{5}-\ln 2+\frac{y}{5}\right|+\left|\ln 2+\frac{y}{5}-\ln 2+\frac{z}{5}\right|+\left|\ln 2+\frac{z}{5}-\ln 2+\frac{x}{5}\right|}{3}]\Big]$$
$$\leqslant \frac{\left|\frac{x}{5}-\frac{y}{5}\right|+\left|\frac{y}{5}-\frac{z}{5}\right|+\left|\frac{z}{5}-\frac{x}{5}\right|}{3}$$
$$+\ln[1+\frac{\left|\frac{x}{5}-\frac{y}{5}\right|+\left|\frac{y}{5}-\frac{z}{5}\right|+\left|\frac{z}{5}-\frac{x}{5}\right|}{3}]$$
$$+\ln\Big[1+\frac{\left|\frac{x}{5}-\frac{y}{5}\right|+\left|\frac{y}{5}-\frac{z}{5}\right|+\left|\frac{z}{5}-\frac{x}{5}\right|}{3}$$
$$+\ln[1+\frac{\left|\frac{x}{5}-\frac{y}{5}\right|+\left|\frac{y}{5}-\frac{z}{5}\right|+\left|\frac{z}{5}-\frac{x}{5}\right|}{3}]\Big]$$
$$\leqslant \frac{1}{5}\widetilde{G}(x,y,z)+\ln[1+\frac{1}{5}\widetilde{G}(x,y,z)]$$
$$+\ln\Big[1+\frac{1}{5}\widetilde{G}(x,y,z)+\ln[1+\frac{1}{5}\widetilde{G}(x,y,z)]\Big]$$
$$\leqslant \frac{9}{10}\widetilde{G}(x,y,z)=\psi(\widetilde{G}(x,y,z))\leqslant \psi(M(x,y,z)),$$

So, from Theorem 2.2.3 f has a fixed point.

Example 2.4.3. Let $\widetilde{G}: X\times X\times X\to\mathbb{R}^+$ be defined on $X=[0,1.5]$ by

$$\widetilde{G}(x,y,z)=e^{\frac{|x-y|+|y-z|+|z-x|}{3}}-1$$

for all $x,y,z\in X$. Then $(X,\widetilde{G})$ is a $\widetilde{G}$-complete $\widetilde{G}$-metric space with $\Omega(t)=e^t-1$.
Define k and $\theta\in\Theta$ by $k=\frac{1}{\sqrt{2}}$ and $\theta(t)=e^{te^t}$. Let X is endowed with the usual order. Let $f: X\to X$ be defined by $fx=\arctan(\frac{x}{16})$. It is easy to see that f is an ordered increasing and continuous self map on X and $0\leqslant f0$. For any $x,y,z\in X$, we have

$$\begin{aligned}\widetilde{G}(fx,fy,fz)&=e^{\frac{|fx-fy|+|fy-fz|+|fz-fx|}{3}}-1\\&=e^{\frac{\left|\arctan\frac{x}{16}-\arctan\frac{y}{16}\right|+\left|\arctan\frac{y}{16}-\arctan\frac{z}{16}\right|+\left|\arctan\frac{z}{16}-\arctan\frac{x}{16}\right|}{3}}-1\\&\leqslant e^{\frac{\left|\frac{x}{16}-\frac{y}{16}\right|+\left|\frac{y}{16}-\frac{z}{16}\right|+\left|\frac{z}{16}-\frac{x}{16}\right|}{3}}-1\\&\leqslant\frac{1}{16}\big(e^{\frac{|x-y|+|y-z|+|z-x|}{3}}-1\big)\\&=\frac{1}{16}\widetilde{G}(fx,fy,fz).\end{aligned}$$

So,

$$\begin{aligned}\Omega[\widetilde{G}(fx,fy,fz)] &= e^{\widetilde{G}(fx,fy,fz)} - 1\\ &\leqslant e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)} - 1\\ &\leqslant \frac{1}{16}\widetilde{G}(fx,fy,fz).\end{aligned}$$

Therefore,

$$\begin{aligned}\theta(\Omega[\widetilde{G}(fx,fy,fz)]) &= e^{\Omega[\widetilde{G}(fx,fy,fz)]e^{\Omega[\widetilde{G}(fx,fy,fz)]}}\\ &\leqslant e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)}}\\ &\leqslant \left[e^{\widetilde{G}(fx,fy,fz)e^{\widetilde{G}(fx,fy,fz)}}\right]^{\frac{1}{\sqrt{2}}} = [\theta(\widetilde{G}(fx,fy,fz))]^{\frac{1}{\sqrt{2}}}.\end{aligned}$$

Thus, (18) is satisfied with $k = \frac{1}{\sqrt{2}}$. Hence, all the conditions of Theorem 2.3.4 are satisfied. We have that 0 is the unique fixed point of f.

2.5 Existence of a solution for an integral equation

We consider the following integral equation:

$$x(t) = \int_a^b K(t,s,x(s))ds + k(t), \tag{28}$$

where $b > a \geqslant 0$. The aim of this section is to present the existence of a solution to (28) that belongs to $X = C[a,b]$ (the set of all continuous real valued functions defined on $[a,b]$) as an application to the Theorem 2.3.4.

The considered problem can be changed as follows.
Let $f : X \to X$ be defined by:

$$fx(t) = \int_a^b (t,s,x(s))ds + k(t),$$

for all $x \in X$ and for all $t \in [a,b]$. Obviously, existence of a solution to (28) is equivalent to the existence of a fixed point of f.
Let,

$$d(u,v) = \max_{t\in[a,b]} |u(t) - v(t)| = ||u - v||_\infty.$$

Let X be equipped with the modified G-metric given by

$$\widetilde{G}(u,v,w) = \xi(\max\{d(u,v), d(v,w), d(w,u)\}),$$

for all $u, v, w \in X$ where $\xi : [0,\infty) \to [0,\infty)$ is a strictly increasing continuous function with $t \leqslant \xi(t)$ for $t \geqslant 0$ and $\xi(0) = 0$ which is a $\widetilde{G}$-complete $\widetilde{G}$-metric space. We endow X with the partial ordered $\preceq$ given by $x \preceq y \iff x(t) \leqslant y(t)$, for all $t \in [a,b]$. It is known that $(X,\preceq)$ has sequential limit comparison property [25, 1].

Now, we will prove the following result.

Theorem 2.5.1. *Suppose that the following hypotheses hold:*

(i) $K : [a,b] \times [a,b] \times R \to R$ *and* $k : [a,b] \to R$ *are continuous;*

(ii) *for all* $s,t \in [a,b]$ *and for all* $x,y \in X$ *with* $x \preceq y$ *we have,*

$$\xi^2\Big(\int_a^b \Big|K(t,r,x(r)) - K(t,r,y(r))\Big|\,dr\Big) \leqslant \frac{\xi\big(\|fx-fy\|_\infty\big)\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}}{\theta\big(\xi(\|fx-fy\|_\infty)\big)},$$

for all $t \in [a,b]$ and $\theta \in \Psi$.

(iii) *There exists continuous function* $\alpha : [a,b] \to \mathbb{R}$ *such that*

$$\alpha(t) \leqslant \int_a^b (t,s,\alpha(s))ds + k(t).$$

Then, the integral equations (28) *has a solution* $x \in X$*.*

Proof. Let $x,y \in X$ be such that $x \succeq y$. From condition (ii), for all $t \in [a,b]$ we have,

$$\begin{aligned}\xi^2\Big(|fx(t) - fy(t)|\Big) &\leqslant \xi^2\Big(\int_a^b |K(t,s,x(s)) - K(t,s,y(s)|ds\Big)\\ &\leqslant \frac{\xi^2\big(\|fx-fy\|_\infty\big)\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}}{\theta\big(\xi^2(\|fx-fy\|_\infty)\big)}.\end{aligned}$$

Hence,

$$\begin{aligned}\xi^2\Big(d(fx,fy)\Big) &= \xi^2\Big(\sup_{t\in[a,b]}|fx(t)-fy(t)|\Big)\\ &\leqslant \frac{\xi^2\Big(\|fx-fy\|_\infty\Big)\theta\Big(\xi(\|x-y\|_\infty)\Big)^{\frac{1}{2}}}{\theta\Big(\xi^2(\|fx-fy\|_\infty)\Big)}.\end{aligned} \tag{29}$$

Hence,

$$\theta\big(\xi^2(\|fx-fy\|_\infty)\big) \leqslant \theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}. \tag{30}$$

Therefore, from (29) we have,

$$\begin{aligned}\theta\Big(\xi\Big(\widetilde{G}(fx,fy,fz)\Big)\Big) &= \theta\Big(\xi\Big(\xi(\max\{d(fx,fy),d(fy,hz),d(fz,fx)\})\Big)\Big)\\ &\leqslant \max\Big\{\theta\Big(\xi^2(d(fx,fy))\Big),\theta\Big(\xi^2(d(fy,fz))\Big),\theta\Big(\xi^2(d(fz,fx))\Big)\Big\}\\ &\leqslant \max\Big\{\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}},\theta\big(\xi(\|y-z\|_\infty)\big)^{\frac{1}{2}},\theta\big(\xi(\|z-x\|_\infty)\big)^{\frac{1}{2}}\Big\}\\ &\leqslant \theta\big(M(x,y,z)\big)^{\frac{1}{2}},\end{aligned}$$

where

$$M(x,y,z) = \max\left\{\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,x,fx)\widetilde{G}(y,y,fy)}{1+\xi[\widetilde{G}(x,y,y)+\widetilde{G}(x,x,fy)]}, \frac{\widetilde{G}(y,y,fy)\widetilde{G}(z,z,fz)}{1+\xi[\widetilde{G}(y,z,z)+\widetilde{G}(y,y,fz)]}, \frac{\widetilde{G}(y,z,z)\widetilde{G}(y,y,z)}{1+\widetilde{G}(y,fy,fy)+\widetilde{G}(z,fz,fz)}\right\},$$

So, from Theorem 2.3.4, there exists $x \in X$, a fixed point of f which is a solution of (28). □

3. Conclusion

Taking $\Omega(x) = sx$, our obtained results coincide with the results in usual G_b-metric spaces and taking $\Omega(x) = x$, our obtained results coincide with the results in usual G-metric spaces.

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