SOME FIXED POINT THEOREMS VIA $\widetilde{G}$-RATIONAL CONTRACTIVE MAPPINGS IN ORDERED MODIFIED G-METRIC SPACES

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Abstract. In this paper, recalling the structure of modified G-metric spaces (as a generalization of both G-metric and G_b-metric spaces), we present the notions of $\widetilde{G}$-rational contractive mappings and investigate the existence of fixed point for such mappings. We also provide examples and an application to illustrate the results presented herein.

1. introduction and preliminaries

There is a large number of generalizations of Banach contraction principle via using different form of contractive conditions in generalized metric spaces. Some of such generalizations are obtained via contractive conditions expressed by rational terms (see, [4], [5], [7], [8], [15], [22] and [27]).

Ran and Reurings initiated the study of fixed point results on partially ordered sets in [26]. Also, many researchers have focused on different contractive conditions in complete metric spaces endowed with a partial order. For more details we refer the reader to [23, 24].

Parvaneh in [17] introduced the concept of an extended b-metric space.

Definition 1.1. [17] Let X be a (nonempty) set. A function $\widetilde{d} : X \times X \to R^+$ is a p-metric iff there exists a strictly increasing continuous function $\Omega : [0,\infty) \to [0,\infty)$ with $\Omega^{-1}(x) \le x \le \Omega(x)$ and $\Omega^{-1}(0) \le 0 \le \Omega(0)$ such that for all $x, y, z \in X$, the following conditions hold:

($\widetilde{d}_1$) $\widetilde{d}(x, y) = 0$ iff $x = y$,
($\widetilde{d}_2$) $\widetilde{d}(x, y) = \widetilde{d}(y, x)$,
($\widetilde{d}_3$) $\widetilde{d}(x, z) \le \Omega(\widetilde{d}(x, y) + \widetilde{d}(y, z))$.

In this case, the pair $(X, \widetilde{d})$ is called a p-metric space, or, an extended b-metric space.

A b-metric [9] is a p-metric, when $\Omega(x) = sx$ while a metric is a p-metric, when $\Omega(x) = x$.

We have the following proposition.

Proposition 1.2. *[17] Let (X, d) be a metric space and let $\widetilde{d}(x, y) = \xi(d(x, y))$ where $\xi : [0,\infty) \to [0,\infty)$ is a strictly increasing function with $x \le \xi(x)$ and $0 = \xi(0)$. In this case, $\widetilde{d}$ is a p-metric with $\Omega(t) = \xi(t)$.*

The above proposition constructs the following example:

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Example 1.3. Let (X, d) be a metric space and let $\widetilde{d}(x, y) = e^{d(x,y)} \sec^{-1}(e^{d(x,y)})$. Then $\widetilde{d}$ is a p-metric with $\Omega(t) = e^t \sec^{-1}(e^t)$.

The concept of a generalized metric space, or a G-metric space, was introduced by Mustafa and Sims. For more details in this field the reader can refer to [28, 29, 30]

Definition 1.4. [18] Let X be a nonempty set and G: $X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ iff $x = y = z$;
(G2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
(G3) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$ with $y \neq z$;
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all three variables);
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Aghajani *et al.* in [1] motivated by the concept of b-metric [9] introduced the concept of generalized b-metric spaces (G_{b}-metric spaces) and then they presented some basic properties of G_{b}-metric spaces.

The following is the definition of modified G-metric spaces which is a proper generalization of the notions of G-metric spaces and G_b-metric spaces [31].

Definition 1.5. Let X be a nonempty set and $\Omega : [0, \infty) \to [0, \infty)$ be a strictly increasing continuous function such that $\Omega^{-1}(x) \leq x \leq \Omega(x)$ for all $x > 0$ and $\Omega^{-1}(0) = 0 = \Omega(0)$. Suppose that a mapping $\widetilde{G} : X \times X \times X \to \mathbb{R}^+$ satisfies:

($\widetilde{G}1$) $\widetilde{G}(x, y, z) = 0$ if $x = y = z$,
($\widetilde{G}2$) $0 < \widetilde{G}(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
($\widetilde{G}3$) $\widetilde{G}(x, x, y) \leq \widetilde{G}(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
($\widetilde{G}4$) $\widetilde{G}(x, y, z) = \widetilde{G}(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),
($\widetilde{G}5$) $G(x, y, z) \leq \Omega[\widetilde{G}(x, a, a) + \widetilde{G}(a, y, z)]$ for all $x, y, z, a \in X$ (rectangle inequality).

Then $\widetilde{G}$ is called a modified G-metric and the pair $(X, \widetilde{G})$ is called a modified G-metric space or a $\widetilde{G}$-metric space.

Each G-metric space is a $\widetilde{G}$-metric space with $\Omega(t) = t$ and every G_b-metric space is a $\widetilde{G}$-metric space with $\Omega(t) = st$.

Proposition 1.6. *[31] Let (X, G) be a G_b-metric space with coefficient $s \geq 1$ and let $\widetilde{G}(x, y, z) = \xi(G(x, y, z))$ where $\xi : [0, \infty) \to [0, \infty)$ is a strictly increasing function with $x \leq \xi(x)$ for all $x > 0$ and $\xi(0) = 0$. We show that $\widetilde{G}$ is a modified G-metric with $\Omega(t) = \xi(st)$.*

For each $x, y, z, a \in X$,

$$\begin{aligned}
&\widetilde{G}(x, y, z) \\
&= \xi(G(x, y, z)) \leq \xi(sG(x, a, a) + sG(a, y, z)) \\
&\leq \xi(s\xi(G(x, a, a)) + s\xi(G(a, y, z)) \\
&= \xi(s\widetilde{G}(x, a, a) + s\widetilde{G}(a, y, z)) \\
&= \Omega(\widetilde{G}(x, a, a) + \widetilde{G}(a, y, z)).
\end{aligned}$$

So, $\widetilde{G}$ is a modified G-metric with $\Omega(t) = \xi(st)$.

The above proposition constructs the following examples:

Example 1.7. [31] Let (X,G) be a G_b-metric space with coefficient $s \geq 1$. Then,

1. $\widetilde{G}(x,y,z) = e^{G(x,y,z)} \sec^{-1}(e^{G(x,y,z)})$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st} \sec^{-1}(e^{st})$.
2. $\widetilde{G}(x,y,z) = [G(x,y,z)+1] \sec^{-1}([G(x,y,z)+1])$ is a $\widetilde{G}$-metric with $\Omega(t) = [st+1]\sec^{-1}([st+1])$.
3. $\widetilde{G}(x,y,z) = e^{G(x,y,z)} \tan^{-1}(e^{G(x,y,z)}-1)$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st}\tan^{-1}(e^{st}-1)$.
4. $\widetilde{G}(x,y,z) = G(x,y,z)\cosh(G(x,y,z))$ is a $\widetilde{G}$-metric with $\Omega(t) = st\cosh(st)$.
5. $\widetilde{G}(x,y,z) = e^{G(x,y,z)} \ln(1+G(x,y,z))$ is a $\widetilde{G}$-metric with $\Omega(t) = e^{st}\ln(1+st)$.
6. $\widetilde{G}(x,y,z) = G(x,y,z)+\ln(1+G(x,y,z))$ is a $\widetilde{G}$-metric with $\Omega(t) = st+\ln(1+st)$.

Definition 1.8. A $\widetilde{G}$-metric $\widetilde{G}$ is said to be symmetric if $\widetilde{G}(x,y,y) = \widetilde{G}(y,x,x)$, for all $x,y \in X$.

Proposition 1.9. *[31] Let X be a $\widetilde{G}$-metric space. Then for each $x,y,z,a \in X$ it follows that:*

(1) if $\widetilde{G}(x,y,z) = 0$ then $x = y = z$,
(2) $\widetilde{G}(x,y,z) \leq \Omega(\widetilde{G}(x,x,y)+\widetilde{G}(x,x,z))$,
(3) $\widetilde{G}(x,y,y) \leq \Omega[2\widetilde{G}(y,x,x)]$,
(4) $\widetilde{G}(x,y,z) \leq \Omega(\widetilde{G}(x,a,z)+\widetilde{G}(a,y,z))$.

Recall that a function f is super-additive if

$$f(x+y) \geq f(x)+f(y)$$

for all $x,y \in D(f)$.

Definition 1.10. Let X be a $\widetilde{G}$-metric space with a super-additive function Ω. We define $\widetilde{d}_{\widetilde{G}}(x,y) = \widetilde{G}(x,y,y)+\widetilde{G}(x,x,y)$, for all $x,y \in X$. It is easy to see that $\widetilde{d}_{\widetilde{G}}$ defines a p-metric $\widetilde{d}$ on X, which we call it the d-metric associated with $\widetilde{G}$.

Definition 1.11. Let X be a $\widetilde{G}$-metric space. A sequence $\{x_n\}$ in X is said to be:

(1) $\widetilde{G}$-Cauchy if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m,n,l \geq n_0, \widetilde{G}(x_n,x_m,x_l) < \varepsilon$;

(2) $\widetilde{G}$-convergent to a point $x \in X$ if, for each $\varepsilon > 0$ there exists a positive integer n_0 such that for all $m,n \geq n_0, \widetilde{G}(x_n,x_m,x) < \varepsilon$.

(3) A $\widetilde{G}$-metric space X is called $\widetilde{G}$-complete, if every $\widetilde{G}$-Cauchy sequence is $\widetilde{G}$-convergent in X.

Proposition 1.12. *Let X be a $\widetilde{G}$-metric space. Then the following are equivalent:*

(1) the sequence $\{x_n\}$ is $\widetilde{G}$-Cauchy.
(2) for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\widetilde{G}(x_n,x_m,x_m) < \varepsilon$ for all $m,n \geq n_0$.

Proposition 1.13. *Let X be a $\widetilde{G}$-metric space. The following are equivalent:*

(1) $\{x_n\}$ is $\widetilde{G}$-convergent to x.
(2) $\widetilde{G}(x_n,x_n,x) \to 0$, as $n \to \infty$.
(3) $\widetilde{G}(x_n,x,x) \to 0$, as $n \to \infty$.

In general, a G_b-metric function $G(x,y,z)$ for $s > 1$ and so a modified G-metric function $\widetilde{G}(x,y,z)$ with nontrivial function Ω is not jointly continuous in all its variables (see [19]).

We will apply the following simple lemma about the $\widetilde{G}$-convergent sequences.

Lemma 1.14. *[31] Let $(X, \widetilde{G})$ be a $\widetilde{G}$-metric space.*

1. Suppose that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are $\widetilde{G}$-convergent to x, y and z, respectively. Then we have

$$(\Omega^{-1})^3[\widetilde{G}(x,y,z)] \leq \liminf_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,z_n) \leq \limsup_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,z_n) \leq \Omega^3[\widetilde{G}(x,y,z)].$$

2. Suppose that $\{x_n\}$ and $\{y_n\}$ are $\widetilde{G}$-convergent to x and y, respectively. Then we have

$$(\Omega^{-1})^2[\widetilde{G}(x,y,\alpha)] \leq \liminf_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,\alpha) \leq \limsup_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,\alpha) \leq \Omega^2[\widetilde{G}(x,y,\alpha)].$$

3. If $\{x_n\}$ be $\widetilde{G}$-convergent to x, then

$$(\Omega^{-1})[\widetilde{G}(x,\alpha,\beta)] \leq \liminf_{n\longrightarrow\infty} \widetilde{G}(x_n,\alpha,\beta) \leq \limsup_{n\longrightarrow\infty} \widetilde{G}(x_n,\alpha,\beta) \leq \Omega[\widetilde{G}(x,\alpha,\beta)].$$

In particular, if $x = y = z$, then we have $\lim_{n\longrightarrow\infty} \widetilde{G}(x_n,y_n,z_n) = 0$.

Proof. 1. Using the rectangle inequality in a $\widetilde{G}$-metric space it is easy to see that,

$$\widetilde{G}(x,y,z) \leq \Omega\Big[\widetilde{G}(x,x_n,x_n) + \Omega\big[\widetilde{G}(y,y_n,y_n) + \Omega[\widetilde{G}(z,z_n,z_n) + \widetilde{G}(x_n,y_n,z_n)]\big]\Big]$$

and

$$\widetilde{G}(x_n,y_n,z_n) \leq \Omega\Big[\widetilde{G}(x_n,x,x) + \Omega\big[G(y_n,y,y) + \Omega[G(z_n,z,z) + G(x,y,z)]\big]\Big].$$

Taking the lower limit as $n \to \infty$ in the first inequality and the upper limit as $n \to \infty$ in the second inequality we obtain the desired result.

2. Using the rectangle inequality we see that,

$$\widetilde{G}(x,y,\alpha) \leq \Omega\Big[\widetilde{G}(x,x_n,x_n) + \Omega\big[\widetilde{G}(y,y_n,y_n) + \widetilde{G}(x_n,y_n,\alpha)\big]\Big]$$

and

$$\widetilde{G}(x_n,y_n,\alpha) \leq \Omega\Big[\widetilde{G}(x_n,x,x) + \Omega\big[G(y_n,y,y) + G(x,y,\alpha)\big]\Big].$$

3. Similarly,

$$\widetilde{G}(x,\alpha,\beta) \leq \Omega\Big[\widetilde{G}(x,x_n,x_n) + \widetilde{G}(x_n,\alpha,\beta)\Big]$$

and

$$\widetilde{G}(x_n,\alpha,\beta) \leq \Omega\Big[\widetilde{G}(x_n,x,x) + G(x,\alpha,\beta)\Big].$$

□

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \to 1 \text{ implies that } \; t_n \to 0, \; \textit{as } n \to \infty.$$

.

In order to generalize the Banach contraction principle, in 1973, Geraghty proved the following.

Theorem 1.15. *[11] Let (X, d) be a complete metric space, and let $f : X \to X$ be a self-map. Suppose that there exists $\beta \in \mathfrak{S}$ such that*

$$d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

holds for all $x, y \in X$. Then f has a unique fixed point $z \in X$ and for each $x \in X$ the Picard sequence $\{f^n x\}$ converges to z.

In 2010, Amini-Harandi and Emami [6] characterized the result of Geraghty in the setting of a partially ordered complete metric space.

In [10], some fixed point theorems for mappings satisfying Geraghty-type contractive conditions are proved in various generalized metric spaces. .

Also, Zabihi and Razani [27] and Shahkoohi and Razani [4] obtained some fixed point results duo to rational Geraghty contractions in b-metric spaces.

Motivated by [5], in this paper we present some fixed point theorems for different rational contractive mappings in partially ordered modified G-metric spaces. Our results extend some existing results in the literature.

2. Main results

2.1. **Fixed point results using $\widetilde{G}$-rational Geraghty contractions.** Let $(X, \widetilde{G})$ be a $\widetilde{G}$-metric space with function Ω and let $\mathcal{F}_\Omega$ denotes the class of all functions $\beta : [0, \infty) \to [0, \Omega^{-1}(1))$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \Omega^{-1}(1) \text{ implies that } \ t_n \to 0, \ as \ n \to \infty.$$

An example of a function in $\mathcal{F}_\Omega$ may be given by $\beta(t) = (\ln 2)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, \ln 2)$ where $\widetilde{G}(x, y, z) = e^{\max(|x-y|,|y-z|,|z-x|)} - 1$ for all $x, y, z \in \mathbb{R}$.

Another example of a function in $\mathcal{F}_\Omega$ may be given by $\beta(t) = W(1)e^{-t}$ for $t > 0$ and $\beta(0) \in [0, W(1))$ where $\widetilde{G}(x, y, z) = \max(|x - y|, |y - z|, |z - x|)e^{\max(|x-y|,|y-z|,|z-x|)}$ for all $x, y, z \in \mathbb{R}$. Note that W is the Lambert W-function (see, e.g., [3])

Definition 2.1. Let $(X, \widetilde{G})$ be an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational Geraghty contraction if, there exists $\beta \in \mathcal{F}_\Omega$ such that,

$$\Omega(\widetilde{G}(fx, fy, fz)) \leq \beta(M(x, y, z))M(x, y, z) \tag{2.1}$$

for all comparable elements $x, y, z \in X$, where

$$M(x, y, z) = \max\Big\{\widetilde{G}(x, y, z), \frac{\widetilde{G}(x, y, fy)[\widetilde{G}(y, z, fz)]^2}{1 + \widetilde{G}(x, fx, f^2x)\widetilde{G}(y, fy, f^2y)}, \\ \frac{\widetilde{G}(x, fx, f^2x)\widetilde{G}(y, fy, f^2y)\widetilde{G}(z, fz, f^2z)}{1 + \widetilde{G}(fx, f^2x, f^3x)\widetilde{G}(fy, f^2y, f^3y)}\Big\}.$$

Recall that a modified G-metric space $(X, \widetilde{G})$ it said to has the s.l.c. property, if whenever $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to u \in X$, one has $x_n \preceq u$ for all $n \in \mathbb{N}$

Theorem 2.2. *Let $(X, \preceq, \widetilde{G})$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational Geraghty contraction. If,*
(I) f is continuous, or,
(II) $(X, \preceq, \widetilde{G})$ has the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$. Since $x_0 \preceq f(x_0)$ and f is increasing, we obtain by induction that

$$x_0 \preceq f(x_0) \preceq f^2(x_0) \preceq ... \preceq f^n(x_0) \preceq f^{n+1}(x_0) \preceq \cdots .$$

We will do the proof in the following steps.

Step 1. We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 2.1 we have

$$\widetilde{G}(x_n, x_{n+1}, x_{n+2}) = \widetilde{G}(fx_{n-1}, fx_n, fx_{n+1}) \leq \beta(M(x_{n-1}, x_n, x_{n+1}))M(x_{n-1}, x_n, x_{n+1}), \tag{2.2}$$

where

$$\begin{aligned}
&M(x_{n-1}, x_n, x_{n+1})\\
&= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \frac{\widetilde{G}(x_{n-1}, x_n, fx_n)\widetilde{G}(x_n, x_{n+1}, fx_{n+1})^2}{1 + \widetilde{G}(x_{n-1}, fx_{n-1}, f^2x_{n-1})\widetilde{G}(x_n, fx_n, f^2x_n)},\\
&\frac{\widetilde{G}(x_{n-1}, fx_{n-1}, f^2x_{n-1})\widetilde{G}(x_n, fx_n, f^2x_n)\widetilde{G}(x_{n+1}, fx_{n+1}, f^2x_{n+1})}{1 + \widetilde{G}(fx_{n-1}, f^2x_{n-1}, f^3x_{n-1})\widetilde{G}(fx_n, f^2x_n, f^3x_n)}\}\\
&= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \frac{\widetilde{G}(x_{n-1}, x_n, x_{n+1})\widetilde{G}(x_n, x_{n+1}, x_{n+2})^2}{1 + \widetilde{G}(x_{n-1}, x_n, x_{n+1})\widetilde{G}(x_n, x_{n+1}, x_{n+2})},\\
&\frac{\widetilde{G}(x_{n-1}, x_n, x_{n+1})\widetilde{G}(x_n, x_{n+1}, x_{n+2})\widetilde{G}(x_{n+1}, x_{n+2}, x_{n+3})}{1 + \widetilde{G}(x_n, x_{n+1}, x_{n+2})\widetilde{G}(x_{n+1}, x_{n+2}, x_{n+3})}\}\\
&\leq \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\}.
\end{aligned}$$

If $\max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\} = \widetilde{G}(x_n, x_{n+1}, x_{n+2})$, then from 2.2 we have,

$$\begin{aligned}
\widetilde{G}(x_n, x_{n+1}, x_{n+2}) &\leq \beta(M(x_{n-1}, x_n, x_{n+1}))\widetilde{G}(x_n, x_{n+1}, x_{n+2})\\
&< \Omega^{-1}(1)\widetilde{G}(x_n, x_{n+1}, x_{n+2})\\
&\leq \widetilde{G}(x_n, x_{n+1}, x_{n+2}),
\end{aligned} \tag{2.3}$$

which is a contradiction.

Hence, $\max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\} = \widetilde{G}(x_{n-1}, x_n, x_{n+1})$. So, from 2.2,

$$\widetilde{G}(x_n, x_{n+1}, x_{n+2}) \leq \beta(M(x_{n-1}, x_n, x_{n+1}))\widetilde{G}(x_{n-1}, x_n, x_{n+1}) < \widetilde{G}(x_{n-1}, x_n, x_{n+1}). \tag{2.4}$$

That is, $\{\widetilde{G}(x_n, x_{n+1}, x_{n+2})\}$ is a decreasing sequence, then there exists $r \geq 0$ such that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) = r$. We will prove that $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$, from 2.4 we have

$$r \leq \lim_{n\to\infty} \beta(M(x_{n-1}, x_n, x_{n+1}))r \leq \Omega^{-1}(1)r,$$

which implies that $\Omega^{-1}(1) \leq 1 \leq \lim_{n\to\infty} \beta(M(x_{n-1}, x_n, x_{n+1})) \leq \Omega^{-1}(1)$. Now, as $\beta \in \mathcal{F}_\Omega$ we conclude that $M(x_{n-1}, x_n, x_{n+1}) \to 0$ which yields that $r = 0$, a contradiction. Hence, the assumption that $r > 0$ is false. That is,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) = 0. \tag{2.5}$$

Consequently,

$$\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.6}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a $\widetilde{G}$-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \geq \varepsilon. \tag{2.7}$$

This means that

$$\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon. \tag{2.8}$$

From the rectangular inequality, we get

$$\varepsilon \leq \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \leq \Omega[\widetilde{G}(x_{m_i}, x_{m_i+1}, x_{m_i+1}) + \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})].$$

Taking the upper limit as $i \to \infty$ and by 2.6, we get

$$\Omega^{-1}(\varepsilon) \leq \limsup_{i\to\infty} \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}). \tag{2.9}$$

From the definition of $M(x, y, z)$ and the above limits,

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) = \limsup_{i\to\infty} \max\{&\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}),\\ &\frac{\widetilde{G}(x_{m_i}, x_{n_i-1}, fx_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})^2}{1 + \widetilde{G}(x_{m_i}, fx_{m_i}, f^2x_{m_i})\widetilde{G}(x_{n_i-1}, fx_{n_i-1}, f^2x_{n_i-1})},\\ &\frac{\widetilde{G}(x_{m_i}, fx_{m_i}, f^2x_{m_i})\widetilde{G}(x_{n_i-1}, fx_{n_i-1}, f^2x_{n_i-1})^2}{1 + \widetilde{G}(fx_{m_i}, f^2x_{m_i}, f^3x_{m_i})\widetilde{G}(fx_{n_i-1}, f^2x_{n_i-1}, f^3x_{n_i-1})}\}\\ &\leq \varepsilon.\end{aligned}$$

Now, from 2.1 and the above inequalities, we have

$$\begin{aligned}\varepsilon &\leq \limsup_{i\to\infty} \Omega(\widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}))\\ &\leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1})) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1})\\ &\leq \varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1}))\end{aligned}$$

which implies that $\Omega^{-1}(1) \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, x_{n_i-1}))$. Now, as $\beta \in \mathcal{F}_\Omega$ we conclude that $M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$ which yields that $\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) \to 0$. Consequently,

$$\widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \leq \Omega[\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) + s\widetilde{G}(x_{n_i-1}, x_{n_i}, x_{n_i})] \to 0,$$

a contradiction to 2.7. Therefore, $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. $\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f.

First, let f is continuous, so, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) holds. Using the assumption on X we have $x_n \preceq u$. Now, by Lemma 1.14,

$$\begin{aligned}(\Omega^{-1})^2[\widetilde{G}(u, u, fu)] &\leq \limsup_{n\to\infty} \widetilde{G}(x_{n+1}, x_{n+1}, fu)\\ &\leq \limsup_{n\to\infty} \beta(M(x_n, x_n, u)) \limsup_{n\to\infty} M(x_n, x_n, u),\end{aligned}$$

where,

$$\lim_{n\to\infty} M(x_n,x_n,u) = \lim_{n\to\infty} \max\{\widetilde{G}(x_n,x_n,u), \frac{\widetilde{G}(x_n,x_n,fx_n)\widetilde{G}(x_n,u,fu)^2}{1+\widetilde{G}(x_n,fx_n,f^2x_n)^2},$$
$$\frac{\widetilde{G}(x_n,fx_n,f^2x_n)^2\widetilde{G}(u,fu,f^2u)}{1+\widetilde{G}(fx_n,f^2x_n,f^3x_n)^2}\}$$
$$= 0.$$

Therefor, we deduce that $\widetilde{G}(u,u,fu) = 0$, so, $u = fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume on contrary that, u and v are two fixed points of f such that $u \neq v$. Then by 2.1, we have

$$\widetilde{G}(u,v,v) = \widetilde{G}(fu,fv,fv) \leq \beta(M(u,v,v))M(u,v,v) = \beta(\widetilde{G}(u,v,v))\widetilde{G}(u,v,v) < \Omega^{-1}(1)\widetilde{G}(u,v,v). \tag{2.10}$$

Because

$$M(u,v,v) = \widetilde{G}(u,v,v).$$

So, we get, $G(u,v,v) < \Omega^{-1}(1)G(u,v,v)$, a contradiction. Hence, $u = v$, and f has a unique fixed point. Conversely, if f has a unique fixed point, then the set of fixed points of f is well ordered. □

2.2. **Fixed point results via comparison functions.** Let Ψ be the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2.3. *If $\psi \in \Psi$, then the following are satisfied.*
(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Definition 2.4. Let $(X, \preceq, \widetilde{G})$ is an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational ψ-contraction if, there exists $\psi \in \Psi$ such that,

$$\Omega(\widetilde{G}(fx,fy,fz)) \leq \psi(M(x,y,z)) \tag{2.11}$$

for all comparable elements $x,y,z \in X$, where

$$M(x,y,z)$$
$$= \max\left\{\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,x,fx)\widetilde{G}(x,x,fy)}{1+\Omega[\widetilde{G}(x,x,fx)+\widetilde{G}(y,y,fy)]},\right.$$
$$\left.\frac{\widetilde{G}(y,y,z)\widetilde{G}(y,y,fz)}{1+\Omega[\widetilde{G}(y,y,fy)+\widetilde{G}(z,z,fz)]}, \frac{\widetilde{G}(x,x,fx)\widetilde{G}(x,x,z)}{1+\widetilde{G}(x,x,fy)+\widetilde{G}(y,y,fx)}\right\}.$$

Theorem 2.5. *Let $(X, \preceq, \widetilde{G})$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational ψ-contractive mapping. If*
(I) f is continuous, or,
(II) $(X, \preceq, \widetilde{G})$ has the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$.

Step I: We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 2.11 we have

$$\begin{aligned} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) &= \widetilde{G}(fx_{n-1}, fx_n, fx_{n+1}) \\ &\leq \psi(M(x_{n-1}, x_n, x_{n+1})) \\ &\leq \psi(\widetilde{G}(x_{n-1}, x_n, x_{n+1})) \\ &< \widetilde{G}(x_{n-1}, x_n, x_{n+1}), \end{aligned} \tag{2.12}$$

because

$$\begin{aligned} &M(x_{n-1}, x_n, x_{n+1}) \\ &= \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \frac{\widetilde{G}(x_{n-1}, x_{n-1}, x_n)\widetilde{G}(x_{n-1}, x_{n-1}, fx_n)}{1 + \Omega[\widetilde{G}(x_{n-1}, x_{n-1}, fx_{n-1}) + \widetilde{G}(x_n, x_n, fx_n)]}, \\ &\frac{\widetilde{G}(x_n, x_n, x_{n+1})\widetilde{G}(x_n, x_n, fx_{n+1})}{1 + \Omega[\widetilde{G}(x_n, x_n, fx_n) + \widetilde{G}(x_{n+1}, x_{n+1}, fx_{n+1})]}, \frac{\widetilde{G}(x_{n-1}, x_{n-1}, fx_{n-1})\widetilde{G}(x_{n-1}, x_{n-1}, x_{n+1})}{1 + \widetilde{G}(x_{n-1}, x_{n-1}, fx_n) + \widetilde{G}(x_n, x_n, fx_{n-1})}\} \\ &\leq \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_{n-1}, x_{n-1}, x_n), \widetilde{G}(x_n, x_n, x_{n+1})\} \\ &\leq \max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\}, \end{aligned}$$

and it is easy to see that $\max\{\widetilde{G}(x_{n-1}, x_n, x_{n+1}), \widetilde{G}(x_n, x_{n+1}, x_{n+2})\} = \widetilde{G}(x_{n-1}, x_n, x_{n+1})$, so from 2.12, we conclude that $\{\widetilde{G}(x_n, x_{n+1}, x_{n+2})\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+2}) = r$.

It is easy to see that $r = \lim_{n\to\infty} \widetilde{G}(x_{n-1}, x_n, x_n) = 0$.

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } \widetilde{G}(x_{m_i}, x_{n_i}, x_{n_i}) \geq \varepsilon. \tag{2.13}$$

This means that

$$\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) < \varepsilon. \tag{2.14}$$

As in the proof of Theorem 2.2, we have,

$$\limsup_{i\to\infty} \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}). \tag{2.15}$$

From the definition of $M(x, y, z)$ and the above limits,

$$\begin{aligned} \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) &= \limsup_{i\to\infty} \max\{\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}), \\ &\frac{\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})\widetilde{G}(x_{m_i}, x_{m_i}, fx_{n_i-1})}{1 + \Omega[\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i}) + \widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]}, \\ &\frac{\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1 + \Omega[\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1}) + \widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]}, \\ &\frac{\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})\widetilde{G}(x_{m_i}, x_{m_i}, x_{n_i-1})}{1 + \widetilde{G}(x_{m_i}, x_{m_i}, fx_{n_i-1}) + \widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{m_i})}\} \\ &\leq \varepsilon. \end{aligned}$$

Now, from 2.11 and the above inequalities, we have

$$\varepsilon \leq \limsup_{i\to\infty} \Omega[\widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})] \leq \limsup_{i\to\infty} \psi(M(x_{m_i}, x_{n_i-1}, x_{n_i-1})) < \varepsilon$$

which is a contradiction. Now, we conclude that $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. $\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f. This step is proved as the proof of step 3 of Theorem 2.2 with some elementary changes. □

If in the above theorem we take $\psi(t) = \sinh t$ and $\widetilde{G}(x, y, z) = \sinh(G(x, y, z))$ then we have the following corollary in the framework of G_b metric spaces.

Corollary 2.6. *Let $(X, G_b, \preceq)$ be an ordered G_b-complete G_b-metric space with coefficient $s > 1$. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$\sinh(s \cdot \sinh(G(fx, fy, fz))) \leq \sinh(M(x, y, z)) \tag{2.16}$$

for all comparable elements $x, y, z \in X$, where

$$M(x, y, z) = \max\left\{\sinh(G(x, y, z)), \frac{\sinh(G(x, x, fx))\sinh(G(y, y, fy))}{1 + \sinh(s \cdot [\sinh(G(x, y, y)) + \sinh(G(x, x, fy))])}, \frac{\sinh(G(y, y, fy))\sinh(G(z, z, fz))}{1 + \sinh(s \cdot [\sinh(G(y, z, z)) + \sinh(G(y, y, fz))])}, \frac{\sinh(G(y, z, z))\sinh(G(y, y, z))}{1 + \sinh(G(y, fy, fy)) + \sinh(G(z, fz, fz))}\right\}.$$

If
(I) f is continuous, or,
(II) $(X, G_b, \preceq)$ enjoys the s.l.c. property,
then f has a fixed point.

2.3. **Fixed point results related to JS-contractions.** Jleli et al. [2] have introduced the class Θ_0 consists of all functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(θ_1) θ is non-decreasing;
(θ_2) for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(θ_3) there exist $r \in (0, 1)$ and $\ell \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
(θ_4) θ is continuous.

They proved the following result:

Theorem 2.7. [2, Corollary 2.1] *Let (X, d) be a complete metric space and let $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta_0$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \implies \theta\big(d(Tx, Ty)\big) \leq \theta\big(d(x, y)\big)^k. \tag{2.17}$$

Then T has a unique fixed point.

From now on, we denote by Θ the set of all functions $\theta : [0, \infty) \to [1, \infty)$ satisfying the following conditions:

θ_1. θ is a continuous strictly increasing function;
θ_2. for each sequence $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;

Remark 2.8. [12] It is clear that $f(t) = e^t$ does not belong to Θ_0, but it belongs to Θ. Other examples are $f(t) = \cosh t$, $f(t) = \frac{2\cosh t}{1+\cosh t}$, $f(t) = 1+\ln(1+t)$, $f(t) = \frac{2+2\ln(1+t)}{2+\ln(1+t)}$, $f(t) = e^{te^t}$ and $f(t) = \frac{2e^{te^t}}{1+e^{te^t}}$, for all $t > 0$.

Definition 2.9. Let $(X, \widetilde{G}, \preceq)$ be an ordered $\widetilde{G}$-metric space. A mapping $f : X \to X$ is called a $\widetilde{G}$-rational JS-contraction if

$$\theta(\Omega[\widetilde{G}(fx, fy, fz)]) \leq \theta(M(x, y, z))^k \tag{2.18}$$

for all comparable elements $x, y, z \in X$, where $\theta \in \Theta$, $k \in [0, 1)$ and

$$\begin{aligned}
&M(x,y,z)\\
&= \max\Big\{\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,x,fx)\widetilde{G}(y,y,fy)}{1+\Omega[\widetilde{G}(x,y,y)+\widetilde{G}(x,x,fy)]},\\
&\frac{\widetilde{G}(y,y,fy)\widetilde{G}(z,z,fz)}{1+\Omega[\widetilde{G}(y,z,z)+\widetilde{G}(y,y,fz)]}, \frac{\widetilde{G}(y,z,z)\widetilde{G}(y,y,z)}{1+\widetilde{G}(y,fy,fy)+\widetilde{G}(z,fz,fz)}\Big\}.
\end{aligned}$$

Theorem 2.10. *Let $(X, \widetilde{G}, \preceq)$ be an ordered $\widetilde{G}$-complete $\widetilde{G}$-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that f be a $\widetilde{G}$-rational JS-contractive mapping. If*
(I) f is continuous, or,
(II) $(X, \widetilde{G}, \preceq)$ enjoys the s.l.c. property,
then f has a fixed point. Moreover, the set of fixed points of f is well ordered if and only if f has one and only one fixed point.

Proof. Put $x_n = f^n(x_0)$.

Step 1. We will show that $\lim_{n\to\infty} \widetilde{G}(x_n, x_{n+1}, x_{n+1}) = 0$. Without any loss of generality, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Since $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N}$, then by 2.28 we have

$$\begin{aligned}
\theta(\widetilde{G}(x_n, x_{n+1}, x_{n+2})) &\leq \theta(\Omega[\widetilde{G}(x_n, x_{n+1}, x_{n+2})])\\
&= \theta(\Omega[\widetilde{G}(fx_{n-1}, fx_n, fx_{n+1})])\\
&\leq \theta(M(x_{n-1}, x_n, x_{n+1}))^k\\
&\leq \theta(\widetilde{G}(x_{n-1}, x_n, x_{n+1}))^k,
\end{aligned} \tag{2.19}$$

because

$$
\begin{aligned}
&M(x_{n-1},x_n,x_{n+1})\\
&=\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\frac{\widetilde{G}(x_{n-1},x_{n-1},fx_{n-1})\widetilde{G}(x_n,x_n,fx_n)}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},fx_n)]},\\
&\frac{\widetilde{G}(x_n,x_n,fx_n)\widetilde{G}(x_{n+1},x_{n+1},fx_{n+1})}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,fx_{n+1})]},\frac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,fx_n,fx_n)+\widetilde{G}(x_{n+1},fx_{n+1},fx_{n+1})}\}\\
&=\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\frac{\widetilde{G}(x_{n-1},x_{n-1},x_n)\widetilde{G}(x_n,x_n,x_{n+1})}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]},\\
&\frac{\widetilde{G}(x_n,x_n,x_{n+1})\widetilde{G}(x_{n+1},x_{n+1},x_{n+2})}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,x_{n+2})]},\frac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_{n+1},x_{n+2},x_{n+2})}\}\\
&\le\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\frac{\widetilde{G}(x_{n-1},x_{n-1},x_n)\Omega[\widetilde{G}(x_n,x_n,x_{n-1})+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]}{1+\Omega[\widetilde{G}(x_{n-1},x_n,x_n)+\widetilde{G}(x_{n-1},x_{n-1},x_{n+1})]},\\
&\frac{\widetilde{G}(x_n,x_n,x_{n+1})\Omega[\widetilde{G}(x_{n+1},x_{n+1},x_n)+\widetilde{G}(x_n,x_n,x_{n+2})]}{1+\Omega[\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_n,x_n,x_{n+2})]},\frac{\widetilde{G}(x_n,x_{n+1},x_{n+1})\widetilde{G}(x_n,x_n,x_{n+1})}{1+\widetilde{G}(x_n,x_{n+1},x_{n+1})+\widetilde{G}(x_{n+1},x_{n+2},x_{n+2})}\}\\
&\le\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\widetilde{G}(x_{n-1},x_{n-1},x_n),\widetilde{G}(x_n,x_n,x_{n+1})\}\\
&\le\max\{\widetilde{G}(x_{n-1},x_n,x_{n+1}),\widetilde{G}(x_n,x_{n+1},x_{n+2})\}.
\end{aligned}
$$

From (2.19) we deduce that,

$$\Theta\big(\widetilde{G}(x_n,x_{n+1},x_{n+2})\big)\le\Theta\big(\widetilde{G}(x_{n-1},x_n,x_{n+1})\big)^k.$$

Therefore,

$$1\le\Theta\big(\widetilde{G}(x_n,x_{n+1},x_{n+2})\big)\le\Theta\big(\widetilde{G}(x_{n-1},x_n,x_{n+1})\big)^k\le\ldots\le\Theta\big(\widetilde{G}(x_0,x_1,x_2)\big)^{k^n}. \tag{2.20}$$

Taking the limit as $n\to\infty$ in (2.20) we have,

$$\lim_{n\to\infty}\Theta\big(\widetilde{G}(x_n,x_{n+1},x_{n+2})\big)=1$$

and since $\Theta\in\Delta_\Theta$ we obtain,

$$\lim_{n\to\infty}\widetilde{G}(x_n,x_{n+1},x_{n+2})=0. \tag{2.21}$$

Therefore, we have,

$$\lim_{n\to\infty}\widetilde{G}(x_n,x_n,x_{n-1})=0. \tag{2.22}$$

Step 2. Now, we prove that the sequence $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. Suppose the contrary, *i.e.*, that $\{x_n\}$ is not a $\widetilde{G}$-Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \text{ and } \widetilde{G}(x_{m_i},x_{n_i},x_{n_i})\ge\varepsilon. \tag{2.23}$$

This means that

$$\widetilde{G}(x_{m_i},x_{n_i-1},x_{n_i-1})<\varepsilon. \tag{2.24}$$

Hence,

$$\widetilde{G}(x_{m_i},x_{m_i},x_{n_i-1})<\Omega(2\varepsilon). \tag{2.25}$$

From the rectangular inequality, we get

$$\varepsilon\le\widetilde{G}(x_{m_i},x_{n_i},x_{n_i})\le\Omega[\widetilde{G}(x_{m_i},x_{m_i+1},x_{m_i+1})+\widetilde{G}(x_{m_i+1},x_{n_i},x_{n_i})].$$

By taking the upper limit as $i \to \infty$, we get

$$\Omega^{-1}(\varepsilon) \leq \limsup_{i\to\infty} \widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i}). \tag{2.26}$$

From the definition of $M(x, y, z)$ and the above limits,

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, x_{n_i-1}) = \limsup_{i\to\infty} \max\{&\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}),\\ &\frac{\widetilde{G}(x_{m_i}, x_{m_i}, fx_{m_i})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{m_i}, x_{n_i-1}, x_{n_i-1}) + \widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{m_i})]},\\ &\frac{\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})}{1+\Omega[\widetilde{G}(x_{n_i-1}, x_{n_i-1}, x_{n_i-1}) + \widetilde{G}(x_{n_i-1}, x_{n_i-1}, fx_{n_i-1})]},\\ &\frac{\widetilde{G}(x_{n_i-1}, x_{n_i-1}, x_{n_i-1})\widetilde{G}(x_{n_i-1}, x_{n_i-1}, x_{n_i-1})}{1+\widetilde{G}(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1}) + \widetilde{G}(x_{n_i-1}, fx_{n_i-1}, fx_{n_i-1})}\}\\ &\leq \varepsilon.\end{aligned}$$

Now, from 2.28 and the above inequalities, we have

$$\begin{aligned}\theta(\Omega[\Omega^{-1}(\varepsilon)]) &\leq \limsup_{i\to\infty} \theta(\Omega[\widetilde{G}(x_{m_i+1}, x_{n_i}, x_{n_i})])\\ &\leq \limsup_{i\to\infty} \theta(M(x_{m_i}, x_{n_i-1}, x_{n_i}))^k\\ &\leq \theta(\varepsilon)^k\end{aligned}$$

which implies that $\varepsilon = 0$, a contradiction. So, we conclude that $\{x_n\}$ is a $\widetilde{G}$-Cauchy sequence. $\widetilde{G}$-Completeness of X yields that $\{x_n\}$ $\widetilde{G}$-converges to a point $u \in X$.

Step 3. u is a fixed point of f.

When f is continuous, the proof is straightforward.

Now, let (II) holds. Using the assumption on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma 1.14,

$$\begin{aligned}\theta((\Omega^{-1})^2[\widetilde{G}(u, u, fu)]) &\leq \limsup_{n\to\infty} \theta(\widetilde{G}(x_{n+1}, x_{n+1}, fu))\\ &\leq \limsup_{n\to\infty} \theta(M(x_n, x_n, u))^k,\end{aligned}$$

where,

$$\begin{aligned}&\lim_{n\to\infty} M(x_n, x_n, u)\\ &= \lim_{n\to\infty} \max\left\{\widetilde{G}(x_n, x_n, u), \frac{\widetilde{G}(x_n, x_n, fx_n)\widetilde{G}(x_n, x_n, fx_n)}{1+\Omega[\widetilde{G}(x_n, x_n, x_n) + \widetilde{G}(x_n, x_n, fx_n)]},\right.\\ &\left.\frac{\widetilde{G}(x_n, x_n, fx_n)\widetilde{G}(u, u, fu)}{1+\Omega[\widetilde{G}(x_n, u, u) + \widetilde{G}(x_n, x_n, fu)]}, \frac{\widetilde{G}(x_n, u, u)\widetilde{G}(x_n, x_n, u)}{1+\widetilde{G}(x_n, fx_n, fx_n) + \widetilde{G}(u, fu, fu)}\right\} = 0.\end{aligned}$$

Therefor, we deduce that $\widetilde{G}(u, u, fu) = 0$, so, $u = fu$.

Finally, suppose that the set of fixed point of f is well ordered. Assume on contrary that, u and v are two fixed points of f such that $u \neq v$. Then by 2.28, we have

$$\theta[\widetilde{G}(u, v, v)] = \theta[\widetilde{G}(fu, fv, fv)] \leq \theta(M(u, v, v))^k = \theta(\widetilde{G}(u, v, v))^k. \tag{2.27}$$

So, we get, $G(u, v, v) = 0$, a contradiction. Hence $u = v$, and f has a unique fixed point. □

If in the above theorem we take $\theta(t)=\frac{2e^{te^t}}{1+e^{te^t}}$ and $\widetilde{G}(x,y,z)=e^{G(x,y,z)}-1$ then we have the following corollary in the framework of G_b metric spaces.

Corollary 2.11. *Let $(X,G_b,\preceq)$ be an ordered G_b-complete G_b-metric space with coefficient $s>1$. Let $f:X\to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in X$ with $x_0\preceq f(x_0)$. Suppose that*

$$\frac{2e^{[e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1]e^{e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1}}}{1+e^{[e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1]e^{e^{s\cdot[e^{G(fx,fy,fz)}-1]}-1}}}\leq\sqrt{\frac{2e^{M(x,y,z)e^{M(x,y,z)}}}{1+e^{M(x,y,z)e^{M(x,y,z)}}}}\tag{2.28}$$

for all comparable elements $x,y,z\in X$, where

$$\begin{aligned}&M(x,y,z)\\&=\max\left\{e^{G(x,y,z)}-1,\frac{[e^{G(x,x,fx)}-1][e^{G(y,y,fy)}-1]}{1+e^{s\cdot[e^{G(x,y,y)}-1+e^{G(x,x,fy)}-1]}-1},\right.\\&\left.\frac{[e^{G(y,y,fy)}-1][e^{G(z,z,fz)}-1]}{1+e^{s\cdot[e^{G(y,z,z)}-1+e^{G(y,y,fz)}-1]}-1},\frac{[e^{G(y,z,z)}-1][e^{G(y,y,z)}-1]}{1+e^{G(y,fy,fy)}-1+e^{G(z,fz,fz)}-1}\right\}.\end{aligned}$$

If
(I) f is continuous, or,
(II) $(X,G_b,\preceq)$ enjoys the s.l.c. property,
then f has a fixed point.

3. Examples

Example 3.1. Let $X=[0,8]$ be equipped with the $\widetilde{G}$-metric

$$\widetilde{G}(x,y,z)=\sinh(\frac{|x-y|+|y-z|+|z-x|}{3})$$

for all $x,y,z\in X$, where $\Omega(x)=\sinh x$ which $\Omega^{-1}(x)=\sinh^{-1}(x)$.

Define a relation $\preceq$ on X by $x\preceq y$ iff $y\leq x$, the function $f:[0,8]\to[0,2]$ by

$$fx=\sqrt{2+\frac{x}{4}}$$

and the function β given by $\beta(t)=\frac{1}{2}<0.88137358702=\Omega^{-1}(1)$.
For all comparable elements $x,y\in X$, we have,

$$\begin{aligned}\Omega(\widetilde{G}(fx,fy,fz))&=\sinh(\sinh(\frac{|\sqrt{2+\frac{x}{4}}-\sqrt{2+\frac{y}{4}}|+|\sqrt{2+\frac{y}{4}}-\sqrt{2+\frac{z}{4}}|+|\sqrt{2+\frac{z}{4}}-\sqrt{2+\frac{x}{4}}|}{3}))\\&\leq\sinh(\sinh(\frac{|\frac{x}{4}-\frac{y}{4}|+|\frac{y}{4}-\frac{z}{4}|+|\frac{z}{4}-\frac{x}{4}|}{3}))\\&\leq\sinh(\frac{\widetilde{G}(x,y,z)}{4})\\&\leq\frac{\widetilde{G}(x,y,z)}{2}=\beta(\widetilde{G}(x,y,z))\widetilde{G}(x,y,z)\leq\beta(M(x,y,z))M(x,y,z),\end{aligned}$$

So, from Theorem 2.2 f has a fixed point.

Example 3.2. Let $X=[0,\infty]$ be equipped with the

$$\widetilde{G}(x,y,z)=\frac{|x-y|+|y-z|+|z-x|}{3}+\ln(\frac{|x-y|+|y-z|+|z-x|}{3})$$

for all $x,y,z\in X$, where $\Omega(x)=x+\ln x$.

Define a relation $\preceq$ on X by $x \preceq y$ iff $y \leq x$, the function $f : X \to X$ by

$$fx = \ln(\frac{x}{5} + 2)$$

and the function ψ given by $\psi(t) = \frac{1}{2}t$. It is obvious that $\psi(t) < t$ for all $t \in X$. For all comparable elements $x, y \in X$, by mean value theorem, we have,

$$\begin{aligned}
&\Omega[\widetilde{G}(fx, fy, fz)] \\
&= \widetilde{G}(fx, fy, fz) + \ln[1 + \widetilde{G}(fx, fy, fz)] + \ln[1 + \widetilde{G}(fx, fy, fz) + \ln[1 + \widetilde{G}(fx, fy, fz)]] \\
&= \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3} \\
&+ \ln[1 + \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3}] \\
&+ \ln\Big[1 + \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3} \\
&+ \ln[1 + \frac{\left|\ln 2 + \frac{x}{5} - \ln 2 + \frac{y}{5}\right| + \left|\ln 2 + \frac{y}{5} - \ln 2 + \frac{z}{5}\right| + \left|\ln 2 + \frac{z}{5} - \ln 2 + \frac{x}{5}\right|}{3}]\Big] \\
&\leq \frac{\left|\frac{x}{5} - \frac{y}{5}\right| + \left|\frac{y}{5} - \frac{z}{5}\right| + \left|\frac{z}{5} - \frac{x}{5}\right|}{3} \\
&+ \ln[1 + \frac{\left|\frac{x}{5} - \frac{y}{5}\right| + \left|\frac{y}{5} - \frac{z}{5}\right| + \left|\frac{z}{5} - \frac{x}{5}\right|}{3}] \\
&+ \ln\Big[1 + \frac{\left|\frac{x}{5} - \frac{y}{5}\right| + \left|\frac{y}{5} - \frac{z}{5}\right| + \left|\frac{z}{5} - \frac{x}{5}\right|}{3} \\
&+ \ln[1 + \frac{\left|\frac{x}{5} - \frac{y}{5}\right| + \left|\frac{y}{5} - \frac{z}{5}\right| + \left|\frac{z}{5} - \frac{x}{5}\right|}{3}]\Big] \\
&\leq \frac{1}{5}\widetilde{G}(x, y, z) + \ln[1 + \frac{1}{5}\widetilde{G}(x, y, z)] \\
&+ \ln\Big[1 + \frac{1}{5}\widetilde{G}(x, y, z) + \ln[1 + \frac{1}{5}\widetilde{G}(x, y, z)]\Big] \\
&\leq \frac{9}{10}\widetilde{G}(x, y, z) = \psi(\widetilde{G}(x, y, z)) \leq \psi(M(x, y, z)),
\end{aligned}$$

So, from Theorem 2.5 f has a fixed point.

Example 3.3. Let $\widetilde{G} : X \times X \times X \to \mathbb{R}^+$ be defined on $X = [0, 1.5]$ by

$$\widetilde{G}(x, y, z) = e^{\frac{|x-y|+|y-z|+|z-x|}{3}} - 1$$

for all $x, y, z \in X$. Then $(X, \widetilde{G})$ is a $\widetilde{G}$-complete $\widetilde{G}$-metric space with $\Omega(t) = e^t - 1$.

Define k and $\theta \in \Theta$ by $k = \frac{1}{\sqrt{2}}$ and $\theta(t) = e^{te^t}$. Let X is endowed with the usual order. Let $f : X \to X$ be defined by $fx = \arctan(\frac{x}{16})$. It is easy to see that f is an ordered increasing and continuous self map on X and $0 \leq f0$. For any $x, y, z \in X$, we have

$$\begin{aligned}
\widetilde{G}(fx,fy,fz) &= e^{\frac{|fx-fy|+|fy-fz|+|fz-fx|}{3}}-1\\
&= e^{\frac{|\arctan\frac{x}{16}-\arctan\frac{y}{16}|+|\arctan\frac{y}{16}-\arctan\frac{z}{16}|+|\arctan\frac{z}{16}-\arctan\frac{x}{16}|}{3}}-1\\
&\le e^{\frac{|\frac{x}{16}-\frac{y}{16}|+|\frac{y}{16}-\frac{z}{16}|+|\frac{z}{16}-\frac{x}{16}|}{3}}-1\\
&\le \frac{1}{16}\big(e^{\frac{|x-y|+|y-z|+|z-x|}{3}}-1\big)\\
&= \frac{1}{16}\widetilde{G}(fx,fy,fz).
\end{aligned}$$

So,

$$\begin{aligned}
\Omega[\widetilde{G}(fx,fy,fz)] &= e^{\widetilde{G}(fx,fy,fz)}-1\\
&\le e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)}-1\\
&\le \frac{1}{16}\widetilde{G}(fx,fy,fz).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\theta(\Omega[\widetilde{G}(fx,fy,fz)]) &= e^{\Omega[\widetilde{G}(fx,fy,fz)]e^{\Omega[\widetilde{G}(fx,fy,fz)]}}\\
&\le e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)e^{\frac{1}{16}\widetilde{G}(fx,fy,fz)}}\\
&\le \big[e^{\widetilde{G}(fx,fy,fz)e^{\widetilde{G}(fx,fy,fz)}}\big]^{\frac{1}{\sqrt{2}}} = [\theta(\widetilde{G}(fx,fy,fz))]^{\frac{1}{\sqrt{2}}}.
\end{aligned}$$

Thus, (2.28) is satisfied with $k=\frac{1}{\sqrt{2}}$. Hence, all the conditions of Theorem 2.11 are satisfied. We have that 0 is the unique fixed point of f.

4. Existence of a solution for an integral equation

We consider the following integral equation:

$$x(t)=\int_a^b K(t,s,x(s))ds+k(t), \tag{4.1}$$

where $b>a\ge 0$. The aim of this section is to present the existence of a solution to 4.1 that belongs to $X=C[a,b]$ (the set of all continuous real valued functions defined on $[a,b]$) as an application to the Theorem 2.11.

The considered problem can be changed as follows.

Let $f:X\to X$ be defined by:

$$fx(t)=\int_a^b (t,s,x(s))ds+k(t),$$

for all $x\in X$ and for all $t\in[a,b]$. Obviously, existence of a solution to 4.1 is equivalent to the existence of a fixed point of f.

Let,

$$d(u,v)=\max_{t\in[a,b]}|u(t)-v(t)|=||u-v||_\infty.$$

Let X be equipped with the modified G-metric given by

$$\widetilde{G}(u,v,w)=\xi(\max\{d(u,v),d(v,w),d(w,u)\}),$$

for all $u,v,w\in X$ where $\xi:[0,\infty)\to[0,\infty)$ is a strictly increasing continuous function with $t\le\xi(t)$ for $t\ge 0$ and $\xi(0)=0$ which is a $\widetilde{G}$-complete $\widetilde{G}$-metric space. We endow

X with the partial ordered $\preceq$ given by $x \preceq y \iff x(t) \leq y(t)$, for all $t \in [a,b]$. It is known that $(X, \preceq)$ has sequential limit comparison property. [23]

Now, we will prove the following result.

Theorem 4.1. *Suppose that the following hypotheses hold:*

(i) $K : [a,b] \times [a,b] \times R \to R$ *and* $k : [a,b] \to R$ *are continuous;*

(ii) for all $s,t \in [a,b]$ *and for all* $x,y \in X$ *with* $x \preceq y$ *we have,*

$$\xi^2\Big(\int_a^b \Big|K(t,r,x(r)) - K(t,r,y(r))\Big|\,dr\Big) \leq \frac{\xi\big(\|fx-fy\|_\infty\big)\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}}{\theta\big(\xi(\|fx-fy\|_\infty)\big)},$$

for all $t \in [a,b]$ *and* $\theta \in \Psi$.

(iii) There exists continuous function $\alpha : [a,b] \to \mathbb{R}$ *such that*

$$\alpha(t) \leq \int_a^b (t,s,\alpha(s))ds + k(t).$$

Then, the integral equations 4.1 has a solution $x \in X$.

Proof. Let $x,y \in X$ be such that $x \succeq y$. From condition (ii), for all $t \in [a,b]$ we have,

$$\begin{aligned}\xi^2\Big(|fx(t)-fy(t)|\Big) &\leq \xi^2\Big(\int_a^b |K(t,s,x(s)) - K(t,s,y(s)|ds\Big)\\ &\leq \frac{\xi^2\big(\|fx-fy\|_\infty\big)\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}}{\theta\big(\xi^2(\|fx-fy\|_\infty)\big)}.\end{aligned}$$

Hence,

$$\begin{aligned}\xi^2\Big(d(fx,fy)\Big) &= \xi^2\Big(\sup_{t\in[a,b]}|fx(t)-fy(t)|\Big)\\ &\leq \frac{\xi^2\Big(\|fx-fy\|_\infty\Big)\theta\Big(\xi(\|x-y\|_\infty)\Big)^{\frac{1}{2}}}{\theta\Big(\xi^2(\|fx-fy\|_\infty)\Big)}.\end{aligned} \tag{4.2}$$

Hence,

$$\theta\big(\xi^2(\|fx-fy\|_\infty)\big) \leq \theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}}. \tag{4.3}$$

Therefore, from 4.3 we have,

$$\begin{aligned}\theta\Big(\xi\Big(\widetilde{G}(fx,fy,fz)\Big)\Big) &= \theta\Big(\xi\Big(\xi(\max\{d(fx,fy),d(fy,hz),d(fz,fx)\})\Big)\Big)\\ &\leq \max\Big\{\theta\Big(\xi^2(d(fx,fy))\Big),\theta\Big(\xi^2(d(fy,fz))\Big),\theta\Big(\xi^2(d(fz,fx))\Big)\Big\}\\ &\leq \max\Big\{\theta\big(\xi(\|x-y\|_\infty)\big)^{\frac{1}{2}},\theta\big(\xi(\|y-z\|_\infty)\big)^{\frac{1}{2}},\theta\big(\xi(\|z-x\|_\infty)\big)^{\frac{1}{2}}\Big\}\\ &\leq \theta\big(M(x,y,z)\big)^{\frac{1}{2}},\end{aligned}$$

where

$$\begin{aligned}M(x,y,z) = \max\Big\{&\widetilde{G}(x,y,z), \frac{\widetilde{G}(x,x,fx)\widetilde{G}(y,y,fy)}{1+\xi[\widetilde{G}(x,y,y)+\widetilde{G}(x,x,fy)]},\\ &\frac{\widetilde{G}(y,y,fy)\widetilde{G}(z,z,fz)}{1+\xi[\widetilde{G}(y,z,z)+\widetilde{G}(y,y,fz)]}, \frac{\widetilde{G}(y,z,z)\widetilde{G}(y,y,z)}{1+\widetilde{G}(y,fy,fy)+\widetilde{G}(z,fz,fz)}\Big\},\end{aligned}$$

So, from Theorem 2.11, there exists $x \in X$, a fixed point of f which is a solution of 4.1. ∎

5. Conclusion

Taking $\Omega(x) = sx$, our obtained results coincide with the results in usual G_b-metric spaces and taking $\Omega(x) = x$, our obtained results coincide with the results in usual G-metric spaces.

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