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On Finite and Infinite Decomposition of Some Hilbert's Type Inequalities

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Abstract. In this work, some Hardy-Hilbert's integral inequalities with the best possible constants are proved. Also, some finite and infinite decompositions of some type Hardy-Hilbert's integral operators are given. Indeed, for a non-negative kernel K, two Kernels K_1 and K_2 are given such that $T_K = T_{K_1} + T_{K_2}$ and $\|T_K\| = \|T_{K_1}\| + \|T_{K_2}\|$ and also, $T_{k_1} \neq cT_{k_2}$ for every constant c. So, the space of bounded linear operators is not strictly convex. Also, as an application of infinite decomposition of some Hardy-Hilbert's integral operators, the convergence of some series of hypergeometric functions are given.

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1. Introduction

David Hilbert in the early 1900s, in his lectures on integral equations, proved a double series inequality. If $\{a_n\}$ and $\{b_m\}$ are two real sequences such that $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{m=1}^{\infty} b_m^2 < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_n b_m}{m+n} < 2\pi\Big\{\sum_{n=1}^{\infty}a_n^2\sum_{m=1}^{\infty}b_m^2\Big\}^{\frac{1}{2}} < \infty.$$

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In 1911, Schour gave a new proof of the inequality which π is the best possible sharp constant. Schour also discovered the integral analogue of the Hilbert's inequality as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \Big\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \Big\}^{\frac{1}{2}}, \quad (1)$$

where f and g are measurable functions such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$.

There are some kinds of Hilbert-type inequalities. For instance, Dongmel Xin in [4] gave the following statement:

suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$ and also $\lambda > 0$, f, $g \geqslant 0$ such that

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)\, dx < \infty,$$

$$0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{\ln(\frac{x}{y}) f(x)g(y)}{x^\lambda - y^\lambda}\, dxdy < \Big[\frac{\pi}{\lambda \sin(\frac{\pi}{r})}\Big]^2 \Big(\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x)\, dx\Big)^{\frac{1}{p}} \times \Big(\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y)\, dy\Big)^{\frac{1}{q}}, \quad (2)$$

where the constant factor is the best possible.

In [2], the above-mentioned statement is generalized as follows:

suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $m \in \mathbb{N}$ and also u, v are two strict increasing differentiable functions, $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, $\lambda > 0$ and f, $g \geqslant 0$,

$$0 < \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1} (u'(x))^{1-p} f^p(x)\, dx < \infty,$$

and

$$0 < \int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1} (v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m-1} f(x)g(y)}{u^\lambda(x)-v^\lambda(y)}\,dxdy$$
$$< K\Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\,dx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{3}$$

where

$$K = C(r,s,\lambda,m) = \frac{\Gamma(2m)}{\lambda^{2m}}\Big(r^{2m}\sum_{k=0}^{\infty}\frac{1}{(rk+1)^{2m}} + s^{2m}\sum_{k=0}^{\infty}\frac{1}{(sk+1)^{2m}}\Big).$$

Also the constant factor is the best possible.
In the same work the following statement is proved.
Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $m\in\mathbb{N}\bigcup\{0\}$ and u,v are two strict increasing differentiable functions, $u(0)=v(0)=0$, $u(\infty)=v(\infty)=\infty$, $\lambda>0$ and $f,\ g\geqslant 0$,

$$0<\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\,dx<\infty,$$

and

$$0<\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\,dy<\infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} f(x)g(y)}{u^\lambda(x)+v^\lambda(y)}\,dxdy$$
$$< K\Big(\int_0^\infty u^{p(1-\frac{\lambda}{r})-1}(u')^{1-p} f^p(x)\,dx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty v^{q(1-\frac{\lambda}{s})-1}(v')^{1-q} g^q(y)\,dy\Big)^{\frac{1}{q}}, \tag{4}$$

where

$$K = C_E(r,s,\lambda,m) = \frac{\Gamma(2m+1)}{\lambda^{2m+1}} \Big(r^{2m+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(rk+1)^{2m+1}} + s^{2m+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(sk+1)^{2m+1}}\Big).$$

Also, the constant factor is the best possible.
A large number of generalizations, extensions and refinements of the above inequality are available in literature such as Hardy *et al.* [1], Mitrinović *et al* [3] and Yang [6].

Recently, Yang [5] by the identity

$$\frac{1}{m+n} = \frac{\max\{m,n\}}{(m+n)^2} + \frac{\min\{m,n\}}{(m+n)^2}, \qquad (m,n \in \mathbb{N}),$$

gave a decomposition of Hilbert's inequality as follows:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\max\{m,n\}}{(m+n)^2}a_m b_n < (\frac{\pi}{2}+1)\Big(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\Big)^{\frac{1}{2}},$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\min\{m,n\}}{(m+n)^2}a_m b_n < (\frac{\pi}{2}-1)\Big(\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\Big)^{\frac{1}{2}}.$$

The sum of two best constant factors is π (the constant factor of the Hilbert's inequality).

The author and Lashkaripour in [2], gave decompositions of some Hilbert's type inequalities.

In this work, decompositions of some Hilbert's type inequalities are given which some of them are infinite forms. In the sence that, for a homogeneous kernel $K(x,y)$ of degree $-\lambda$, by finding a sequence $\{K_n(x,y)\}$ of homogeneous kernels with the same degree and proving inequalities of the form

$$\int_0^{\infty}\int_0^{\infty}K(x,y)f(x)g(y)dxdy < C\Big\{\int_0^{\infty}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^{\infty}g^q(y)dy\Big\}^{\frac{1}{q}},$$

and

$$\int_0^{\infty}\int_0^{\infty}K_n(x,y)f(x)g(y)dxdy < C_n\Big\{\int_0^{\infty}f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^{\infty}g^q(y)dy\Big\}^{\frac{1}{q}},$$

such that the constants are best possible and $C = \sum_{n=1}^{\infty} C_n$, the first inequality decompose infinitely.

In this section, by the following identities, three pairs of new Hilbert-type inequalities which respectively decompose inequalities (3), (4) and (5) are given.

$$\frac{1}{x^\lambda - y^\lambda} = \frac{x^\lambda}{x^{2\lambda} - y^{2\lambda}} + \frac{y^\lambda}{x^{2\lambda} - y^{2\lambda}}.$$

$$\frac{1}{x^\lambda + y^\lambda} = \frac{x^\lambda}{x^{2\lambda} - y^{2\lambda}} - \frac{y^\lambda}{x^{2\lambda} - y^{2\lambda}}.$$

$$\frac{1}{\Big(A\min\{x,y\} + B\max\{x,y\}\Big)^\lambda} = \frac{A\min\{x,y\}}{\Big(A\min\{x,y\} + B\max\{x,y\}\Big)^{\lambda+1}} + \frac{B\max\{x,y\}}{(A\min\{x,y\} + B\max\{x,y\})^{\lambda+1}}.$$

Theorem 1.1. *Suppose that* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\frac{1}{r}+\frac{1}{s} = 1$, $m \in \mathbb{N}\cup\{0\}$ *and* u, v *are two strict increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, $u > v$, $\lambda > 0$ *and also* f, $g \geqslant 0$,

$$0 < \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$

and

$$0 < \int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} u^\lambda(x) f(x) g(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\, dxdy < K\Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\, dx\Big)^{\frac{1}{p}} \times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\, dy\Big)^{\frac{1}{q}},$$

where

$$K = K(r,s,\lambda,m) = \frac{\Gamma(2m+1)}{\lambda^{2m+1}} \Big(-r^{2m+1}\sum_{k=0}^{\infty}\frac{1}{((2k+1)r+1)^{2m+1}} + s^{2m+1}\sum_{k=0}^{\infty}\frac{1}{(2sk+1)^{2m+1}}\Big).$$

Also, the constant factor is the best possible.

Proof. Let $f(x) = F(x)(u'(x))^{\frac{1}{q}}$ and $g(y) = G(y)(v'(y))^{\frac{1}{p}}$. Then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} u^\lambda(x) f(x) g(y)}{u^{2\lambda}(x) - v^{2\lambda}(y)} dxdy$$
$$= \int_0^\infty \int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} F(x)G(y)u^\lambda(x)(u'(x))^{\frac{1}{q}}(v'(y))^{\frac{1}{p}}}{u^{2\lambda}(x) - v^{2\lambda}(y)} dxdy$$
$$= \int_0^\infty \int_0^\infty \Big(\frac{\left(\ln(\frac{u}{v})\right)^{2m} u^\lambda}{u^{2\lambda} - v^{2\lambda}}\Big)^{\frac{1}{p}} \times \frac{u^{\frac{(1-\frac{\lambda}{r})}{q}}}{v^{\frac{(1-\frac{\lambda}{s})}{p}}} F(x)(v')^{\frac{1}{p}}$$
$$\times \Big(\frac{\left(\ln(\frac{u}{v})\right)^{2m} u^\lambda}{u^{2\lambda} - v^{2\lambda}}\Big)^{\frac{1}{q}} \times \frac{v^{\frac{(1-\frac{\lambda}{s})}{p}}}{u^{\frac{(1-\frac{\lambda}{r})}{q}}} G(y)(u')^{\frac{1}{q}}\, dxdy$$
$$\leqslant \Big(\int_0^\infty \int_0^\infty \frac{\left(\ln(\frac{u}{v})\right)^{2m} u^\lambda}{u^{2\lambda} - v^{2\lambda}} \frac{u^{(p-1)(1-\frac{\lambda}{r})}}{v^{(1-\frac{\lambda}{s})}} F^p(x)v'(y)\, dxdy\Big)^{\frac{1}{p}}$$
$$\times \Big(\int_0^\infty \int_0^\infty \frac{\left(\ln(\frac{u}{v})\right)^{2m} u^\lambda}{u^{2\lambda} - v^{2\lambda}} \frac{v^{(q-1)(1-\frac{\lambda}{s})}}{u^{(1-\frac{\lambda}{r})}} G^q(y)u'(x)\, dxdy\Big)^{\frac{1}{q}}$$
$$= M^{\frac{1}{p}} N^{\frac{1}{q}}.$$

Note that

$$M = \int_0^\infty \Big(\int_0^\infty \frac{\left(\ln(\frac{u}{v})\right)^{2m}}{u^\lambda\left(1 - \left(\frac{v}{u}\right)^{2\lambda}\right)} \frac{u^{(p-1)(1-\frac{\lambda}{r})}}{v^{(1-\frac{\lambda}{s})}} v'(y)\, dy\Big) F^p(x)\, dx.$$

By substituting $e^{\frac{z}{\lambda}} = \frac{v(y)}{u(x)}$, one obtains

$$M = \frac{1}{\lambda^{2m+1}} \Big(\int_{-\infty}^\infty \frac{z^{2m} e^{\frac{z}{s}}}{1 - e^{2z}} dz\Big) \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1} (u'(x))^{1-p} f^p(x)\, dx$$
$$= \frac{1}{\lambda^{2m+1}} \Big(\int_0^\infty \frac{z^{2m} e^{\frac{-z}{s}}}{1 - e^{-2z}} dz - \int_0^\infty \frac{z^{2m} e^{-z(1+\frac{1}{r})}}{1 - e^{-2z}} dz\Big)$$
$$\times \Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1} (u'(x))^{1-p} f^p(x)\, dx\Big)$$
$$= K(r, s, \lambda, m) \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1} (u'(x))^{1-p} f^p(x)\, dx,$$

where $K(r,s,\lambda,m) = \frac{\Gamma(2m+1)}{\lambda^{2m+1}}$
$\Big(-r^{2m+1}\sum_{k=0}^{\infty}\frac{1}{((2k+1)r+1)^{2m+1}} + s^{2m+1}\sum_{k=0}^{\infty}\frac{1}{(2sk+1)^{2m+1}}\Big)$.

By the same way, one obtains

$$N = K(r,s,\lambda,m)\int_0^{\infty}(v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy.$$

This proves the inequality.

If the inequality mentioned in Theorem 1.1 takes the form of equality, then there exist constants c_1 and c_2 such that ${c_1}^2+{c_2}^2\neq 0$ and

$$\begin{aligned} c_1 f^p(x)(u'(x))^{-p}(u(x))^{p(1-\frac{\lambda}{r})} &= c_2 g^q(y)(v'(y))^{-q}(v(y))^{1-\frac{\lambda}{s})} \\ &= c \quad \text{a.e. in}(0,\infty)\times(0,\infty), \end{aligned}$$

where c is constant. Without loss of generality, suppose $c_1\neq 0$. One has

$$f^p(x) = \frac{c}{c_1}(u'(x))^p(u(x))^{p(\frac{\lambda}{r}-1)} \text{ a.e. in } (0,\infty).$$

Now, we have

$$\int_0^{\infty}(u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx = \frac{c}{c_1}\int_0^{\infty}\frac{du}{u},$$

which is in contradiction with

$$\int_0^{\infty}(u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx < \infty.$$

If the constant factor is not the best possible, then there is a positive number C with $C < K(r,s,\lambda,m)$ such that

$$\begin{aligned} &\int_0^{\infty}\int_0^{\infty}\frac{\Big(\ln\Big(\frac{u(x)}{v(y)}\Big)\Big)^{2m}u^{\lambda}f(x)g(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy \\ &\qquad < C\Big(\int_0^{\infty}(u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \\ &\qquad \times\Big(\int_0^{\infty}(v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}. \end{aligned}$$

Assume that $0 < \epsilon < (\frac{s+1}{s})\lambda q$ and

$$f_\epsilon(x) = \begin{cases} 0 & 0 \leqslant x < u^{-1}(1), \\ (u(x))^{\frac{\lambda}{r}-1-\frac{\epsilon}{p}} u'(x) & x \geqslant u^{-1}(1), \end{cases}$$

and

$$g_\epsilon(y) = \begin{cases} 0 & 0 \leqslant y < v^{-1}(1), \\ (v(y))^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}} v'(y) & y \geqslant v^{-1}(1). \end{cases} .$$

One can show that

$$\begin{aligned} \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\, dx &= \int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\, dy \\ &= \frac{1}{\epsilon}. \end{aligned}$$

On the other hand, we have

$$\int_0^\infty \int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} u^\lambda f_\epsilon(x) g_\epsilon(y)}{u^{2\lambda}(x) - v^{2\lambda}(y)}\, dxdy$$

$$\begin{aligned} &= \int_{u^{-1}(1)}^\infty u^{-1-\epsilon} u' \Big(\int_{\frac{1}{u}}^\infty \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}}{1-t^{2\lambda}}\, dt \Big)\, dx \\ &= \int_1^\infty w^{-1-\epsilon} \Big(\int_{\frac{1}{w}}^1 \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}}{1-t^{2\lambda}}\, dt \\ &+ \int_1^\infty \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}}{1-t^{2\lambda}}\, dt \Big) dw \\ &= \int_0^1 \Big(\int_{\frac{1}{t}}^\infty w^{-1-\epsilon} dw \Big) \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}}{1-t^{2\lambda}} dt \\ &+ \int_1^\infty \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1-\frac{\epsilon}{q}}}{1-t^{2\lambda}}\, dt \Big) w^{-1-\epsilon} dw \\ &= \frac{1}{\epsilon} \Big(\int_0^1 \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1}}{1-t^{2\lambda}}\, dt + o_1(1) \\ &+ \int_1^\infty \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1}}{1-t^{2\lambda}}\, dt + o_2(1) \Big) \qquad (\epsilon \to 0) \\ &= \frac{1}{\epsilon} \Big(\int_0^\infty \frac{-\big(\ln(t)\big)^{2m} t^{\frac{\lambda}{s}-1}}{1-t^{2\lambda}}\, dt + o(1) \Big) \quad (\epsilon \to 0) \\ &= \frac{K(r,s,\lambda,m)}{\epsilon} + o(1). \quad (\epsilon \to 0) \end{aligned}$$

We deduce that $C > K(r, s, \lambda, m)$ as ϵ tends to zero. $\square$

By the same way, one may obtain the following theorem.

Theorem 1.2. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $m \in \mathbb{N}$ *and* u, v *are two strict increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, $\lambda > 0$ *and also* f, $g \geqslant 0$,

$$0 < \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$

and

$$0 < \int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} v^\lambda(x) f(x) g(y)}{u^{2\lambda}(x) - v^{2\lambda}(y)}\, dxdy$$

$$< K(s, r, \lambda, m)\Big(\int_0^\infty u^{p(1-\frac{\lambda}{r})-1}(u')^{1-p} f^p(x)\, dx\Big)^{\frac{1}{p}}$$

$$\times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q} g^q(y)\, dy\Big)^{\frac{1}{q}}.$$

Also the constant factor is the best possible.

Remark 1.3. *By the identity,*

$$K(r, s, \lambda, m) + K(s, r, \lambda, m) = C_E(r, s, \lambda, m),$$

the sum of the two best constant factors in Theorems 1.1 and 1.2 is the best constant factor in inequality (4). So, two theorems mentioned above are a decomposition of inequality (4).

By similar computations, one may obtain the following two theorems:

Theorem 1.4. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r} + \frac{1}{s} = 1$, $m \in N$ *and* u, v *are two strict increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, $\lambda > 0$ *and* f, $g \geqslant 0$,

$$0 < \int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$

and

$$0<\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy<\infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m-1}u^\lambda(x)f(x)g(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy$$
$$<K\Big(\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}},$$

where

$$K=K(r,s,\lambda,m)=\frac{\Gamma(2m)}{\lambda^{2m}}\Big(r^{2m}\sum_{k=0}^\infty\frac{1}{((2k+1)r+1)^{2m}}+s^{2m}\sum_{k=0}^\infty\frac{1}{(2sk+1)^{2m}}\Big).$$

Also the constant factor is the best possible.

Theorem 1.5. *Suppose that* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{r}+\frac{1}{s}=1$, $m\in\mathbb{N}$ *and* u, v *are two strict increasing differentiable functions,* $u(0)=v(0)=0$, $u(\infty)=v(\infty)=\infty$, $\lambda>0$ *and also* f, $g\geqslant 0$,

$$0<\int_0^\infty (u(x))^{p(1-\frac{\lambda}{r})-1}(u'(x))^{1-p}f^p(x)\,dx<\infty,$$

and

$$0<\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy<\infty.$$

Then

$$\int_0^\infty\int_0^\infty \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m-1}v^\lambda(x)f(x)g(y)}{u^{2\lambda}(x)-v^{2\lambda}(y)}\,dxdy$$
$$<K(s,r,\lambda,m)\Big(\int_0^\infty u^{p(1-\frac{\lambda}{r})-1}(u')^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^\infty (v(y))^{q(1-\frac{\lambda}{s})-1}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

Also, the constant factor is the best possible.

Remark 1.6. *By the identity*

$$K(r,s,\lambda,m)+K(s,r,\lambda,m)=C(r,s,\lambda,m),$$

sum of the best constant factors in Theorem 1.4 and 1.5 is the best constant factor in inequality (3). So the above-mentioned two theorems are a decomposition of inequality (3). Also, note that by taking $m = 1$ in Theorems 1.4 and 1.5, we have

$$\begin{aligned} K(r,s,\lambda,1) &= \frac{\Gamma(2)}{\lambda^2}\Big(r^2\sum_{k=0}^{\infty}\frac{1}{((2k+1)r+1)^2}+s^2\sum_{k=0}^{\infty}\frac{1}{(2sk+1)^2}\Big) \\ &= \frac{1}{4\lambda^2}\Bigg(\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{r+1}{2r}\right)^2}+\sum_{k=0}^{\infty}\frac{1}{\left(k+\frac{r-1}{2r}\right)^2} \\ &= \frac{1}{4\lambda^2}\Big(\psi'(\frac{r+1}{2r})+\psi'(\frac{r-1}{2r})\Big) \\ &= \left[\frac{\pi}{2\lambda\sin(\frac{\pi}{2s})}\right]^2, \end{aligned}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, satisfies in the identity

$$\psi'(x) + \psi'(1-x) = (\pi\csc(\pi x))^2.$$

Also, we have

$$\begin{aligned} K(s,r,\lambda,1) &= \frac{\Gamma(2)}{\lambda^2}\Big(s^2\sum_{k=0}^{\infty}\frac{1}{((2k+1)s+1)^2}+r^2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\Big). \\ &= \left[\frac{\pi}{2\lambda\sin(\frac{\pi}{2r})}\right]^2. \end{aligned}$$

These correspond to the constant factors of inequalities mentioned in Theorems 2.5 and 2.6 in [2], respectively.

In the sequel, we prove a pair of Hilbert-type inequality which decompose the following inequality:
if $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $0 \leqslant \frac{A}{B} < 1$, A and B are nonnegative and u, v are two strict increasing differentiable functions, $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, and f, $g \geqslant 0$ and also

$$0 < \int_0^{\infty} (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$

$$0 < \int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(A\min\{u,v\}+B\max\{u,v\})^\lambda}\, dxdy$$
$$< K\Big(\int_0^\infty u^{1-\lambda}(u')^{1-p} f^p(x)\, dx\Big)^{\frac{1}{p}}$$
$$\times \Big(\int_0^\infty (v(y))^{1-\lambda}(v(y)')^{1-q} g^q(y)\, dy\Big)^{\frac{1}{q}}, \quad (5)$$

where

$$\Psi(p,q,\lambda) = \beta(1,\phi_p(\lambda))F(\lambda,\phi_p(\lambda);1+\phi_p(\lambda);\frac{-A}{B})+\beta(1,\phi_q(\lambda))F(\lambda,\phi_q(\lambda);1+\phi_q(\lambda);\frac{-A}{B}),$$

such that $\phi_r(\lambda) = 1 - \frac{2-\lambda}{r}$ and $F(\alpha,\beta;\gamma;z)$ denotes the hypergeometric function defined by

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\, dt,$$

providing $\gamma > \beta > 0$ and $|z| < 1$. The constant factor is the best possible(cf. [2] Theorem 2.9).

Theorem 1.7. *Suppose that* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda > 2-\min\{p,q\}$, $0 \leqslant \frac{A}{B} < 1$, *A and B are nonnegative and u, v are two strict increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, *and f, g* $\geqslant 0$ *and also*

$$0 < \int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$
$$0 < \int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{\min\{u,v\}f(x)g(y)}{(A\min\{u,v\}+B\max\{u,v\})^{\lambda+1}}\, dxdy < B^{-(\lambda+1)}\Psi_1(p,q,\lambda)$$
$$\times \Big(\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\, dx\Big)^{\frac{1}{p}}$$
$$\times \Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g^q(y)\, dy\Big)^{\frac{1}{q}},$$

where

$$\Psi_1(p,q,\lambda) = \beta(1, 1+\phi_q(\lambda))F\Big(\lambda+1, 1+\phi_q(\lambda); 2+\phi_q(\lambda); \frac{-A}{B}\Big)$$
$$+\beta(1, 1+\phi_p(\lambda))F\Big(\lambda+1, 1+\phi_p(\lambda); 2+\phi_p(\lambda); \frac{-A}{B}\Big).$$

The constant factor is the best possible.

Proof. Let $f(x) = F(x)(u'(x))^{\frac{1}{q}}$ and $g(y) = G(y)(v'(y))^{\frac{1}{p}}$. Then

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{\min\{u(x)+v(y)\}f(x)g(y)}{(A\min\{u,v\}+B\max\{u,v\})^{\lambda+1}}\,dxdy \\
&= \int_0^\infty\int_0^\infty \frac{\min^{\frac{1}{p}}\{u,v\}F(x)(v')^{\frac{1}{p}}}{(A\min\{u,v\}+B\max\{u,v\})^{\frac{\lambda+1}{p}}}\Big(\frac{u}{v}\Big)^s \\
&\times \frac{\min^{\frac{1}{q}}\{u,v\}G(y)(u')^{\frac{1}{q}}}{(A\min\{u,v\}+B\max\{u,v\})^{\frac{\lambda+1}{q}}}\Big(\frac{v}{u}\Big)^s\,dxdy \\
&\leqslant \Big(\int_0^\infty\int_0^\infty \frac{\min\{u,v\}F^p(x)v'\left(\frac{u}{v}\right)^{sp}}{(A\min\{u,v\}+B\max\{u,v\})^{\lambda+1}}\,dxdy\Big)^{\frac{1}{p}} \\
&\times \Big(\int_0^\infty\int_0^\infty \frac{\min\{u,v\}G^q(y)u'\left(\frac{v}{u}\right)^{sq}}{(A\min\{u,v\}+B\max\{u,v\})^{\lambda+1}}\,dxdy\Big)^{\frac{1}{q}} \\
&= M^{\frac{1}{p}}N^{\frac{1}{q}}.
\end{aligned}$$

Note that

$$\begin{aligned}
M &= \int_0^\infty\int_0^\infty \frac{\min\{1,\frac{v}{u}\}F^p(x)v'\left(\frac{u}{v}\right)^{sp}}{u^\lambda(A\min\{1,\frac{v}{u}\}+B\max\{1,\frac{v}{u}\})^{\lambda+1}}\,dxdy \\
&= \int_0^\infty u^{1-\lambda}F^p(x)\Big(\int_0^\infty \frac{\min\{1,t\}t^{-sp}}{(A\min\{1,t\}+B\max\{1,t\})^{\lambda+1}}\,dt\Big)\,dx \\
&= \int_0^\infty u^{1-\lambda}F^p(x)\Big(\int_0^1 \frac{t^{1-sp}}{(At+B)^{\lambda+1}}\,dt + \int_1^\infty \frac{t^{-sp}}{(A+Bt)^{\lambda+1}}\,dt\Big) \\
&= B^{-(\lambda+1)}\Big(\beta(1,2-sp))F\Big(\lambda+1, 2-sp; 3-sp; -\frac{A}{B}\Big) \\
&+ \beta(1,\lambda+sp))F\Big(\lambda+1,\lambda+sp;\lambda+sp+1;-\frac{A}{B}\Big)\Big)\Big(\int_0^\infty u^{1-\lambda}F^p(x)\,dx\Big),
\end{aligned}$$

and,

$$N = B^{-(\lambda+1)}\Big(\beta(1,2-sq))F\Big(\lambda+1,2-sq;3-sq;-\frac{A}{B}\Big) + \beta(1,\lambda+sq))F\Big(\lambda+1,\lambda+sq;\lambda+sq+1;-\frac{A}{B}\Big)\Big)\Big(\int_0^\infty v^{1-\lambda}G^q(y)\,dy\Big).$$

Now one may find that the factor in the right-hand side attains its minimum at $s=\frac{2-\lambda}{pq}$. This completes the proof.
If inequality in Theorem 1.7 takes the form of equality, then there exist constants c_1 and c_2 such that ${c_1}^2+{c_2}^2\neq 0$ and

$$\begin{aligned} c_1 f^p(x)(u'(x))^{-p}(u(x))^{2-\lambda} &= c_2 g^q(y)(v'(y))^{-q}(v(y))^{2-\lambda} \\ &= c \text{ a.e. in } \quad (0,\infty)\times(0,\infty), \end{aligned}$$

where c is constant. Without loss of generality, suppose that $c_1\neq 0$. One has
$f^p(x)=\frac{c}{c_1}(u'(x))^p(u(x))^{\lambda-2}$ a.e. in $(0,\infty)$.

Now, we have

$$\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx = \frac{c}{c_1}\int_0^\infty \frac{du}{u},$$

which is in contradiction with

$$0<\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx<\infty.$$

If the constant factor is not the best possible, then there is a positive number K with

$$K<B^{-(\lambda+1)}\Psi_1(p,q,\lambda),$$

such that

$$\int_0^\infty\int_0^\infty \frac{\min\{u,v\}f(x)g(y)}{(A\min\{u,v\}+B\max\{u,v\})^{\lambda+1}}\,dxdy$$

$$< K\Big(\int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p}f^p(x)\,dx\Big)^{\frac{1}{p}} \times\Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q}g^q(y)\,dy\Big)^{\frac{1}{q}}.$$

Assume that

$$f_\epsilon(x) = \begin{cases} 0 & 0 \leqslant x < u^{-1}(1) \\ (u(x))^{-\frac{2+\epsilon-\lambda}{p}} u'(x) & x \geqslant u^{-1}(1), \end{cases}$$

and

$$g_\epsilon(y) = \begin{cases} 0 & 0 \leqslant y < v^{-1}(1) \\ (v(y))^{-\frac{2+\epsilon-\lambda}{q}} v'(y) & y \geqslant v^{-1}(1), \end{cases}$$

where $0 < \epsilon < \lambda + q - 2$. One can show that

$$\begin{aligned} \int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f_\epsilon^p(x)\, dx &= \int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g_\epsilon^q(y)\, dy \\ &= \frac{1}{\epsilon}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{\min\{u,v\} f_\epsilon(x) g_\epsilon(y)}{(A\min\{u,v\} + B\max\{u,v\})^{\lambda+1}}\, dxdy \\ &\qquad > B^{-(\lambda+1)} \Bigg(\int_{u^{-1}(1)}^\infty \Big(\int_0^1 \frac{t^{\beta+1}}{(1+\frac{A}{B}t)^{\lambda+1}} dt \\ &\qquad - \int_0^{\frac{1}{u(x)}} t^{\beta+1} dt + \int_0^1 \frac{t^{-1-\beta+\lambda}}{(1+\frac{A}{B}t)^{\lambda+1}} dt \Big) u^{-1-\epsilon} u' dx \Bigg) \\ &\qquad = \frac{1}{\epsilon}\Big(B^{-(\lambda+1)} \Psi(p,q,\lambda) + o(1) \Big) \\ &\qquad + \frac{B^{-(\lambda+1)}}{(\beta+2)(\epsilon+\beta-2)}, \end{aligned}$$

where $\alpha = \frac{\lambda-2-\epsilon}{p}$ and $\beta = \frac{\lambda-2-\epsilon}{q}$. By tending ϵ to zero, a contradiction is obtained. □

Theorem 1.8. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 2 - \min\{p, q\}$, $0 \leqslant \frac{A}{B} < 1$, *A and B are nonnegative and u, v are two strict increasing differentiable functions,* $u(0) = v(0) = 0$, $u(\infty) = v(\infty) = \infty$, *and* f, $g \geqslant 0$ *and also*

$$0 < \int_0^\infty (u(x))^{1-\lambda}(u'(x))^{1-p} f^p(x)\, dx < \infty,$$

$$0 < \int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty \frac{\max\{u, v\} f(x) g(y)}{(A \min\{u, v\} + B \max\{u, v\})^{\lambda+1}}\, dxdy$$
$$< B^{-(\lambda+1)} \Psi_2(p, q, \lambda) \Big(\int_0^\infty u^{1-\lambda}(u')^{1-p} f^p(x)\, dx \Big)^{\frac{1}{p}}$$
$$\times \Big(\int_0^\infty (v(y))^{1-\lambda}(v'(y))^{1-q} g^q(y)\, dy \Big)^{\frac{1}{q}},$$

where

$$\Psi_2(p, q, \lambda) = \beta(1, \phi_q(\lambda)) F(\lambda{+}1, \phi_q(\lambda); 1{+}\phi_q(\lambda); \frac{-A}{B}) + \beta(1, \phi_p(\lambda)) F(\lambda{+}1, \phi_p(\lambda); 1{+}\phi_p(\lambda); \frac{-A}{B}).$$

The constant factor is the best possible.

Remark 1.9. *By the identity*

$$F(a+1, b+1; c+1; z) = \frac{c}{bz}\Big(F(a+1, b; c; z) - F(a, b; c; z) \Big),$$

one may prove that

$$B^{-(\lambda+1)}\Big(A\Psi_1(p, q, \lambda) + B\Psi_2(p, q, \lambda) \Big) = B^{-\lambda}\Psi(p, q, \lambda).$$

So ,Theorems 1.7 and 1.8 give a decomposition of inequality (5).

1.1 Some infinite decompositions

In 1934, Hardy et *al*. [1] published the following statement.

Theorem 1.1.1. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $K(x, y)$ *is a non-negative homogeneous function of degree -1 in* $\mathbb{R}^2_+$. *If* $f(x), g(y) \geqslant 0$, $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$, $k = \int_0^\infty K(u, 1)u^{-\frac{1}{p}}du$ *is finite. Then, we have* $k = \int_0^\infty K(1, u)u^{-\frac{1}{q}}du$ *and*

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < k\Big\{\int_0^\infty f^p(x)dx\Big\}^{\frac{1}{p}}\Big\{\int_0^\infty g^q(y)dy\Big\}^{\frac{1}{q}}, \tag{6}$$

$$\int_0^\infty \Big(\int_0^\infty K(x,y)f(x)dx\Big)^p dy < k^p \int_0^\infty f^p(x)dx. \tag{7}$$

The constant factor k is the best possible.

In special case, by taking $K(x, y) = \frac{\ln(\frac{x}{y})}{x-y}$ they proved the following two pairs of equivalent inequalities: If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and f, $g \geqslant 0$ such that

$$0 < \int_0^\infty f^p(x)\, dx < \infty,$$
$$0 < \int_0^\infty g^q(y)\, dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{\ln(\frac{x}{y})f(x)g(y)}{x-y}dxdy < \Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^2\Big(\int_0^\infty f^p(x)dx\Big)^{\frac{1}{p}} \times \Big(\int_0^\infty g^q(y)dy\Big)^{\frac{1}{q}}, \tag{8}$$

$$\int_0^\infty \Big(\int_0^\infty \frac{\ln(\frac{x}{y})}{x-y}f(x)dx\Big)^p dy \leqslant k^p \int_0^\infty f^p(x)dx. \tag{9}$$

Now, by the above theorem, some infinite decompositions of some Hardy-Hilbert's type inequalities are given.

Theorem 1.1.2. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* f, $g \geqslant 0$ *such that*

$$0 < \int_0^\infty f^p(x)\,dx < \infty,$$
$$0 < \int_0^\infty g^q(y)\,dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty K_n(x,y)f(x)g(y)dxdy < k_n\Big(\int_0^\infty f^p(x)dx\Big)^{\frac{1}{p}} \times \Big(\int_0^\infty g^q(y)dy\Big)^{\frac{1}{q}}, \tag{10}$$

where

$$K_n(x,y) = \frac{\ln(\frac{x}{y})\min^n\{x,y\}}{sgn(x-y)\max^{n+1}\{x,y\}},$$
$$k_n = \frac{1}{(n+\frac{1}{p})^2} + \frac{1}{(n+\frac{1}{q})^2}.$$

Proof. K_n is a non-negative homogeneous function of degree -1 in $\mathbb{R}_+^2$ and so by Theorem 1.1.1, we have

$$k_n = \int_0^\infty K_n(x,1)x^{-\frac{1}{p}}dx = \frac{1}{(n+\frac{1}{p})^2} + \frac{1}{(n+\frac{1}{q})^2}.$$

Also, the constant factors k_n are the best possible. □

Remark 1.1.3. *Note that*

$$\sum_{n=0}^\infty k_n = \psi'(\frac{1}{p}) + \psi'(\frac{1}{q}) = \Big[\frac{\pi}{\sin(\frac{\pi}{p})}\Big]^2.$$

So, inequalities (10) decompose infinitly inequality (8).

Theorem 1.1.4. *Suppose* f, $g \geqslant 0$ *such that*

$$0 < \int_0^\infty f^2(x)\,dx < \infty,$$

$$0 < \int_0^\infty g^2(y)\,dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty K_n(x,y) f(x) g(y) dx dy < k_n \Big(\int_0^\infty f^2(x) dx \Big)^{\frac{1}{2}} \times \Big(\int_0^\infty g^2(y) dy \Big)^{\frac{1}{2}}, \tag{11}$$

where

$$K_n(x,y) = \frac{(-1)^n) \min^n\{x,y\}}{\max^{n+1}\{x,y\}},$$

$$k_n = \frac{(-4)^n}{2n+1}.$$

Proof. Applying Theorem 1.1.1 completes the proof. □

Remark 1.1.5. *Note that*

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{2n+1} = \pi.$$

So, inequalities (11) decompose infinitly inequality (1).

In 2002, Yang [6] gave a generalization of inequality (1) by introducing a parameter $\lambda > 0$ as:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda \sin(\frac{\pi}{p})} \Big\{ \int_0^\infty t^{(p-1)(1-\lambda)} f^p(t) dt \Big\}^{\frac{1}{p}} \times \Big\{ \int_0^\infty t^{(q-1)(1-\lambda)} g^q(t) dt \Big\}^{\frac{1}{q}}, \tag{12}$$

where the constant factor $\frac{\pi}{\lambda \sin(\frac{\pi}{p})}$ is the best possible.

In the following, ifinite decomposition of the above inequality is given.

Theorem 1.1.6. *Suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$ *and* f, $g \geqslant 0$ *such that*

$$0 < \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x)\,dx < \infty,$$

$$0 < \int_0^\infty y^{(q-1)(1-\lambda)} g^q(y)\,dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty K_n(x,y) f(x) g(y) dx dy < k_n \Big(\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx \Big)^{\frac{1}{p}} \times \Big(\int_0^\infty y^{(q-1)(1-\lambda)} g^q(y) dy \Big)^{\frac{1}{q}}, \tag{13}$$

where

$$K_n(x,y) = \frac{(-1)^n \min^n\{x^\lambda, y^\lambda\}}{\max^{n+1}\{x^\lambda, y^\lambda\}},$$

$$k_n = \frac{(-1)^n}{\lambda}\Big[\frac{1}{n+\frac{1}{p}} + \frac{1}{n+\frac{1}{q}}\Big].$$

The constant factors k_n *are the best possible.*

Proof. By applying the Holder's inequality, we have

$$\begin{aligned}\int_0^\infty \int_0^\infty K_n(x,y) f(x) g(y) dx dy &= \int_0^\infty \int_0^\infty \Big\{ K_n^{\frac{1}{p}}(x,y) \frac{x^{\frac{(1-\frac{\lambda}{q})}{q}}}{y^{\frac{(1-\frac{\lambda}{p})}{p}}} f(x) \Big\} \\ &\quad \times \Big\{ K_n^{\frac{1}{q}}(x,y) \frac{y^{\frac{(1-\frac{\lambda}{p})}{p}}}{x^{\frac{(1-\frac{\lambda}{q})}{q}}} g(y) \Big\} dx dy \\ &\leqslant M^{\frac{1}{p}} N^{\frac{1}{q}},\end{aligned}$$

where, by taking $y = xu^{\frac{1}{\lambda}}$,

$$M = \frac{(-1)^n}{\lambda} \int_0^\infty \Big\{ \int_0^\infty \frac{\min^n\{1,u\}}{\max^{n+1}\{1,u\}} u^{-\frac{1}{q}} du \Big\} x^{(p-1)(1-\lambda)} f^p(x) dx$$
$$= \frac{(-1)^n}{\lambda} \Big[\frac{1}{n+\frac{1}{p}} + \frac{1}{n+\frac{1}{q}} \Big] \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx.$$

Simillarly, one may obtain

$$N = \frac{(-1)^n}{\lambda} \int_0^\infty \Big\{ \int_0^\infty \frac{\min^n\{1,u\}}{\max^{n+1}\{1,u\}} u^{-\frac{1}{p}} du \Big\} x^{(q-1)(1-\lambda)} g^q(y) dy$$
$$= \frac{(-1)^n}{\lambda} \Big[\frac{1}{n+\frac{1}{p}} + \frac{1}{n+\frac{1}{q}} \Big] \int_0^\infty y^{(q-1)(1-\lambda)} g^q(y) dy.$$

If (13) takes the form of equality, then there exist constants A and B such that $A^2 + B^2 \neq 0$ and

$$Ax^{(p-1)(1-\lambda)} f^p(x) = By^{(q-1)(1-\lambda)} = c \quad (a.e).$$

In the case $A \neq 0$, we have

$$\int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) = \int_0^\infty \frac{c}{A} dx = \infty,$$

which is a contradiction.

If the constant factor k_n is not the best possible, then there exist a positive constant C (with $C < k_n$) and $a > 0$ such that

$$\int_a^\infty \int_a^\infty K_n(x,y) f(x) g(y) dx dy < C \Big(\int_a^\infty x^{(p-1)(1-\lambda)} f^p(x) dx \Big)^{\frac{1}{p}} \times \Big(\int_a^\infty y^{(q-1)(1-\lambda)} g^q(y) dy \Big)^{\frac{1}{q}}. \tag{14}$$

For $\varepsilon > 0$ small enough ($\varepsilon < \lambda(p-1)$) and $0 < b < a$, setting f_ε and g_ε as:

$$f_\varepsilon(x) = g_\varepsilon(x) = 0, \quad x \in (0,b);$$
$$f_\varepsilon(x) = x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{q}}, \quad g_\varepsilon(x) = x^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{p}}, \quad x \in [b,\infty).$$

Then,

$$\int_a^\infty\int_b^\infty K_n(x,y)f_\varepsilon(x)g_\varepsilon(y)dxdy = \int_a^\infty\int_b^\infty K_n(x,y)x^{-1-\frac{\varepsilon}{p}+\frac{\lambda}{q}}y^{-1-\frac{\varepsilon}{q}+\frac{\lambda}{p}}dxdy.$$

Let $b\to 0^+$, by (14) and taking $y = xu^{\frac{1}{\lambda}}$, we have

$$\int_a^\infty\int_0^\infty K_n(x,y)f_\varepsilon(x)g_\varepsilon(y)dxdy = \frac{1}{\lambda\varepsilon a}\Big(\int_0^\infty \frac{(-1)^n\min^n\{1,u\}}{\max^{n+1}\{1,u\}}u^{-\frac{1}{q}(1+\frac{\varepsilon}{\lambda})}du \leqslant \frac{C}{a\varepsilon}.$$

For $\varepsilon\to 0^+$, we have $k_n\leqslant C$. □

Remark 1.1.7. *Note that*

$$\sum_{n=0}^\infty k_n = \frac{1}{\lambda}[pF(1,\frac{1}{p},1+\frac{1}{p};-1)+qF(1,\frac{1}{q},1+\frac{1}{q};-1)] = \frac{1}{\lambda}\Big[\int_0^1\frac{dx}{x^{\frac{1}{q}}(x+1)}+\int_0^1\frac{dx}{x^{\frac{1}{p}}(x+1)}\Big].$$

By taking $x=\frac{1}{t}$ in the second integral, we have

$$\sum_{n=0}^\infty k_n = \frac{1}{\lambda}\Big[\int_0^1\frac{dx}{x^{\frac{1}{q}}(x+1)}+\int_1^\infty\frac{dx}{x^{\frac{1}{q}}(x+1)}\Big] = \frac{\pi}{\lambda\sin(\frac{\pi}{p})}.$$

This means that inequalities (13) infinitly decompose (12).

Yang [7] gave an extension of the Hardy-Hilbert's inequality as: Suppose that $\lambda > 2-\min\{p,q\}$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < \beta(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}) \times\Big\{\int_0^\infty x^{1-\lambda}f^p(x)dx\Big\}^{\frac{1}{p}} \times\Big\{\int_0^\infty y^{1-\lambda}g^q(y)dy\Big\}^{\frac{1}{q}}. \tag{15}$$

By the above methods, one my prove the following theorem:

Theorem 1.1.8. *suppose that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $f,\, g \geqslant 0$ *such that*

$$0 < \int_0^\infty x^{1-\lambda} f^p(x)\, dx < \infty,$$

$$0 < \int_0^\infty y^{1-\lambda} g^q(y)\, dy < \infty.$$

Then

$$\int_0^\infty \int_0^\infty K_n(x,y)dxdy < k_n \times \Big\{ \int_0^\infty x^{1-\lambda} f^p(x)dx \Big\}^{\frac{1}{p}} \times \Big\{ \int_0^\infty y^{1-\lambda} g^q(y)dy \Big\}^{\frac{1}{q}}, \tag{16}$$

where,

$$K_n(x,y) = \frac{(-1)^n \min^n\{x,y\}}{\max^{n+1}\{x,y\}(x+y)^{\lambda-1}} \ ,$$

and

$$k_n = (-1)^n \Big[\frac{1}{n+1-\frac{2-\lambda}{q}} F(\lambda-1, n+1-\frac{2-\lambda}{q}; n+2-\frac{2-\lambda}{q}; -1) + \frac{1}{n-1+\lambda+\frac{2-\lambda}{q}} F(\lambda-1, n-1+\lambda+\frac{2-\lambda}{q}; n+\lambda+\frac{2-\lambda}{q}; -1)\Big].$$

Remark 1.1.9. *Note that*

$$\begin{aligned}
\sum_{n=0}^\infty k_n &= \sum_{n=0}^\infty (-1)^n \Big[\int_0^1 \frac{x^{n-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx + \int_1^\infty \frac{x^{-n-1-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx \Big] \\
&= \lim_{\varepsilon\to 0^+} \sum_{n=0}^\infty (-1)^n \Big[\int_0^{1-\varepsilon} \frac{x^{n-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx + \int_{1-\varepsilon}^\infty \frac{x^{-n-1-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx \Big] \\
&= \lim_{\varepsilon\to 0^+} \Big[\int_0^{1-\varepsilon} \sum_{n=0}^\infty (-1)^n \frac{x^{n-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx + \int_{1-\varepsilon}^\infty \sum_{n=0}^\infty (-1)^n \frac{x^{-n-1-\frac{2-\lambda}{q}}}{(x+1)^{\lambda-1}} dx \Big] \\
&= \int_0^\infty \frac{x^{\frac{\lambda-2}{q}}}{(x+1)^\lambda} d = \beta(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}).
\end{aligned}$$

In other words, inequalities (16) infinitly decompose (15).
Note that by taking $\lambda = 2$, $p = q = 2$, *we have*

$$\sum_{n=0}^{\infty} \frac{2(-1)^n}{n+1} F(1, n+1; n+2; -1) = \beta(1,1) = 1.$$

Also, we have $F(1, n; n+1; -1) = 1 - na_n$, *where* $a_n = \int_0^1 \frac{x^n}{x+1} dx$. *One may prove that* $a_n + a_{n-1} = \frac{1}{n}$, $n = 1, 2, 3, ...$ *and* $a_0 = \ln 2$, $a_1 = 1 - \ln 2$. *Hence, we have*

$$a_n = (-1)^{n-1}[1 - \frac{1}{2} + \frac{1}{3} - \cdot + \frac{(-1)^{n-1}}{n}] - \ln 2.$$

This implies that

$$F(1, n+1; n+2; -1) = (n+1)\Big[(-1)^{n-1}\Big(1 - \frac{1}{2} + \frac{1}{3} - \cdot + \frac{(-1)^{n-1}}{n}\Big) - \ln 2\Big].$$

On the other hand, we have

$$\begin{aligned} a_n = \int_0^1 \frac{x^n}{x+1} dx &= \int_0^1 \sum_{m=0}^{\infty} (-1)^m x^{m+n} dx \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{n+m+1}. \end{aligned}$$

So,

$$\begin{aligned} F(1, n+1; n+2; -1) &= 1 - \sum_{m=0}^{\infty} \frac{(-1)^m (n+1)}{n+m+2} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m (n+1)}{n+m+1}. \end{aligned}$$

Hence, we have

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{m+n+1} = \frac{1}{2}.$$

1.2 Decomposition of the Hardy-Hilbert's operator

The Hilbert's integral operator is defined as: $T : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, for any $f \in L^2(\mathbb{R}^+)$, there exists a function $h = Tf \in L^2(\mathbb{R}^+)$ satisfying

$$h(y) = (Tf)(y) = \int_0^\infty \frac{f(x)}{x+y}dx, \quad y \in (0, \infty).$$

For any $g \in L^2(\mathbb{R}^+)$, an inner product of Tf and g is defined as follows:

$$(Tf, g) = \int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}dx \Big) g(y)dy = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy.$$

Setting the norm of f as $\|f\|_2 = \{\int_0^\infty f^2(x)dx\}^{\frac{1}{2}}$, if $\|f\|_2, \|g\|_2 > 0$, then the Hardy-Hilbert's inequality may be rewritten as

$$(Tf, g) < \pi \|f\|_2 \|g\|_2.$$

It follow that $\|T\| = \pi$.
Simillarly, the Hardy-Hilbert's integral operator $T_p : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, is defined as

$$(T_p f)(y) = \int_0^\infty \frac{f(x)}{x+y}dx, \quad y \in (0, \infty).$$

Also, for any non-negative $g \in L^q(\mathbb{R}^+)$, the formal inner product of $T_p f$ and g is defined as follows;

$$(T_p f, g) = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy.$$

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $L^r(\mathbb{R}^+) \quad (r = p, q)$ are real normed linear spaces and $k(x, y)$ is a non-negative symmetric measurable function in $\mathbb{R})^2_+$ satisfying

$$\int_0^\infty k(x, t)(\frac{x}{t}^{\frac{1}{r}})dt = k_0(r) \in \mathbb{R} \quad (x > 0).$$

Yang ([6]) defined an integral operator as $T : L^r(\mathbb{R}^+) \to L^r(\mathbb{R}^+) \quad (r = p, q)$, for any $f(\geqslant 0) \in L^p(\mathbb{R}^+$, there exists $h = Tf \in L^p(\mathbb{R}^+)$, such that

$$(Tf)(y) = h(y) := \int_0^\infty k(x, y)f(x)dx \quad (y > 0).$$

Or, for any $g(\geqslant 0) \in L^q(\mathbb{R}^+)$, there exists $\tilde{h} = Tg \in L^q(\mathbb{R}^+)$, such that

$$(Tg)(x) = \tilde{h}(x) := \int_0^\infty k(x,y)g(y)dy \quad (x > 0).$$

Yang proved that the operator T is bounded with $\|T\| \leqslant k_0(p)$. He proved that if $\varepsilon > 0$, is small enough and the integral $\int_0^\infty k(x,t)(\frac{x}{t})^{\frac{1+\varepsilon}{r}} dt$ $(r = p, q; x > 0)$ is convergent to a constant $k_\varepsilon(p)$ independent of x satisfying $k_\varepsilon(p) = k_0(p) + o(1)(\varepsilon \to 0^+)$, then $\|T\| = k_0(p)$. If $\|T\| > 0$, $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$, $\|f\|_p, \|g\|_q > 0$, then

$$(Tf, g) < \|T\| \|f\|_p \|g\|_q.$$

By the above notations, taking

$$K_1(x,y) = \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} u^\lambda(x)}{u^{2\lambda}(x) - v^{2\lambda}(y)},$$

$$K_2(x,y) = \frac{\left(\ln\left(\frac{u(x)}{v(y)}\right)\right)^{2m} v^\lambda(y)}{u^{2\lambda}(x) - v^{2\lambda}(y)},$$

$$K(x,y) = K_1(x,y) + K_2(x,y),$$

and $T_M : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+) \quad (M = K, K_1, K_2)$, for any $f(\geqslant 0) \in L^p(\mathbb{R}^+)$,

$$(T_M f)(y) = \int_0^\infty M(x,y) f(x) dx,$$

we have $T_K = T_{K_1} + T_{K_2}$ and by Theorems 1.1 and 1.2,

$$\|T_K\| = \|T_{K_1}\| + \|T_{K_2}\|.$$

Note that a Banach space $(V, \|.\|)$ is strictly convex if and only if $x \neq 0$ and $y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ together imply that $x = cy$ for some constant $c > 0$. So, by the above decomposition, the space of bounded linear operators is not strictly convex.

By taking

$$K(x,y) = \frac{1}{x^\lambda + y^\lambda},$$

$$K_n(x,y) = \frac{(-1)^n \min^n\{x^\lambda, y^\lambda\}}{\max^{n+1}\{x^\lambda, y^\lambda\}},$$

we have $T_K = \sum_{n=0}^{\infty} T_{K_n}$ and Remark 1.1.7 says that

$$\|T_K\| = \sum_{n=0}^{\infty} \|T_{K_n}\|.$$

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