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Some properties of ℓEQ-algebras

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Abstract. In this paper, first we investigate some properties of $(\ell EQ)EQ$-algebras. Furthermore, we study some characterizations of ℓEQ-algebras. It is proved that every linearly ordered EQ-algebra and every EQ-algebra with three and four elements are ℓEQ-algebras. Finally we characterize 5-element ℓEQ-algebras and find EQ-algebras that are not ℓEQ-algebras.

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1. Introduction

Fuzzy logic can be defined as a means of studying uncertainty and approximate reasoning [11,12]. For instance residuated logic[1], BL-logic [6], Lukasiewicz-logic [1] and MTL-logic [4] are some of the fuzzy logics which are adopted to generalize the Boolean truth functions on $\{0,1\}$.

Fuzzy type theory which has a fuzzy equality connective has been developed as a counterpart of the classical higher-order logic [9].

A specific algebra known as EQ-algebra has been introduced by V. Novák whose goal is to achieve the algebra of truth values for fuzzy type theory(FTT) [7]. Residuated lattices are special cases of EQ-algebras.

An EQ-algebra has three binary operations (meet, multiplication and a fuzzy equality) and a top element. A binary operation implication is derived from

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fuzzy equality by $x \to y = (x \wedge y) \sim x$. This implication and multiplication are not closely tied by the adjunction any more, therefore EQ-algebras generalize residuated lattices.

A special class of EQ-algebras which is called ℓEQ-algebras is defined in [10] and properties of them are investigated in [2].

In studies conducted on EQ-algebras, we were looking for those EQ-algebras which are not ℓEQ-algebras. In different finite examples we couldn‘t find such example. This results in this research in which we investigate ℓEQ-algebras with order of 3,4 and 5. Finally, in this article we found an EQ-algebra with order of 5 which is not an ℓEQ-algebra. We hope these characterizations open the door of finding more ℓEQ-algebras.

This paper is organized as follows: In Section 2, we review the basic definitions, properties and theorems. In Section 3, we give some properties of EQ-algebra with three, four and five elements. In Section 4, we show that every linearly ordered EQ-algebra is an ℓEQ-algebra and also EQ-algebra with three and four elements are ℓEQ-algebra. Finally we give some characterizations of a 5-element EQ-algebra.

2. Preliminaries

An algebra $\xi = (E, \wedge, \otimes, \sim, 1)$ of type $(2,2,2,0)$ is called an EQ-algebra where for all $a, b, c, d \in E$:

$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We set $a \leqslant b$ iff $a \wedge b = a$,

$(E2)$ $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leqslant b$,

$(E3)$ $a \sim a = 1$, (reflexivity axiom)

$(E4)$ $((a \wedge b) \sim c) \otimes (d \sim a) \leqslant c \sim (d \wedge b)$, (substitution axiom)

$(E5)$ $(a \sim b) \otimes (c \sim d) \leqslant (a \sim c) \sim (b \sim d)$, (congruence axiom)

$(E6)$ $(a \wedge b \wedge c) \sim a \leqslant (a \wedge b) \sim a$, (monotonicity axiom)

$(E7)$ $a \otimes b \leqslant a \sim b$, for all $a, b, c \in E$.[2,10]

The binary operations $\wedge$, $\otimes$ and $\sim$ are called meet, multiplication and fuzzy equality, respectively.

We introduce the unary and binary operations $\widetilde{\ }$ and $\to$ for $a, b \in E$, by

$$\widetilde{a} = a \sim 1 \text{ and } a \to b = (a \wedge b) \sim a$$

If E is a nonempty set with three binary operations $\wedge, \otimes, \sim$ such that $(E, \wedge, 1)$ is a $\wedge$-semilattice, $(E, \otimes, 1)$ is a monoid and $\otimes$ is isotone with respect to $\leqslant$, then $(E, \otimes, \wedge, \sim, 1)$ is an EQ-algebra, where $a \sim b = 1$, for all $a, b \in E$.

Lemma 2.1. [2, 10] *Let ξ be an EQ-algebra. Then the following properties hold for all $a, b, c, d \in E$:*

(e_1) $a \sim b = b \sim a$,
(e_2) $(a \sim b) \otimes (b \sim c) \leqslant (a \sim c)$,
(e_3) $(a \to b) \otimes (b \to c) \leqslant (a \to c)$ *and* $(b \to c) \otimes (a \to b) \leqslant (a \to c)$,
(e_4) $a \sim d \leqslant (a \wedge b) \sim (d \wedge b)$,
(e_5) $(a \sim d) \otimes ((a \wedge b) \sim c) \leqslant (d \wedge b) \sim c$,
(e_6) $(a \wedge b) \sim a \leqslant (a \wedge b \wedge c) \sim (a \wedge c)$,
(e_7) $a \otimes b \leqslant a \wedge b \leqslant a, b$,
(e_8) $b \leqslant \widetilde{b} \leqslant a \to b$,
(e_9) $(a \to b) \otimes (b \to a) \leqslant a \sim b \leqslant (a \to b) \wedge (b \to a)$,
(e_{10}) *If* $a \leqslant b$, *then* $a \to b = 1$, $b \to a = a \sim b$, $\widetilde{a} \leqslant \widetilde{b}$, $c \to a \leqslant c \to b$ *and* $b \to c \leqslant a \to c$,
(e_{11}) *If* $a \leqslant b \leqslant c$, *then* $a \sim c \leqslant a \sim b$ *and* $a \sim c \leqslant b \sim c$,
(e_{12}) $a \otimes (a \sim b) \leqslant \widetilde{b}$.

Definition 2.2. [2, 10] *Let ξ be an EQ-algebra. We say that it is*

(i) *good, if for all $a \in E$, $\widetilde{a} = a$,*

(ii) *lattice-ordered EQ-algebra if it is a lattice,*

(iii) *lattice EQ-algebra(ℓEQ-algebra) if it is a lattice-ordered EQ-algebra in which the following substitution axiom holds, for all $a, b, c, d \in E$:*

$$(E8)\ ((a \vee b) \sim c) \otimes (d \sim a) \leqslant c \sim (d \vee b).$$

(iv) *prelinear if for all $a, b \in E$, 1 is the unique upper bound in E of the set $\{(a \to b), (b \to a)\}$.*

It is clear, every finite EQ-algebra is a lattice-ordered.
The following theorem gives some properties of ℓEQ-algebras:

Theorem 2.3. [2, 3, 10] *Let ξ be an ℓEQ-algebra. For all $a, b, c, d \in E$ the following hold:*
(a) $a \to b = (a \vee b) \to b = (a \vee b) \sim b$,
(b) $(a \sim d) \otimes ((a \vee b) \sim c) \leqslant ((d \vee b) \sim c)$,
(c) $(a \sim b) \otimes (c \sim d) \leqslant (a \vee c) \sim (b \vee d)$,
(d) $a \to b \leqslant (a \vee c) \to (b \vee c)$.

Theorem 2.4. [2] *The following properties are equivalent in every lattice-ordered EQ-algebra ξ:*
(a) *ξ is an ℓEQ-algebra,*
(b) *ξ satisfies, for all $a, b, c \in E$:*

$$a \sim b \leqslant (a \vee c) \sim (b \vee c).$$

Theorem 2.5. [2] Every prelinear and good EQ-algebra ξ is an ℓEQ-algebra, whereby the join operation is given by for all $a, b \in E$:

$$a \vee b = ((a \to b) \to b) \wedge ((b \to a) \to a).$$

3. Some Properties of EQ-Algebras

In this section, we have the following new properties of EQ-algebras.

Theorem 3.1. *Let ξ be an EQ-algebra. Then the following hold for all $a, b, c, d \in E$:*
(a) $((a \wedge b) \to c) \otimes (d \sim a) \leqslant (d \wedge b) \to c,$
(b) $(d \to c) \otimes (a \sim d) \leqslant a \to c.$

Proof. (a) By axiom $(E6)$ and using (e_1), (e_2) and (e_4) we obtain

$$\begin{aligned} ((a \wedge b) \to c) \otimes (d \sim a) &\leqslant ((a \wedge b \wedge c) \sim (a \wedge b)) \otimes ((d \wedge b) \sim (a \wedge b)) \\ &\leqslant (a \wedge b \wedge c) \sim (d \wedge b) \\ &\leqslant (a \wedge b \wedge c \wedge d) \sim (d \wedge b) \\ &\leqslant (c \wedge d \wedge b) \sim (d \wedge b) \\ &= (d \wedge b) \to c. \end{aligned}$$

(b) Using (a), we find

$$\begin{aligned} (d \to c) \otimes (a \sim d) &= ((d \wedge 1) \to c) \otimes (a \sim d) \\ &\leqslant (a \wedge 1) \to c \\ &= a \to c. \quad \square \end{aligned}$$

By the two following theorems, we investigate some characterizations of 4 and 5-element prelinear EQ-algebras:

Theorem 3.2. *Every non-linearly ordered EQ-algebra with four elements is a prelinear EQ-algebra.*

Proof. Let $E = \{0, a, b, 1\}$ be a non-linearly ordered EQ-algebra. Then $0 \leqslant a, b \leqslant 1$ and so by $a \leqslant b \to a$ and $b \leqslant a \to b$, we can easily check that $(a \to b) \vee (b \to a) = 1$, for all $a, b \in E$. $\square$

Theorem 3.3. *Let $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ be a non-linear ordered 5-element EQ-algebra. Then under each of the following conditions ξ is a prelinear EQ-algebra:*

(i) $0 \leqslant c \leqslant a, b \leqslant 1$ *and* a, b *are incomparable,*
(ii) $0 \leqslant a \leqslant c \leqslant 1$, $0 \leqslant b \leqslant 1$ *and elements* $\{a, b\}$ *and* $\{b, c\}$ *are pairwise incomparable,*
(iii) $0 \leqslant a, b, c \leqslant 1$ *and elements* $\{a, b\}$, $\{a, c\}$ *and* $\{b, c\}$ *are pairwise incomparable.*

Proof. *(i)* We show that $(a \to b) \vee (b \to a) = 1$, for all $a, b \in \xi$. By $c \leqslant a$ and $c \leqslant b$, we imply $c \to a = 1$ and $c \to b = 1$. So $(c \to a) \vee (a \to c) = 1$ and $(c \to b) \vee (b \to c) = 1$. By $1 = a \vee b \leqslant (a \to b) \vee (b \to a)$, we get $(a \to b) \vee (b \to a) = 1$. Then the proof is complete.
The proof of *(ii)*, *(iii)* is similar to part *(i)*. □

Lemma 3.4. *Let* $\xi = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra with the bottom element 0. If* $a \wedge b = 0$, $a \sim 1 = a$ *and* $b \sim 1 = b$, *for* $a, b \in E$, *then* $0 \sim 1 = 0$.

Proof. We have $0 \sim 1 \leqslant a \sim 1 = a$ and $0 \sim 1 \leqslant b \sim 1 = b$, (by (e_{11})). Then we get $0 \sim 1 \leqslant a \wedge b = 0$, hence $0 \sim 1 = 0$. □

Lemma 3.5. *Let* $\xi = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra with the bottom element 0. If* $b \sim 1 = c \sim 1 = 1$ *and* $b \wedge c = 0$, *for* $b, c \in E$, *then* $a \sim 1 = 1$, *for every* $a \in E$.

Proof. Using (e_4), we have $1 = c \sim 1 \leqslant (c \wedge b) \sim (1 \wedge b) = 0 \sim b$, thus $0 \sim b = 1$. Now $1 = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1$ implies $0 \sim 1 = 1$. Hence by $1 = 0 \sim 1 \leqslant a \sim 1$, we get that $a \sim 1 = 1$, for every $a \in E$. □

Lemma 3.6. *Let* $\xi = (E, \wedge, \otimes, \sim, 1)$ *be an EQ-algebra. If* $a \sim 1 = 1$, $b \sim 1 = c$, *then* $a \sim b = c$, *for* $a, b, c \in E$.

Proof. From (e_1) and (e_2), we obtain

$$\begin{aligned} c = 1 \otimes c &= (a \sim 1) \otimes (b \sim 1) \leqslant a \sim b \\ &= (a \sim b) \otimes (a \sim 1) \leqslant b \sim 1 = c. \quad \square \end{aligned}$$

Theorem 3.7. *Let* $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ *be a non-linearly ordered 5-element EQ-algebra where* $0 \leqslant a, b \leqslant c \leqslant 1$ *and* a, b *are incomparable. If* $b \sim c = b$, *then* $0 \sim a = b$.

Proof. From $b = b \sim c \leqslant (b \wedge a) \sim (c \wedge a) = 0 \sim a$ and $0 \sim c \leqslant b \sim c = b$, we get that $0 \sim a \in \{b, c, 1\}$ and $0 \sim c \in \{0, b\}$, respectively. Assume $0 \sim a = 1$. Now if $0 \sim c = 0$, then $a \leqslant a \sim 1 \leqslant a \sim c = 1 \otimes (a \sim c) = (0 \sim a) \otimes (a \sim c) \leqslant 0 \sim c = 0$, which is a contradiction. If $0 \sim c = b$, then $b = 0 \sim c \leqslant a \sim c$. On the other hand from (e_{11}), we have $a = 1 \otimes a \leqslant 1 \sim a \leqslant a \sim c$, so $c = a \vee b \leqslant a \sim c$, implies $a \sim c \in \{c, 1\}$. If $a \sim c = c$, then $c = 1 \otimes c = (0 \sim a) \otimes (a \sim c) \leqslant 0 \sim c = b$, that is a contradiction. If $a \sim c = 1$, then $1 = a \sim c \leqslant (a \wedge b) \sim (c \wedge b) = 0 \sim b$, implies $0 \sim b = 1$. By $1 = (a \sim 0) \otimes (0 \sim b) \leqslant a \sim b$, we get that $a \sim b = 1$. Therefore $1 = (c \sim a) \otimes (a \sim b) \leqslant c \sim b$, that is $b \sim c = 1$, which is a contradiction. If $0 \sim a = c$, then similar to the proof of the above way, we get a contradiction. Therefore $0 \sim a = b$. □

The following theorem gives some properties of ℓEQ-algebras:

Theorem 3.8. *The following properties hold in every ℓEQ-algebra ξ, for all $a, b, c, d \in E$:*
(a) $a \sim d \leqslant ((a \vee b) \sim c) \sim ((d \vee b) \sim c)$,
(b) $(c \to (a \vee b)) \otimes (d \sim a) \leqslant c \to (d \vee b)$,
(c) $((a \vee b) \to c) \otimes (d \sim a) \leqslant (d \vee b) \to c$,
(d) $(c \to (a \vee b)) \otimes (a \to d)^2 \leqslant c \to (d \vee b)$.

Proof. (a) By axiom $(E5)$ and Theorem 2.4, we obtain

$$\begin{aligned} a \sim d &\leqslant (a \vee b) \sim (d \vee b) \\ &= ((a \vee b) \sim (d \vee b)) \otimes (c \sim c) \\ &\leqslant ((a \vee b) \sim c) \sim ((d \vee b) \sim c). \end{aligned}$$

(b) Using Theorem 2.4 and axiom $(E4)$, we get

$$\begin{aligned} (c \to (a \vee b)) \otimes (d \sim a) &= ((c \wedge (a \vee b)) \sim c) \otimes (d \sim a) \\ &\leqslant ((c \wedge (a \vee b)) \sim c) \otimes ((d \vee b) \sim (a \vee b)) \\ &\leqslant ((c \wedge (d \vee b)) \sim c) \\ &= c \to (d \vee b). \end{aligned}$$

(c) By Theorem 2.3(a) and $(E8)$, we find

$$\begin{aligned} ((a \vee b) \to c) \otimes (d \sim a) &= ((a \vee b \vee c) \sim c) \otimes (d \sim a) \\ &\leqslant (d \vee b \vee c) \sim c \\ &= (d \vee b) \to c. \end{aligned}$$

(d) By $(E8)$ and Theorems 2.3 and 2.4(a), we obtain

$$\begin{aligned} (c \to (a \vee b)) \otimes (a \to d)^2 &= ((c \vee a \vee b) \sim (a \vee b)) \otimes ((a \vee d) \sim d) \otimes (a \to d) \\ &\leqslant ((c \vee a \vee b \vee d) \sim (a \vee b \vee d)) \otimes ((a \vee d) \sim d) \otimes (a \to d) \\ &\leqslant ((c \vee b \vee d) \sim (a \vee b \vee d)) \otimes ((a \vee d) \sim d) \\ &\leqslant (c \vee b \vee d) \sim (d \vee b) = c \to (d \vee b). \quad \square \end{aligned}$$

4. Some Properties of ℓEQ-Algebras with Three, Four and Five Elements

In this section, we characterize ℓEQ-algebras with order of 3,4 and 5.

Theorem 4.1. *Every linearly ordered EQ-algebra is an ℓEQ-algebra.*

Proof. Let E be a linearly ordered EQ-algebra. Then E is a lattice-ordered. We show that E is an ℓEQ-algebra. By Theorem 2.4, it is sufficient to show that $a \sim b \leqslant (a \vee c) \sim (b \vee c)$, for $a, b, c \in E$.
Let $a, b, c \in E$ and $a \leqslant b$. Then $b \leqslant c$, implies $a \leqslant b \leqslant c$, so $a \sim b \leqslant 1 = (a \vee c) \sim (b \vee c)$. If $c \leqslant b$, we have $a \leqslant c$ or $c \leqslant a$, then $a \leqslant c \leqslant b$ or $c \leqslant a \leqslant b$. Therefore by (e_{11}), $a \sim b \leqslant c \sim b = (a \vee c) \sim (b \vee c)$ or $a \sim b \leqslant a \sim b = (a \vee c) \sim (b \vee c)$. $\square$

Since every EQ-algebra with three elements is a linearly ordered EQ-algebra, so we have the following corollary:

Corollary 4.2. *Every EQ-algebra with three elements is an ℓEQ-algebra.*

Theorem 4.3. *Every non-linearly ordered EQ-algebra $\xi = (E, \wedge, \otimes, \sim, 1)$ with four elements $\{0, a, b, 1\}$ is an ℓEQ-algebra.*

Proof. According to Theorem 2.4, it is sufficient to show that

$$a \sim b \leqslant (a \vee c) \sim (b \vee c), for \quad a, b, c \in E \qquad (I)$$

By properties of EQ-algebras, all possible inequalities of the (I) hold, except for the following inequalities:
(a) $0 \sim a \leqslant (0 \vee b) \sim (a \vee b) = b \sim 1$,
(b) $0 \sim b \leqslant (0 \vee a) \sim (b \vee a) = a \sim 1$,
(c) $a \sim b \leqslant (a \vee a) \sim (b \vee a) = a \sim 1$,
(d) $a \sim b \leqslant (a \vee b) \sim (b \vee b) = 1 \sim b$.

Now we show that the above inequalities also hold. We have $a \otimes 1 \leqslant a \sim 1$ and $b \otimes 1 \leqslant b \sim 1$, so $a \sim 1 \in \{a, 1\}$ and $b \sim 1 \in \{b, 1\}$. We consider the following cases:

Case(1). $a \sim 1 = 1$, $b \sim 1 = 1$. In this case the inequalities $(a) - (d)$ hold.

Case(2). $a \sim 1 = a$, $b \sim 1 = 1$. By $b \sim 1 = 1$, the inequalities (a) and (d) hold. Using Lemma 3.6, we get $a \sim b = a$. Now from $a = a \sim b \leqslant (a \wedge b) \sim (b \wedge b) = 0 \sim b$(by (e_4)), we obtain $0 \sim b \in \{a, 1\}$. If $0 \sim b = a$, then the inequalities (b) and (c) hold. If $0 \sim b = 1$, then $0 \sim 1 = 1$ because $1 = (0 \sim b) \otimes (b \sim 1) \leqslant (0 \sim 1)$. From (e_{11}), we have $1 = 0 \sim 1 \leqslant a \sim 1$. Thus $a \sim 1 = 1$, that is a contradiction.

Case(3). $a \sim 1 = 1$, $b \sim 1 = b$. The proof of this case is similar to the case(2).

Case(4). $a \sim 1 = a$, $b \sim 1 = b$. Using Lemma 3.4, we get $0 \sim 1 = 0$ and so E is a good EQ-algebra. Moreover by Theorem 3.2, E is a prelinear EQ-algebra. Thus from Theorem 2.5, E is an ℓEQ-algebra. $\square$

Theorem 4.4. *Every non-linearly ordered 5-element EQ-algebra $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ in which $0 \leqslant a \leqslant c \leqslant 1$ and elements $\{a, b\}$ and $\{b, c\}$ are pairwise incomparable, is an ℓEQ-algebra.*

Proof. We show that the relation (I) hold for $a, b, c \in E$. By properties of EQ-algebras, all possible inequalities of the relation (I) hold, except for the following inequalities:
(a) $0 \sim a \leqslant (0 \vee b) \sim (a \vee b) = b \sim 1$,
(b) $0 \sim b \leqslant (0 \vee a) \sim (b \vee a) = a \sim 1$,
(c) $0 \sim c \leqslant (0 \vee b) \sim (c \vee b) = b \sim 1$,
(d) $0 \sim b \leqslant (0 \vee c) \sim (b \vee c) = c \sim 1$,
(e) $a \sim b \leqslant (a \vee a) \sim (b \vee a) = a \sim 1$,
(f) $a \sim b \leqslant (a \vee b) \sim (b \vee b) = 1 \sim b$,
(g) $a \sim b \leqslant (a \vee c) \sim (b \vee c) = c \sim 1$,
(h) $b \sim c \leqslant (b \vee a) \sim (c \vee a) = 1 \sim c$,
(i) $b \sim c \leqslant (b \vee b) \sim (c \vee b) = b \sim 1$.

We show that the above inequalities $(a) - (d)$ hold.

We have $a \otimes 1 \leqslant a \sim 1$, $b \otimes 1 \leqslant b \sim 1$ and $c \otimes 1 \leqslant c \sim 1$. Then $a \sim 1 \in \{a, c, 1\}$, $b \sim 1 \in \{b, 1\}$ and $c \sim 1 \in \{c, 1\}$. We consider the following cases:

Case(1). $a \sim 1 = a$, $b \sim 1 = b$ and $c \sim 1 = c$. By Lemma 3.4, we get $0 \sim 1 = 1$, hence E is a good EQ-algebra. Using Theorem 3.3(ii), E is a prelinear EQ-algebra. Therefore E is an ℓEQ-algebra(by Theorem 2.5).

Case(2). $a \sim 1 = a$, $b \sim 1 = b$ and $c \sim 1 = 1$. We have $1 = c \sim 1 \leqslant (c \wedge b) \sim (1 \wedge b) = 0 \sim b$, thus $0 \sim b = 1$. Then $b = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1 \leqslant b \sim 1 = b$, that is $0 \sim 1 = b$. So $b = 0 \sim 1 \leqslant a \sim 1 = a$, which is a contradiction.

Case(3). $a \sim 1 = a$, $b \sim 1 = 1$ and $c \sim 1 = c$. By $1 = b \sim 1 \leqslant (b \wedge c) \sim (1 \wedge c) = 0 \sim c$, we get that $0 \sim c = 1$. Now $0 \leqslant a \leqslant c$ implies $1 = 0 \sim c \leqslant a \sim c$, that is $a \sim c = 1$. We get a contradiction, because $c = 1 \otimes c = (a \sim c) \otimes (c \sim 1) \leqslant a \sim 1 = a$.

Case(4). $a \sim 1 = a$, $b \sim 1 = 1$ and $c \sim 1 = 1$. By hypothesis, we have $b \wedge c = 0$, then from Lemma 3.5, we get that $a \sim 1 = 1$, that is a contradiction.

Case(5). $a \sim 1 = 1$, $b \sim 1 \in \{b, 1\}$ and $c \sim 1 = c$. By $a \leqslant c \leqslant 1$ and (e_{11}), we have $1 = a \sim 1 \leqslant c \sim 1 = c$, which is a contradiction.

Case(6). $a \sim 1 = 1$, $b \sim 1 = 1$, and $c \sim 1 = 1$. In this case the inequalities $(a) - (d)$ hold, hence E is an ℓEQ-algebra.

Case(7). $a \sim 1 = 1$, $b \sim 1 = b$ and $c \sim 1 = 1$. From $a \sim 1 = 1$ and $c \sim 1 = 1$, the inequalities (b) and (d) hold. By Lemma 3.6, we get $a \sim b = b$. Now $b = a \sim b \leqslant (a \wedge a) \sim (b \wedge a) = a \sim 0$ implies $a \sim 0 \in \{1, b\}$. If $a \sim 0 = 1$, then we get $0 \sim 1 = 1$, because $1 = (a \sim 0) \otimes (a \sim 1) \leqslant 0 \sim 1$. So $1 = 0 \sim 1 \leqslant b \sim 1 = b$, which is a contradiction, therefore $0 \sim a = b$. Hence the inequality (a) holds. Similar to the above we get $0 \sim c = b$, thus the inequality (c) holds.

Case(8). $a \sim 1 = c$, $b \sim 1 = b$ and $c \sim 1 = c$. From $b \wedge c = 0$ and $0 \sim 1 \leqslant a \sim 1 = c$, $0 \sim 1 \leqslant b \sim 1 = b$, we get $0 \sim 1 = 0$. By $c = a \sim 1 \leqslant (a \wedge b) \sim (1 \wedge b) = 0 \sim b$, we obtain $0 \sim b \in \{1, c\}$. If $0 \sim b = 1$, then $b = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1 = 0$, that is a contradiction. Therefore $0 \sim b = c$, hence the inequalities (b) and (d) hold. From $b = b \sim 1 \leqslant (b \wedge c) \sim (1 \wedge c) = 0 \sim c$, we get $0 \sim c \in \{b, 1\}$. If $0 \sim c = 1$, then $c = (c \sim 0) \otimes (c \sim 1) \leqslant 0 \sim 1 = 0$ implies $c = 0$, that is a contradiction. Thus $0 \sim c = b$. By $b = 0 \sim c \leqslant 0 \sim a$, we obtain $0 \sim a \in \{b, 1\}$. If $0 \sim a = 1$, then $c = (0 \sim a) \otimes (a \sim 1) \leqslant 0 \sim 1 = 0$, which is a contradiction. Hence $0 \sim a = b$ and so the inequalities (a) and (c) hold.

Case(9). $a \sim 1 = c$, $b \sim 1 = b$ and $c \sim 1 = 1$. In this case, we get a contradiction because, by $1 = c \sim 1 \leqslant (c \wedge b) \sim (1 \wedge b) = 0 \sim b$, we obtain $0 \sim b = 1$. Thus $b = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1 \leqslant b \sim 1 = b$, so $0 \sim 1 = b$. On the other hand from (e_{11}), we obtain $0 \sim 1 \leqslant b \wedge c = 0$ that is $0 \sim 1 = 0$.

Case(10). $a \sim 1 = c$, $b \sim 1 = 1$ and $c \sim 1 = c$. By $b \sim 1 = 1$, the inequalities (a), (c) hold. Now by (e_4), $c = a \sim 1 \leqslant (a \wedge b) \sim (1 \wedge b) = 0 \sim b$ implies $0 \sim b \in \{1, c\}$. If $0 \sim b = 1$, then $1 = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1$ that is $0 \sim 1 = 1$. On the other hand $1 = 0 \sim 1 \leqslant a \sim 1 = c$, so $c = 1$, which is a contradiction. Hence $0 \sim b = c$ and therefore the inequalities (b) and (d) hold.

Case(11). $a \sim 1 = c$, $b \sim 1 = 1$ and $c \sim 1 = 1$. Similar to the proof of case(4), we get a contradiction.

Now we prove the inequality (e). By (e_4), we have $a \sim b \leqslant (a \wedge b) \sim (b \wedge b) = 0 \sim b$, and so by (b) we get that $a \sim b \leqslant a \sim 1$.

Similar to the proof of (e), we can easily check that the inequalities $(f) - (i)$ hold, by (e_4) and the inequalities $(a) - (d)$. □

Theorem 4.5. *Every non-linearly ordered 5-element EQ-algebra $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ in which $0 \leqslant c \leqslant a, b \leqslant 1$ and a, b are incomparable, is an ℓEQ-algebra.*

Proof. We show that the relation (I) hold, for $a, b, c \in E$. By properties of

EQ-algebras, all possible inequalities of the relation (I) hold, except for the following inequalities:
(a) $0 \sim a \leqslant (0 \vee b) \sim (a \vee b) = b \sim 1$,
(b) $0 \sim b \leqslant (0 \vee a) \sim (b \vee a) = a \sim 1$,
(c) $b \sim c \leqslant (b \vee a) \sim (c \vee a) = 1 \sim a$,
(d) $a \sim c \leqslant (a \vee b) \sim (c \vee b) = 1 \sim b$,
(e) $a \sim b \leqslant (a \vee a) \sim (b \vee a) = a \sim 1$,
(f) $a \sim b \leqslant (a \vee b) \sim (b \vee b) = 1 \sim b$.

We show that the above inequalities $(a) - (d)$ hold. we can easily check that the inequalities $(e), (f)$ can be obtained from (e_4) and the inequalities (c) and (d).

We have $a \otimes 1 \leqslant a \sim 1$ and $b \otimes 1 \leqslant b \sim 1$, then $a \sim 1 \in \{a, 1\}$ and $b \sim 1 \in \{b, 1\}$. We consider the following cases:

Case(1). $a \sim 1 = 1$, $b \sim 1 = 1$. In this case all of the inequalities hold.

Case(2). $a \sim 1 = 1$, $b \sim 1 = b$. From $a \sim 1 = 1$, the inequalities (b) and (c) hold. Also by Lemma 3.6, we obtain $a \sim b = b$. By $b = a \sim b \leqslant (a \wedge a) \sim (b \wedge a) = a \sim c$, we get that $a \sim c \in \{b, 1\}$. Now if $a \sim c = 1$, then $1 = (c \sim a) \otimes (a \sim 1) \leqslant c \sim 1$, that is $c \sim 1 = 1$. On the other hand, $c \leqslant b \leqslant 1$ implies $1 = c \sim 1 \leqslant b \sim 1$ (by (e_{11})), thus $b \sim 1 = 1$, which is a contradiction. Hence $a \sim c = b$ and the inequality (d) holds. From $0 \leqslant c \leqslant a$, we obtain $0 \sim a \leqslant a \sim c = b = b \sim 1$, therefore the inequality (a) holds.

Case(3). $a \sim 1 = a$, $b \sim 1 = 1$. The proof is similar to the proof of case(2).

Case(4). $a \sim 1 = a$, $b \sim 1 = b$. By $c \leqslant a, b \leqslant 1$, we obtain $c \sim 1 \leqslant a \sim 1 = a$ and $c \sim 1 \leqslant b \sim 1 = b$. Therefore $1 \sim c \leqslant a \wedge b = c \leqslant 1 \sim c$, that is $1 \sim c = c$. From $0 \sim 1 \leqslant a \sim 1 = a$ and $0 \sim 1 \leqslant b \sim 1 = b$, we get $0 \sim 1 \in \{0, c\}$. Let $0 \sim 1 = 0$. Then E is a good EQ-algebra. Also by Theorem 3.3(i), E is a prelinear EQ-algebra, hence E is an ℓEQ-algebra(by Theorem 2.5).

Now let $0 \sim 1 = c$. Then by $b = b \sim 1 \leqslant (b \wedge a) \sim (1 \wedge a) = c \sim a$, we get that $c \sim a \in \{1, b\}$. If $c \sim a = 1$, then $a = 1 \otimes a = (c \sim a) \otimes (a \sim 1) \leqslant c \sim 1 = c$, that is a contradiction. Thus $a \sim c = b$ and the inequality (d) holds. Now $a = a \sim 1 \leqslant (a \wedge b) \sim (1 \wedge b) = c \sim b$, implies $c \sim b \in \{1, a\}$. If $c \sim b = 1$, we get a contradiction, because $b = 1 \otimes b = (c \sim b) \otimes (b \sim 1) \leqslant c \sim 1 = c$. Thus $c \sim b = a$ and so the inequality (c) holds. From $0 \leqslant c \leqslant a, b$, we obtain $0 \sim a \leqslant c \sim a = b = b \sim 1$ and $0 \sim b \leqslant c \sim b = a = a \sim 1$, therefore the inequalities (a) and (b) hold. $\square$

Theorem 4.6. *Every non-linearly ordered 5-element EQ-algebra* $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ *in which* $0 \leqslant a, b, c \leqslant 1$*, and elements* $\{a, b\}$*,*

$\{a,c\}$ *and* $\{b,c\}$ *are pairwise incomparable, is an* ℓ*EQ-algebra.*

Proof. By $c = c \otimes 1 \leqslant c \sim 1$, we have two cases:

Case(1). $c \sim 1 = 1$. From $1 = c \sim 1 \leqslant (c \wedge a) \sim (1 \wedge a) = 0 \sim a$ and $1 = c \sim 1 \leqslant (c \wedge b) \sim (1 \wedge b) = 0 \sim b$, we get that $0 \sim a = 0 \sim b = 1$. Therefore $1 = (0 \sim a) \otimes (0 \sim b) \leqslant a \sim b$, implies $a \sim b = 1$. On the other hand, by $b \sim 1 = (a \sim b) \otimes (b \sim 1) \leqslant a \sim 1$ and $a \sim 1 = (b \sim a) \otimes (a \sim 1) \leqslant b \sim 1$, we obtain $a \sim 1 = b \sim 1$. Also we have $a = a \otimes 1 \leqslant a \sim 1 = b \sim 1$ and $b = b \otimes 1 \leqslant b \sim 1 = a \sim 1$, then $a \sim 1 = b \sim 1 = 1$. By $1 = (a \sim 1) \otimes (1 \sim c) \leqslant a \sim c$, we get that $a \sim c = 1$. Similarly we can prove $b \sim c = 1$. Now $1 = b \sim c \leqslant (b \wedge c) \sim (c \wedge c) = 0 \sim c$ and $1 = (0 \sim a) \otimes (a \sim 1) \leqslant 0 \sim 1$, imply $0 \sim c = 0 \sim 1 = 1$. Therefore $a \sim b = 1$, for every $a, b \in E$, that is E is an ℓEQ-algebra.

Case(2). $c \sim 1 = c$. From the case(1) and $a \leqslant a \sim 1$, $b \leqslant b \sim 1$, we get that $a \sim 1 = a$ and $b \sim 1 = b$. Then by Theorem 3.4, $0 \sim 1 = 0$. On the other hand, by Theorem 3.3(iii), E is a prelinear EQ-algebra. Therefore E is an ℓEQ-algebra(by Theorem 2.5). □

In the following theorems, we denote ξ_5 by $\xi = (\{0, a, b, c, 1\}, \wedge, \otimes, \sim, 1)$ a non-linearly ordered 5-element EQ-algebra where $0 \leqslant a, b \leqslant c \leqslant 1$ and a, b are incomparable.

Theorem 4.7. ξ_5 *is an* ℓ*EQ-algebra if and only if* $0 \sim a = b \sim c$ *and* $0 \sim b = a \sim c$.

Proof. Let ξ_5 be an ℓEQ-algebra. Then by Theorem 2.4, $0 \sim a \leqslant (0 \vee b) \sim (a \vee b) = b \sim c$. Also from (e_4), we have $b \sim c \leqslant (b \wedge a) \sim (c \wedge a) = 0 \sim a$. Hence $0 \sim a = b \sim c$. Similarly we can prove $0 \sim b = a \sim c$.
Conversely, let $0 \sim a = b \sim c$ and $0 \sim b = a \sim c$. Then we show that ξ_5 is an ℓEQ-algebra. Using Theorem 2.4, it is sufficient to show that the relation (I) hold, for $a, b, c \in E$. By properties of EQ-algebras, all possible inequalities of the relation

(I) hold, except for the following inequalities:
(a) $0 \sim a \leqslant (0 \vee b) \sim (a \vee b) = b \sim c$,
(b) $0 \sim b \leqslant (0 \vee a) \sim (b \vee a) = a \sim c$,
(c) $a \sim b \leqslant (a \vee a) \sim (b \vee a) = a \sim c$,
(d) $a \sim b \leqslant (a \vee b) \sim (b \vee b) = b \sim c$.
By assumption the inequalities (a) and (b) hold. From (e_4), we obtain $a \sim b \leqslant (a \wedge b) \sim (b \wedge b) = 0 \sim b = a \sim c$ and $a \sim b \leqslant (a \wedge a) \sim (b \wedge a) = a \sim 0 = b \sim c$.

Then the inequalities (c) and (d) hold and the proof is complete. □

In ξ_5, we have $b \leqslant c \to b = b \sim c$, and so $b \sim c \in \{b, c, 1\}$.

Theorem 4.8. *If $b \sim c = 1$, for $b, c \in \xi_5$, then ξ_5 is an ℓEQ-algebra.*

Proof. By $1 = b \sim c \leqslant (b \wedge a) \sim (c \wedge a) = 0 \sim a$, we get that $b \sim c = 0 \sim a = 1$. From (e_8) and (e_{10}), we have $a \leqslant c \to a = a \sim c$, so we obtain $a \sim c \in \{a, c, 1\}$. Now we have the following cases:

Case(1). $a \sim c = 1$. By $1 = a \sim c \leqslant (a \wedge b) \sim (c \wedge b) = 0 \sim b$, we get that $a \sim c = 0 \sim b$. Therefore by Theorem 4.7, E is an ℓEQ-algebra.

Case(2). $a \sim c = a$. From (e_2) and (e_{11}), we get $a = 1 \otimes a \leqslant (0 \sim a) \otimes (a \sim c) \leqslant 0 \sim c \leqslant a \sim c = a$, thus $0 \sim c = a$. By $a = a \sim c \leqslant 0 \sim b$, we get that $0 \sim b \in \{a, c, 1\}$. Now assume $0 \sim b = 1$, then $1 = (0 \sim b) \otimes (b \sim c) \leqslant 0 \sim c = a$, which is a contradiction. If $0 \sim b = c$, then $1 = (0 \sim 0) \otimes (b \sim c) \leqslant (0 \sim b) \sim (0 \sim c) = c \sim a = a$, that is a contradiction. Therefore $0 \sim b = a$. Hence by Theorem 4.7, E is an ℓEQ-algebra.

Case(3). $a \sim c = c$. Similar to the proof of case(2), E is an ℓEQ-algebra. □

Theorem 4.9. *In ξ_5, if $b \sim c = b$, and $a \sim c \in \{a, 1\}$, then ξ_5 is an ℓEQ-algebra.*

Proof. By Theorem 3.7, we have $0 \sim a = b$. Let $a \sim c = a$. Then by (e_{11}), we get that $0 \sim c \leqslant a \sim c = a$ and $0 \sim c \leqslant b \sim c = b$. So $0 \sim c \leqslant a \wedge b = 0$, that is $0 \sim c = 0$. Also by $a \sim c \leqslant (a \wedge b) \sim (c \wedge b) = 0 \sim b$, we obtain $0 \sim b \in \{a, c, 1\}$. Now if $0 \sim b = 1$, then $b = 1 \otimes b = (0 \sim b) \otimes (b \sim c) \leqslant 0 \sim c = 0$, which is a contradiction. If $0 \sim b = c$, then $b = 1 \otimes b = (0 \sim 0) \otimes (b \sim c) \leqslant (0 \sim b) \sim (0 \sim c) = c \sim 0 = 0$, that is a contradiction. Then $0 \sim b = a$ and so by Theorem 4.7, ξ_5 is an ℓEQ-algebra. Assume $a \sim c = 1$, then by $a \sim c \leqslant 0 \sim b$,we obtain $0 \sim b = 1$. Thus $0 \sim b = a \sim c$, and by Theorem 4.7, ξ_5 is an ℓEQ-algebra. □

In ξ_5, we know $a \sim c \leqslant 0 \sim b$ and so $a \sim c = c$ implies $0 \sim b \in \{c, 1\}$, now we have the following theorems in ξ_5:

Theorem 4.10. *Let $b \sim c = b$, $a \sim c = c$ and $0 \sim b = c$. Then ξ_5 is an ℓEQ-algebra.*

Proof. From $b \sim c = b$ and Theorem 3.7, we get that $0 \sim a = b$. Therefore $0 \sim a = b \sim c = b$ and $a \sim c = 0 \sim b = c$, ξ_5 is an ℓEQ-algebra, by Theorem 4.7. □

Theorem 4.11. (i) *Let $b \sim c = b$, $a \sim c = c$ and $0 \sim b = 1$. Then we get two fuzzy equalities such that ξ_5 is not an ℓEQ-algebra.*
(ii) *Let $b \sim c = c$ in ξ_5. Then we get two fuzzy equalities such that ξ_5 is not an ℓEQ-algebra.*

Proof. (i) By $0 \leqslant a, b \leqslant c$ and (e_{11}), we get $0 \sim c \leqslant b \sim c = b$ and $0 \sim c \leqslant a \sim c = c$, so $0 \sim c \leqslant b \wedge c = b$, therefore $0 \sim c \in \{0, b\}$. Now if $0 \sim c = 0$, then by (e_2) $b = 1 \otimes b = (0 \sim b) \otimes (b \sim c) \leqslant 0 \sim c = 0$, which is a contradiction. Thus $0 \sim c = b$. By $(E7)$ and (e_{11}), we obtain $b = b \otimes 1 \leqslant b \sim 1 \leqslant b \sim c = b$ that is $b \sim 1 = b$. From $b = 1 \otimes b = (0 \sim b) \otimes (b \sim 1) \leqslant 0 \sim 1 \leqslant b \sim 1 = b$, we obtain $0 \sim 1 = b$. By $0 \leqslant a \leqslant 1$ and (e_{11}), we have $b = 0 \sim 1 \leqslant a \sim 1$.On the other hand $a = a \otimes 1 \leqslant a \sim 1$, therefore $c = a \vee b \leqslant a \sim 1$, thus $a \sim 1 \in \{c, 1\}$. Now if $a \sim 1 = 1$, then from $a \leqslant c \leqslant 1$ and (e_{11}), we get $1 = a \sim 1 \leqslant a \sim c = c$, which is a contradiction. Hence $a \sim 1 = c$. By $b \sim c = b$ and Theorem 3.7, we get that $0 \sim a = b$. From (e_4), we have $a \sim b \leqslant (a \wedge a) \sim (b \wedge a) = a \sim 0 = b$, so $a \sim b \leqslant b$. Thus by $b = 1 \otimes b = (b \sim 0) \otimes (0 \sim a) \leqslant b \sim a \leqslant b$, we get that $a \sim b = b$. We have $c = c \otimes 1 \leqslant c \sim 1$, then $c \sim 1 \in \{c, 1\}$. Hence we can form two fuzzy equalities, say $\sim_1$ and $\sim_2$ such that ξ_5 with either $\sim_1$ or $\sim_2$ is not an ℓEQ-algebra, because $1 = 0 \sim b \nleqslant (0 \vee a) \sim (b \vee a) = a \sim c = c$:

$\sim_1$	0	a	b	c	1
0	1	b	1	b	b
a	b	1	b	c	c
b	1	b	1	b	b
c	b	c	b	1	c
1	b	c	b	c	1

$\sim_2$	0	a	b	c	1
0	1	b	1	b	b
a	b	1	b	c	c
b	1	b	1	b	b
c	b	c	b	1	1
1	b	c	b	1	1

(ii) Let $b \sim c = c$. Then by (e_4), we have $c = b \sim c \leqslant (b \wedge a) \sim (c \wedge a) = 0 \sim a$, therefore $0 \sim a \in \{c, 1\}$. If $0 \sim a = c$, then similar to the proof of Theorem 4.10, ξ_5 is an ℓEQ-algebra. If $0 \sim a = 1$, similar to the proof of (i) we can obtain two the following fuzzy equalities $\sim_3$ and $\sim_4$ such that ξ_5 with either $\sim_3$ or $\sim_4$ is not an ℓEQ-algebra: $\square$

$\sim_3$	0	a	b	c	1
0	1	1	a	a	a
a	1	1	a	a	a
b	a	a	1	c	c
c	a	a	c	1	c
1	a	a	c	c	1

$\sim_4$	0	a	b	c	1
0	1	1	a	a	a
a	1	1	a	a	a
b	a	a	1	c	c
c	a	a	c	1	1
1	a	a	c	1	1

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