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Non-Abelian Tensor Absolute Centre of A Group

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Abstract. In 1904, Schur proved his famous result which says that if the central factor group of a given group is finite then so is its derived subgroup. In 1994, Hegarty showed that if the absolute central factor group, $G/L(G)$, is finite then so is its auto-commutator subgroup, $K(G)$. Using the notion of non-abelian tensor product, we introduce the concept of tensor absolute centre, $L^{\otimes}(G)$, and $K^{\otimes}(G) = G \otimes \mathrm{Aut}(G)$. Then under some condition we prove that the finiteness of $G/L^{\otimes}(G)$ implies that $K^{\otimes}(G)$ is also finite.

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1. Introduction

Let G and H be two groups which act on each other on the right h^g and g^h, for all $g \in G$ and $h \in H$. Clearly, every group acts on itself by conjugation. In 1987, Brown et. al. [3] introduced the non-abelian tensor product of groups, which is denoted by $G \otimes H$ and it is the group generated by the symbols $g \otimes h$, satisfying the following relations:

$$gg' \otimes h = (g^{g'} \otimes h^{g'})(g' \otimes h),$$

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$$g \otimes hh' = (g \otimes h')(g^{h'} \otimes h^{h'}),$$

for all $g, g' \in G$ and $h, h' \in H$. The actions are compatible in the sense that:

$$g'^{h^g} = (g')^{g^{-1}hg},$$

$$h'^{g^h} = (h')^{h^{-1}gh}.$$

In the special case, $G \otimes G$ is said to be the *tensor square* of G. Recall that, the set of all right n-Engel elements of a group G is defined by $R_n(G) = \{g \in G \mid [g, {}_n a] = 1, \ \forall a \in G\}$, where $[g, a] = g^{-1}a^{-1}ga$, and $[g, {}_n a] = [[g, {}_{n-1} a], a] = [g, \underbrace{a, \ldots, a}_{n}]$.

Using the non-abelian tensor square, the set of all right $n_\otimes$-Engel elements of G is defined as follows:

$$R_n^\otimes(G) = \{g \in G \mid [g, {}_{n-1} a] \otimes a = 1_\otimes, \ \forall a \in G\}.$$

See also [2] for the case $n = 2$. Biddle and Kappe in [2], proved that $R_2^\otimes(G)$ is always a characteristic subgroup of G contained in $R_2(G)$. Moravec in [9] determined some further information on $R_2^\otimes(G)$ and introduced the concept of $2_\otimes$-Engel groups. A group G is called $2_\otimes$-Engel when $[g, a] \otimes a = 1_\otimes$, for all $g, a \in G$.

For each element g of a given group G, and an automorphism α of $\mathrm{Aut}(G)$,

$$[g, \alpha] = g^{-1}\alpha(g) = g^{-1}g^\alpha,$$

is called the *autocommutator* of g and α. For the group G,

$$L(G) = \{g \in G : [g, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\},$$

and

$$K(G) = [G, \mathrm{Aut}(G)] = \langle [g, \alpha] : g \in G, \alpha \in \mathrm{Aut}(G) \rangle,$$

are called the *absolute centre* and *autocommutator subgroup* of G, respectively. The concepts of absolute centre and autocommutator subgroup of a group ascend to the works of Baer [1] and Hegarty [5, 6]. Clearly, they are both characteristic subgroups and if the automorphism α runs over the inner automorphisms, then one gets the centre, $Z(G)$, and the commutator subgroup, G', respectively.

Moghaddam and Sadeghifard in [8], introduced the concept of $2_{\otimes}$-auto Engel set as follows:

$$AR_2^{\otimes}(G) = \{g \in G \mid [g, \alpha] \otimes \alpha = 1_{\otimes}, \ \forall \alpha \in \mathrm{Aut}(G)\}.$$

They proved that $AR_2^{\otimes}(G)$ is a characteristic subgroup of G contained in $R_2^{\otimes}(G)$.

A smallest positive integer n is called the *exponent* of G, if $g^n = 1$, for all $g \in G$ and denoted by $\exp(G)$.

In 1904, Schur [12] was the first mathematician who studied the connection between the central factor group of a group G and its derived subgroup, G'. He is credited with the following well-known result and known as Schur's Theorem which says that if $G/Z(G)$ is finite then so is its derived subgroup.

The converse of Schur's result is also an interesting problem. It is studied by many authors and show that it is held under some conditions. In fact, it is not true in general and one may consider the infinite extra special p-group for any odd prime p as a counterexample.

In the present article we introduce a new notion of *tensor absolute centre* and *tensor autocommutator* of a given group and finally the analogue of Schur's result is also proved.

2. Preliminary Results

In this section we state the following facts, which are needed in proving our main results.

Proposition 2.1. *([3], Proposition 2) Let G and H be groups equipped with compatible actions on each other. Then*

(i) the groups G and H act on $G \otimes H$ so that

$$(g' \otimes h)^g = g'^g \otimes h^g, \quad (g \otimes h')^h = g^h \otimes h'^h,$$

for all $g, g' \in G$, $h, h' \in H$.

(ii) *there are group homomorphisms $\lambda : G \otimes H \to G$ and $\lambda' : G \otimes H \to H$ such that*

$$\lambda(g \otimes h) = g^{-1}g^h, \quad \lambda'(g \otimes h) = (h^{-1})^g h,$$

for all $g \in G$ and $h \in H$.

Proposition 2.2. *([4], Proposition 3) Let G and H be groups equipped with compatible actions on each other. Then the following identities are satisfied:*

(i) $(g^{-1} \otimes h)^g = (g \otimes h)^{-1} = (g \otimes h^{-1})^h$;

(ii) $(g' \otimes h')^{g \otimes h} = (g' \otimes h')^{[g,h]}$;

(iii) $(g^{-1}g^h) \otimes h' = (g \otimes h)^{-1}(g \otimes h)^{h'}$;

(iv) $g' \otimes h^{-1g}h = (g \otimes h)^{-g'}(g \otimes h)$;

(v) $g^{-1}g^h \otimes (h'^{-1})^{g'}h' = [g \otimes h, g' \otimes h']$,

for all $g, g' \in G$ and $h, h' \in H$.

In 2015, Moghaddam and Sadeghifard [8] introduced the action of a group G on $\mathrm{Aut}(G)$ given by $\alpha^g = \alpha^{\varphi_g} = \varphi_{g^{-1}} \circ \alpha \circ \varphi_g$ and the action of $\mathrm{Aut}(G)$ on G given by $g^\alpha = (g)\alpha$, for all $g \in G$, $\alpha \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$. They also showed that the above actions are compatible.

So, the non-abelian tensor product $G \otimes \mathrm{Aut}(G)$ generated by the symbols $g \otimes \alpha$ such that

$$gg' \otimes \alpha = (g^{g'} \otimes \alpha^{g'})(g' \otimes \alpha), \quad g \otimes \alpha\beta = (g \otimes \beta)(g^\beta \otimes \alpha^\beta),$$

for all $g, g' \in G$ and $\alpha, \beta \in \mathrm{Aut}(G)$.

Lemma 2.3. *([8], Lemma 2.3) The above actions are well-defined and compatible.*

Proposition 2.4. *([8], Lemma 3.2) Let G be a group, then* *(i)* $(g \otimes \alpha)^{-1} = g \otimes \alpha^{-1}$;

(ii) $([g, \beta] \otimes \alpha)([g, \alpha] \otimes \beta) = 1_{\otimes}$;

(iii) $[g, \alpha] \otimes \varphi_g = 1_{\otimes}$,

for all $g \in AR_2^{\otimes}(G)$ *and* $\alpha, \beta \in \mathrm{Aut}(G)$.

The set of all 2-auto Engel elements of a given group G is defined as follows (see also [11]).

$$AR_2(G) = \{g \in G \mid [g,_2 \alpha] = [g, \alpha, \alpha] = 1, \forall \alpha \in \mathrm{Aut}(G)\}.$$

Lemma 2.5. *The set of all* $2_{\otimes}$*-auto Engel elements of a given group G contains* $L(G)$ *and being contained in the set of* 2*-auto Engel elements of* G.

Proof. It is obvious that $L(G)$ is a subset of $AR_2^{\otimes}(G)$. Now, by Proposition 2. 1(ii), there exists a homomorphism $\gamma : G \otimes \mathrm{Aut}(G) \to K(G)$ given by $\gamma(g \otimes \alpha) = g^{-1}g^{\alpha} = [g, \alpha]$, from which the claim follows. □

The next proposition gives some useful properties of $AR_2^{\otimes}(G)$, which is needed in proving our main results.

Proposition 2.6. (*[8], Proposition* 3.5) *Let G be a group. Then*

(i) $g^{-1} \otimes \alpha = (g \otimes \alpha)^{-1}$;

(ii) $[g, \alpha]^n \otimes \beta = ([g, \alpha] \otimes \beta)^n$;

(iii) $g \otimes \alpha^n = (g \otimes \alpha)^n$;

(iv) $[g, \alpha] \otimes [\beta, \gamma] = 1_{\otimes}$;

(v) $g \otimes [\alpha, \beta] = ([g, \alpha] \otimes \beta)^2$,

for all $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$, $g \in AR_2^{\otimes}(G)$ *and* $n \in \mathbb{N}$.

The following corollary is an immediate consequence of the above proposition.

Corollary 2.7. *Let G be any group. Then*

$$[g_1, g_2] \otimes [\varphi_{g_1}, \alpha] = 1_{\otimes},$$

for all g_1, g_2 in $AR_2^{\otimes}(G)$ and $\alpha \in \mathrm{Aut}(G)$.

Proof. By Proposition 2.6(iv), we have

$$[g_1, g_2] \otimes [\varphi_{g_1}, \alpha] = [g_1, \varphi_{g_2}]\ otimes[\varphi_{g_1}, \alpha] = 1_{\otimes}. \quad \square$$

The following proposition is an immediate consequence of Propositions 2.4, and 2.6, see also [8].

Proposition 2.8. *Let G be a $2_{\otimes}$-auto Engel group. Then*

(i) $G \otimes \mathrm{Aut}(G)$ is abelian;

(ii) $C_G^{\otimes}(\alpha) = \{g \in G \mid g \otimes \alpha = 1_{\otimes},\ \forall \alpha \in \mathrm{Aut}(G)\}$ is a characteristic subgroup of G.

Proposition 2.9. (*[11], Theorem 3.8*) *Let G be a 2-auto Engel group. Then $\mathrm{Aut}(G)$ is nilpotent of class at most 2.*

Proof. Lemma 3.1(d), (f) in [11] imply that $[x, [\alpha, \beta, \gamma]] = [x, [\alpha, \beta], \gamma]^2 = 1$, for all $x \in G$ and $\alpha, \beta, \gamma \in \mathrm{Aut}(G)$. Therefore $[\alpha, \beta, \gamma] = id_G$ and hence $\mathrm{Aut}(G)$ is nilpotent of class at most 2. $\square$

The following lemma is used in the next section.

Lemma 2.10. (*Dicman*) *Let $\{x_1, ..., x_n\}$ be a finite normal subset of a group G where $|x_i|$ is finite for each $1 \leqslant i \leqslant n$. Then $X = \langle x_1, ..., x_n \rangle$ is a finite normal subgroup of G and $|X| \leqslant \prod_{i=1}^{n} |x_i|$.*

Proof. See [10], page 425. $\square$

Definition 2.11. *Let G be any group, then*

$$L^{\otimes}(G) = \{g \in G \mid g \otimes \alpha = 1_{\otimes},\ \forall \alpha \in \mathrm{Aut}(G)\},$$

is called tensor absolute centre of G. Also,

$$K^{\otimes}(G) = G \otimes \mathrm{Aut}(G) = \langle g \otimes \alpha \mid g \in G,\ \alpha \in \mathrm{Aut}(G) \rangle,$$

is called tensor autocommutator of G.

Lemma 2.12. *Let G be an arbitrary group. Then $L^{\otimes}(G)$ is characteristic subgroup of G.*

Proof. By the above definition, for all $g \in L^{\otimes}(G)$ and $\alpha, \beta \in \mathrm{Aut}(G)$, we have

$$g^{\beta} \otimes \alpha = (g \otimes \alpha^{\beta^{-1}})^{\beta} = 1_{\otimes}.$$

Hence $g^{\beta} \in L^{\otimes}(G)$ and so it is characteristic in G. $\square$

Note that, Proposition 2.1(ii) implies that $L^{\otimes}(G) \leqslant L(G) \leqslant Z(G)$.

3. Main Results

Using the property of $L^{\otimes}(G)$ of a given group G, we may define the following map

$$\psi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(G/L^{\otimes}(G))$$

$$\alpha \longmapsto \bar{\alpha},$$

where $\bar{\alpha} : G/L^{\otimes}(G) \longrightarrow G/L^{\otimes}(G)$ given by $\bar{\alpha}(gL^{\otimes}(G)) = g^{\alpha}L^{\otimes}(G)$, for all $g \in G$. Clearly, $\bar{\alpha}$ and ψ are well-defined and they are automorphism and homomorphism, respectively. Now we define

$$\mathrm{Aut}_{L^{\otimes}(G)}(G) = \{\alpha \in \mathrm{Aut}(G) \mid \ [g, \alpha] \in L^{\otimes}(G), \ \forall g \in G\},$$

which is a normal subgroup of $\mathrm{Aut}(G)$, as we have

$$\begin{aligned}
\mathrm{Ker}(\psi) &= \{\alpha \in \mathrm{Aut}(G) \mid \psi(\alpha) = \overline{id}\} \\
&= \{\alpha \in \mathrm{Aut}(G) \mid \bar{\alpha}(gL^{\otimes}(G)) = gL^{\otimes}(G), \ \forall g \in G\} \\
&= \{\alpha \in \mathrm{Aut}(G) \mid g^{\alpha}L^{\otimes}(G) = gL^{\otimes}(G), \ \forall g \in G\} \\
&= \{\alpha \in \mathrm{Aut}(G) \mid [g, \alpha] \in L^{\otimes}(G), \ \forall g \in G\} \\
&= \mathrm{Aut}_{L^{\otimes}(G)}(G).
\end{aligned}$$

The following lemma is very useful for our further investigations.

Lemma 3.1. *If the factor group $G/L^{\otimes}(G)$ is finite. Then* $\mathrm{Aut}(G)$ *is finite if and only if* $\mathrm{Aut}_{L^{\otimes}(G)}(G)$ *is finite.*

Proof. Suppose that $G/L^{\otimes}(G)$ is finite, then by the above discussion $\mathrm{Aut}(G)/\mathrm{Aut}_{L^{\otimes}(G)}(G)$ is also finite. Thus the result can be obtained easily. $\square$

The following theorem is helpful in proving our main results in this section.

Theorem 3.2. *Let G be any group, Then*

$$g_1g_2 \otimes \alpha = (g_1 \otimes \alpha)(g_2 \otimes \alpha)([g_1, g_2] \otimes \alpha),$$

for all $g_1, g_2 \in AR_2^{\otimes}(G)$, and $\alpha \in \mathrm{Aut}(G)$.

Proof. One can easily check that $\varphi_{[g_1,g_2]}(x) = [\varphi_{g_2^{-1}}, \varphi_{g_1^{-1}}](x)$, for all x in G. Proposition 2.6 and Corollary 2.7 imply that

$$\begin{aligned} g_1g_2 \otimes \alpha = & \ (g_1^{g_2} \otimes \alpha^{g_2})(g_2 \otimes \alpha) \\ & = (g_1[g_1, g_2] \otimes \alpha^{\varphi_{g_2}})(g_2 \otimes \alpha) \\ & = (g_1 \otimes \alpha^{\varphi_{g_2}})^{[g_1,g_2]}([g_1, g_2] \otimes \alpha^{\varphi_{g_2}})(g_2 \otimes \alpha) \\ & = (g_1 \otimes \alpha^{\varphi_{g_2}})([g_1, g_2] \otimes \alpha^{\varphi_{g_2}})(g_2 \otimes \alpha) \\ & = (g_1 \otimes \alpha[\alpha, \varphi_{g_2}])([g_1, g_2] \otimes \alpha[\alpha, \varphi_{g_2}])(g_2 \otimes \alpha) \\ & = (g_1 \otimes \alpha)(g_2 \otimes \alpha)([g_1, g_2] \otimes \alpha). \quad \square \end{aligned}$$

Corollary 3.3. *Let G be any group, then*

$$g^n \otimes \alpha = (g \otimes \alpha)^n,$$

for all $g \in AR_2^{\otimes}(G)$, $\alpha \in \mathrm{Aut}(G)$ and $n \in \mathbb{N}$.

Proof. The result is trivially true for $n = 1$. Using induction on n, and assume the result holds for $n - 1$, then by the above theorem

$$\begin{aligned} g^n \otimes \alpha = & \ g^{n-1} g \otimes \alpha \\ & = (g^{n-1} \otimes \alpha)(g \otimes \alpha)([g, g^{n-1}] \otimes \alpha) \\ & = (g \otimes \alpha)^n. \quad \square \end{aligned}$$

Theorem 3.4. *For any finite $2_{\otimes}$-auto Engel group G, $exp(K^{\otimes}(G))$ divides $exp(G)$ and $exp(\mathrm{Aut}(G))$.*

Proof. Assume $exp(G) = n$ then $g^n = 1$, for all $g \in G$. On the other hand, Corollary 3.3 implies that $1_\otimes = g^n \otimes \alpha = (g \otimes \alpha)^n$, for every α in $\mathrm{Aut}(G)$. Therefore $exp(K^\otimes(G)) \mid exp(G)$.

Now assume $exp(\mathrm{Aut}(G)) = m$, then $\alpha^m = id$, for all $\alpha \in \mathrm{Aut}(G)$. So by Proposition 2.6(iii), we have $1_\otimes = g \otimes \alpha^m = (g \otimes \alpha)^m$. Thus $exp(K^\otimes(G)) \mid exp(\mathrm{Aut}(G))$. □

Lemma 3.5. *Let G be a $2_\otimes$-auto Engel group with $\mid G/L^\otimes(G) \mid = n$. Then the exponent of $K^\otimes(G)$ divides n.*

Proof. Take any generator $g \otimes \alpha$ of $K^\otimes(G)$, where $g \in G$ and $\alpha \in \mathrm{Aut}(G)$. By the assumption, $g^n \in L^\otimes(G)$. Hence using Corollary 3.3, $1_\otimes = g^n \otimes \alpha = (g \otimes \alpha)^n$, which gives the result. □

Proposition 3.6. *For any $2_\otimes$-auto Engel group G, there exists a monomorphism from $\mathrm{Aut}_{L^\otimes(G)}(G)$ into $\mathrm{Hom}(G/L^\otimes(G), L^\otimes(G))$.*

Proof. Consider the map

$$\psi : \mathrm{Aut}_{L^\otimes(G)}(G) \longrightarrow \mathrm{Hom}(G/L^\otimes(G), L^\otimes(G))$$

$$\alpha \longmapsto \alpha^*,$$

where $\alpha^* : G/L^\otimes(G) \longrightarrow L^\otimes(G)$ given by $\alpha^*(gL^\otimes(G)) = [g, \alpha]$, for all $g \in G$. Clearly, α^* is well-defined homomorphism, since for all g_1 and g_2 in G, if $g_1 L^\otimes(G) = g_2 L^\otimes(G)$ then $g_1 g_2^{-1} \in L^\otimes(G) \leqslant Z(G)$, which implies that $[g_1, g_2^{-1}] = 1$. By the definition of $L^\otimes(G)$ and Theorem 3.2, $g_1 \otimes \alpha = g_2 \otimes \alpha$ and so $\alpha^*(g_1 L^\otimes(G)) = \alpha^*(g_2 L^\otimes(G))$. On the other hand, α^* is a homomorphism, as for all $g_1, g_2 \in G$,

$$\alpha^*(g_1 L^\otimes(G) g_2 L^\otimes(G)) = \alpha^*(g_1 g_2 L^\otimes(G)) = [g_1 g_2, \alpha] = [g_1, \alpha][g_2, \alpha].$$

Clearly, the map ψ is a well-defined homomorphism,

$$\psi(\alpha_1\alpha_2) = (\alpha_1\alpha_2)^*(gL^\otimes(G)) = [g, \alpha_1\alpha_2] = [g, \alpha_2][g, \alpha_1]^{\alpha_2} = [g, \alpha_1][g, \alpha_2],$$

for all $\alpha_1, \alpha_2 \in \mathrm{Aut}_{L^\otimes(G)}(G)$. One can easily check that ψ is a monomorphism. □

Now, we are in a position to prove the following result.

Theorem 3.7. *Let the tensor absolute central factor group $G/L^{\otimes}(G)$ of a $2_{\otimes}$-auto Engel group G is finite. Then $K^{\otimes}(G)$ is finite if and only if $\mathrm{Aut}_{L^{\otimes}(G)}(G)$ is finite.*

Proof. Suppose that $K^{\otimes}(G)$ is finite, and $\mathrm{Aut}_{L^{\otimes}(G)}(G)$ is infinite. Proposition 3.6 implies that the image $\alpha^*(G/L^{\otimes}(G))$ must be infinite subgroup of $L^{\otimes}(G)$. Hence, $L^{\otimes}(G)$ contains infinite number elements of the form $[g, \alpha]$.

Now, we remind the well known epimorphism

$$\kappa : K^{\otimes}(G) = G \otimes \mathrm{Aut}(G) \to K(G),$$

given by $\kappa(g \otimes \alpha) = [g, \alpha]$. Therefore $K^{\otimes}(G)$ has infinite number of generators of the form $g \otimes \alpha$, which contradicts the assumption.

Conversely, assume the subgroup $\mathrm{Aut}_{L^{\otimes}(G)}(G)$ is finite, then by Lemma 3.1, $\mathrm{Aut}(G)$ is also finite. Now, the finiteness property of $G/L^{\otimes}(G)$ implies that the set $\mathcal{K} = \{g \otimes \alpha \mid g \in G,\ \alpha \in \mathrm{Aut}(G)\}$ is finite. Thus, Proposition 2.2(ii) and Corollary 3.3 imply that the set $\mathcal{K}$ is a normal subset of $K^{\otimes}(G)$ and each element of $\mathcal{K}$ is of finite order. Hence, by Dicman's Lemma $K^{\otimes}(G) = \langle \mathcal{K} \rangle$ is finite. □

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