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On Kuhn-Tucker Problem Related to η-Convex Functions

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Abstract. Using the concept of η-convex functions as generalization of convex functions, we inquiry about the relation between minimization problem and Kuhn-Tucker problem with new settings and give sufficient and necessary optimality condition. Also the relation between minimization problem and it's Mond-Weir dual problem in η-convex case is investigated.

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1. Introduction and Preliminaries

The following convention for equalities and inequalities will be used.

Ordering relations The relations $=, <, <=, \leqslant$ defined below are called ordering relations (in $\mathbb{R}^n$). If $x, y \in \mathbb{R}^n$, then
$x = y \Leftrightarrow x_i = y_i, \quad i = 1, ..., n$
$x < y \Leftrightarrow x_i < y_i, \quad i = 1, ..., n$
$x <= y \Leftrightarrow x_i <= y_i, \quad i = 1, ..., n$
$x \leqslant y \Leftrightarrow x <= y, \ \ and \ \ x \neq y.$
Consider the minimization problem as the following.

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The Minimization Problem (MP)
Find $\bar{x}$, if it exists, such that

$$\begin{cases} f(\bar{x}) = \min_{x\in X} f(x) \\ \bar{x} \in X = \{x \in X_0, g(x) <= 0\}, \end{cases}$$

where $X_0 \subseteq \mathbb{R}^n$ and two functions $f : X_0 \to \mathbb{R}$ and $g : X_0 \to \mathbb{R}^m$ are differentiable. The set X is called the *feasible region*, $\bar{x}$ the *solution*, and $f(\bar{x})$ the *minimum*. All points x in the feasible region X are called *feasible points*.

It is known that the convexity of f and g is equivalent with inequalities

$$f(x) - f(\bar{x}) >= \nabla f(\bar{x})(x - \bar{x}),$$

$$g(x) - g(\bar{x}) >= \nabla g(\bar{x})(x - \bar{x}),$$

for any $x, \bar{x} \in X$.

In 1981, Hanson [4] considered (MP) where there exists a function $\eta : X \times X \to \mathbb{R}^n$ such that for any $x, \bar{x} \in X$

$$f(x) - f(\bar{x}) >= \nabla f(\bar{x})\eta(x, \bar{x}),$$

$$g(x) - g(\bar{x}) >= \nabla g(\bar{x})\eta(x, \bar{x}),$$

and proved that (MP) with this conditions also satisfies the following properties.

(i) Every feasible Kuhn-Tucker point is a minimum point (Theorem 2.1 in [4]),

(ii) Duality holds between (MP) and its related dual problem, where the dual problem is

$$\begin{cases} \max_{(x,u)} f(x) + ug(x) \\ \bigtriangledown \mathrm{f(x)} + u \bigtriangledown \mathrm{g(x)} = 0 \\ u >= 0, \end{cases}$$

for $x \in X_0$ and $u \in \mathbb{R}^m$.

In fact Hanson observed that we can consider the function $\eta(x, \bar{x})$ instead of $x - \bar{x}$ and then establish properties (i) and (ii) again in scalar convex programming. For more generalizations and results see [5, 6, 10].

Motivated by [4], in this paper we consider the function $\eta(f(x), f(\bar{x}))$ instead of $f(x) - f(\bar{x})$ in the definition of a convex function. This kind of function is called *η-convex*. We investigate relation between minimization problem, Kuhn-Tucker problem, sufficient and necessary optimality conditions. In fact it is shown that under some special conditions we can establish properties (i) and (ii) in above for η-convex functions. Also we show that duality holds between minimization problem and it's Mond-Weir dual problem. We generally use [7] to achieve our expected results.

Definition 1.1. *Suppose that X_0 is an arbitrary subset of $\mathbb{R}^n$ and $\eta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a bifunction. A function $f : X_0 \to \mathbb{R}^m$ is called convex with respect to η (briefly η-convex) on $\bar{x}$, if*

$$\left.\begin{array}{l} y \in X_0, \\ \lambda \in [0,1], \\ \lambda y + (1-\lambda)\bar{x} \in X_0, \end{array}\right\} \longrightarrow f\big(\lambda y + (1-\lambda)\bar{x}\big) <= f(\bar{x}) + \lambda \eta\big(f(y), f(\bar{x})\big).$$

Geometrically above definition is equivalent with the fact that if a function is η-convex on a convex set X_0, then it's graph between any $x, y \in X_0$ is under or on the path starting from $\big(y, f(y)\big)$ and ending at $\Big(x, f(y) + \eta\big(f(x), f(y)\big)\Big)$. If the end point of the path should be $f(x)$, for every $x, y \in X_0$, then we should have $\eta(x, y) = x - y$ and the function reduces to a convex one. If in (MP), X_0 is a convex set and f is an η-convex function on X_0 then it is called *η-convex programming.*
Note that the scalar version of an η-convex functions introduced in [2] (firstly named by φ-convex function) and the authors achieved some results and inequalities for real η-convex functions as well. For more results see [3, 11, 12]. There exist some examples about η-convexity of a function.

Example 1.2. [12] (1) Define $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} -x, & x \geqslant 0; \\ x, & x < 0. \end{cases}$$

and consider a bifunction η as $\eta(x,y) = -x-y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is easy to check that f is an η-convex function but not a convex one.

(2) Consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1; \\ 1, & x > 1. \end{cases}$$

and define the bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\eta(x,y) = \begin{cases} x+y, & x \leqslant y; \\ 2(x+y), & x > y. \end{cases}$$

Then f is η-convex whereas it is not convex.

From now we consider the functions f, g defined from X_0 to $\mathbb{R}^m$ and the bifunction η defind from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^m$, unless otherwise be stated.

2. Basic Results

In this section as a lemma we give an inequality related to the gradient of an η-convex function. Also we investigate about the relation between minimization problem and local minimization problem.

Lemma 2.1. *Let X_0 be open and f be differentiable at $\bar{x} \in X_0$. If f is η-convex at $\bar{x}$ then*

$$\eta\big(f(x), f(\bar{x})\big) >= \bigtriangledown f(\bar{x})(x - \bar{x}),$$

for each $x \in X_0$.

Proof. For any $x \in X_0$ and $0 < \lambda \leqslant 1$

$$f\big(\lambda x + (1-\lambda)\bar{x}\big) <= f(\bar{x}) + \lambda \eta\big(f(x), f(\bar{x})\big),$$

or

$$\frac{f\big(\bar{x} + \lambda(x-\bar{x})\big) - f(\bar{x})}{\lambda} <= \eta\big(f(x), f(\bar{x})\big).$$

It follows that

$$(x-\bar{x})\frac{f\big(\bar{x} + \lambda(x-\bar{x})\big) - f(\bar{x})}{\lambda(x-\bar{x})} <= \eta\big(f(x), f(\bar{x})\big).$$

Letting $\lambda \to 0^+$, we get

$$(x - \bar{x}) \bigtriangledown f(\bar{x}) <= \eta\big(f(x), f(\bar{x})\big),$$

for any $x \in X_0$. $\square$

Example 2.2. Consider the functions f defined in Example 1.2, part (2), and $\bar{x} \in (0, 1) \cup (1, \infty)$. If $0 < \bar{x} < 1$, then in the case that $x \leqslant \bar{x}$ we have

$$\eta\big(f(x), f(\bar{x})\big) = \eta(x, \bar{x}) = x + \bar{x} \geqslant 0 \geqslant (x - \bar{x}) = \bigtriangledown f(\bar{x})(x - \bar{x}).$$

In the case that $x > \bar{x}$ we have

$$\eta\big(f(x), f(\bar{x})\big) = \eta(x, \bar{x}) = 2x + 2\bar{x} \geqslant 0 \geqslant (x - \bar{x}) = \bigtriangledown f(\bar{x})(x - \bar{x}).$$

If $\bar{x} > 1$, then in any case

$$\eta\big(f(x), f(\bar{x})\big) \geqslant 0 = 0 \cdot (x - \bar{x}).$$

Definition 2.3. (**condition** A)
The bifunction η satisfies condition A, if $\eta(x, y) >= 0$ $\big(\eta(x, y) > 0\big)$ implies $x >= y$ $\big(x > y\big)$ or if $\eta(x, y) <= 0$ $\big(\eta(x, y) < 0\big)$ implies $x <= y$ $\big(x < y\big)$.

Corollary 2.4. *Let X_0 be open and f be a differentiable η-convex function at $\bar{x} \in X_0$. If f satisfies condition A and $\bigtriangledown f(\bar{x}) = 0$, then $\bar{x}$ is a minimum point of f.*

Proposition 2.5. *Let X_0 be convex and let f be an η-convex function such that for each $x \in X_0$, $\eta(x, x) \leqslant 0$. The set of solutions of (MP) is convex.*

Proof. Let x_1 and x_2 be solutions of (MP). So

$$f(x_1) = f(x_2) = \min_{x \in X} f(x).$$

For $0 <= \lambda <= 1$, we have $\lambda x_1 + (1 - \lambda)x_2 \in X_0$ and

$$f\big(\lambda x_1 + (1 - \lambda)x_2\big) <= f(x_2) + \lambda\eta\big(f(x_1), f(x_2)\big) =$$
$$f(x_2) + \lambda\eta\big(f(x_2), f(x_2)\big) <= f(x_2) = \min_{x \in X} f(x).$$

Hence $\lambda x_1 + (1-\lambda)x_2$ is also a solution of (MP). □

Under special condition there exists relation between minimization problem and local minimization problem.

The local minimization problem (LMP)
Find $\bar{x}$ in X, if it exists, such that for some open neighborhood $N_\delta(\bar{x})$ around $\bar{x}$ with radius $\delta > 0$,

$$x \in N_\delta(\bar{x}) \cap X \Rightarrow f(x) >= f(\bar{x}).$$

Lemma 2.6. *If $\bar{x}$ is a solution of (MP), then it is also a solution of (LMP). The converse is true if X is convex and f is η-convex at $\bar{x}$ where η satisfies condition A.*

Proof. If $\bar{x}$ solves (MP), then $\bar{x}$ solves (LMP) for any $\delta > 0$. To prove the converse suppose that $\bar{x}$ solves (LMP) for some $\delta > 0$, and let X be convex and f be η-convex at $\bar{x}$. Let $\bar{y}$ be any point in X distinct from $\bar{x}$. Since X is convex, $(1-\lambda)\bar{x} + \lambda\bar{y} \in X$ for $0 < \lambda \leqslant 1$. By choosing λ small enough, that is, $0 < \lambda < \delta / \parallel \bar{y} - \bar{x} \parallel$ and $\lambda \leqslant 1$, we have that

$$\bar{x} + \lambda(\bar{y} - \bar{x}) = (1-\lambda)\bar{x} + \lambda\bar{y} \in N_\delta(\bar{x}) \cap X.$$

Hence since $\bar{x}$ solves (LMP) and f is η-convex,

$$f(\bar{x}) <= f\big(\bar{x} + \lambda(\bar{y} - \bar{x})\big) <= f(\bar{x}) + \lambda\eta\big(f(\bar{y}), f(\bar{x})\big).$$

So

$$\eta(f(\bar{y}), f(\bar{x})) >= 0,$$

for any $\bar{y} \in X$. Condition A implies that

$$f(\bar{y}) >= f(\bar{x}),$$

for any $\bar{y} \in X$. Then $\bar{x}$ solves (MP). □

3. Main Results

In this section we investigate relation between minimization problem and Kuhn-Tucker problem with new settings and give sufficient and necessary optimality condition.

The Kuhn-Tucker problem (KTP)
Find $\bar{x} \in X_0, \bar{u} \in \mathbb{R}^m$ if they exist, such that

$$\begin{cases} \bigtriangledown f(\bar{x}) + \bar{u} \bigtriangledown g(\bar{x}) = 0 \\ g(\bar{x}) <= 0 \\ \bar{u} g(\bar{x}) = 0 \\ \bar{u} >= 0. \end{cases}$$

It is implicit in the above statement that f and g are differentiable at $\bar{x}$.

Theorem 3.1. (Sufficient optimality condition for (MP))
Let X_0 be open and f, g be differentiable and η-convex at $\bar{x}$. Suppose that η satisfies condition A and $(\bar{x}, \bar{u})$ is a solution of (KTP) such that $f + \bar{u}g$ is an η-convex function. Then $\bar{x}$ is a solution of (MP).

Proof. Suppose that x is a feasible point of (MP) and $(\bar{x}, \bar{u})$ is a solution of (KTP). Since $f + \bar{u}g$ is η-convex then

$$f\big(\lambda x + (1-\lambda)\bar{x}\big) + \bar{u}g\big(\lambda x + (1-\lambda)\bar{x}\big) = (f + \bar{u}g)\big(\lambda x + (1-\lambda)\bar{x}\big) <= \big(f(\bar{x}) + \bar{u}g(\bar{x})\big) + \lambda\eta\big((f + \bar{u}g)(x), (f + \bar{u}g)(\bar{x})\big),$$

for $\lambda > 0$. So

$$\frac{f\big(\lambda x + (1-\lambda)\bar{x}\big) + \bar{u}g\big(\lambda x + (1-\lambda)\bar{x}\big) - f(\bar{x}) - \bar{u}g(\bar{x})}{\lambda} <= \eta\big((f + \bar{u}g)(x), (f + \bar{u}g)(\bar{x})\big).$$

Letting $\lambda \to 0^+$ we get

$$\bigtriangledown f(\bar{x}) + \bar{u} \bigtriangledown g(\bar{x}) <= \eta\big((f + \bar{u}g)(x), (f + \bar{u}g)(\bar{x})\big).$$

From the facts that η satisfies condition A and $\bar{u}g(\bar{x}) = 0$ we have

$$f(x) + \bar{u}g(x) >= f(\bar{x}).$$

It is clear that $\bar{u} >= 0$ and $g(x) <= 0$ which imply that $\bar{u}g(x) <= 0$. Hence

$$f(x) >= f(x) + \bar{u}g(x) >= f(\bar{x}). \quad \square$$

For necessary optimality condition we need some background.

Definition 3.2. *[7] A matrix A is said to be nonvacuous if it contains at least one element A_{ij}. An $m \times n$ matrix A with $m >= 1$ and $n >= 1$ is nonvacuous even if all its elements $A_{ij} = 0$.*

Denote the transpose of the matrix A by A^T.

Theorem 3.3. [7](**Motzkin's theorem of alternative**) *Let A, B, C be given matrices, with A being nonvacuous. Then*
either
$Ax > 0 \quad Bx >= 0 \quad Cx = 0$ *has a solution x,*
or the system

$$\begin{cases} A^T y_1 + B^T y_2 + C^T y_3 = 0 \\ y_1 \geqslant 0,\ y_2 >= 0, \end{cases}$$

has a solution y_1, y_2, y_3,
but never both.

The following lemma is a consequence of Linearization Lemma in [1].

Lemma 3.4. *Let $\bar{x}$ is a solution of (LMP), let f and g be differentiable at $\bar{x}$ and let $I = \{i \mid g_i(\bar{x}) = 0\}$. Then the system*

$$\begin{cases} \bigtriangledown f(\bar{x})z < 0 \\ \bigtriangledown g_I(\bar{x})z <= 0, \end{cases}$$

has no solution.

Definition 3.5. *Let X_0 be a convex set. The η-convex function g on X_0 which defines the feasible region*

$$X = \{x | x \in X_0, g(x) <= 0\},$$

is said to satisfies generalized Slater's condition (briefly g-Slater's condition) *if there exists an $x' \in X_0$ such that $g(x') < 0$.*

Theorem 3.6. (necessary optimality condition for (MP))
Let X_0 be open and $\bar{x}$ solves (MP). Suppose that f, g are differentiable and η-convex at $\bar{x}$ such that η satisfies the reverse of condition A and g satisfies g-Slater's condition on X_0. Then there exists a $\bar{u} \in \mathbb{R}^m$ such that $(\bar{x}, \bar{u})$ solves (KTP).

Proof. Let $\bar{x}$ solves (MP). Let $I = \{i|g_i(\bar{x}) = 0\}$ and $J = \{i|g_i(\bar{x}) < 0\}$. From Lemma 2.6 and Lemma 3.4 we have that the system

$$\begin{cases} \bigtriangledown f(\bar{x})z < 0 \\ \bigtriangledown g_I(\bar{x})z <= 0, \end{cases}$$

has no solution $z \in \mathbb{R}^n$. By Motzkin's theorem, there exist $\bar{r}_0$, $\bar{r}_I$ such that

$$\bar{r}_0 \bigtriangledown f(\bar{x}) + \bar{r}_I \bigtriangledown g_I(\bar{x}) = 0, \qquad (\bar{r}_0, \bar{r}_I) \geqslant 0,\ \bar{r}_I >= 0.$$

If we define $\bar{r}_J = 0$ and $\bar{r} = (\bar{r}_I, \bar{r}_J)$, then since $g_I(\bar{x}) = 0$ we have

$$\begin{cases} \bar{r}g(\bar{x}) = \bar{r}_I g_I(\bar{x}) + \bar{r}_J g_J(\bar{x}) = 0 \\ \bar{r}_0 \bigtriangledown f(\bar{x}) + \bar{r} \bigtriangledown g(\bar{x}) = 0 \\ (\bar{r}_0, \bar{r}_I) \geqslant 0,\ \bar{r}_I >= 0. \end{cases}$$

Also since $\bar{x}$ is in X, then $g(\bar{x}) \leqslant 0$.

Now if we show that $\bar{r}_0 > 0$, then $\frac{\bar{r}}{\bar{r}_0}$ is required vector $\bar{u}$ for (KTP) condition and the proof is completed.

If I is empty ($\bar{r}_I = 0$), Since $(\bar{r}_0, \bar{r}_I) \geqslant 0$ then we have $\bar{r}_0 > 0$. If I is nonempty, by contrary suppose that $\bar{r}_0 = 0$. Then since $\bar{r}_J = 0$ we have that

$$\bar{r}_I \bigtriangledown g_I(\bar{x}) = 0, \quad \bar{r}_I >= 0.$$

On the other hand since g satisfies g-slater's condition on X_0, then there exists $x' \in X_0$ such that $g(x') < 0$. Particularly for I, $g_I(x') < 0$ and so from Lemma 2.1 and the reverse of condition A we have

$$(x' - \bar{x})_I \bigtriangledown g_I(\bar{x}) \leqslant \eta\big(g_I(x'), g_I(\bar{x})\big) = \eta\big(g_I(x'), 0\big) < 0.$$

So for $\bar{z} = \bar{x} - x'$ we have $\bigtriangledown g_I(\bar{x})z > 0$. Multiplying this inequality by $\bar{r}_I$ gives

$$\bar{r}_I \bigtriangledown g_I(\bar{x})\bar{z} > 0, \quad \bar{r}_I >= 0,$$

which contradicts the fact that $\bar{r}_I \bigtriangledown g_I(\bar{x}) = 0$. Hence $\bar{r}_0 > 0$. $\square$

There exists a simple example satisfying conditions of Theorems (3.1) and (3.6).

Example 3.7. Consider $a \in \mathbb{R}^+ \cup \{0\}$ and $k \in [1, +\infty]$. Define the function $f : [a-k, +\infty) \to [-k, k]$ as

$$f(x) = \begin{cases} x-a, & a-k \leqslant x \leqslant a+k; \\ k, & x > a+k, \end{cases}$$

and the bifunction $\eta_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\eta_1(x,y) = \begin{cases} x+y, & x \leqslant y, x > a; \\ 2x+2y, & x > y, x > a. \\ -x-y, & a-k \leqslant x \leqslant a. \end{cases}$$

Also consider the function $g : (-\infty, a+k] \to [-k^2, k^2]$ as

$$g(x) = \begin{cases} k(-x+a), & a-k \leqslant x \leqslant k+a; \\ k^2, & x < a-k. \end{cases}$$

with

$$\eta_2(x,y) = \begin{cases} x+y, & x < y, a \leqslant x \leqslant a+k \text{ or } x \geqslant y, x < a; \\ x-y, & x \geqslant y, a \leqslant x \leqslant a+k \text{ or } x < y, x < a. \end{cases}$$

The functions f and g are respectively η_1-convex and η_2-convex. Also both of them are differentiable in $\bar{x} = a$. If we consider $X = \Big\{x \in (-\infty, a+k] \;\Big|\; g(x) \leqslant 0\Big\}$, then $\bar{x} = a \in X$. Now if we set $(\bar{x}, \bar{u}) = (a, \frac{1}{k})$, then we have

$$\begin{cases} \bigtriangledown f(a) + \frac{1}{k} \bigtriangledown g(a) = 0, \\ g(a) <= 0, \\ \frac{1}{k} g(a) = 0, \\ \frac{1}{k} >= 0. \end{cases}$$

which implies that $(\bar{x}, \bar{u}) = (a, \frac{1}{k})$ satisfy the (KTP). Furthermore we can see that the point $\bar{x} = a$ is a solution for (MP).

4. Mond-Weir Duality

In 1961, Wolf [13] extended the duality theory to convex nonlinear programming problems with convex constraints. He considered the problem of weak duality as the following.

Find $\bar{x} \in X_0$ and $\bar{u} \in \mathbb{R}^m$ if they exist, such that

$$\begin{cases} f(\bar{x}) + \bar{u}g(\bar{x}) = \min\limits_{(x,u)} f(x) + ug(x) \\ \bigtriangledown f(\bar{x}) + \bar{u} \bigtriangledown g(\bar{x}) = 0 \qquad\qquad (WD) \\ \bar{u} >= 0, \end{cases}$$

assuming that f and g are convex. He also showed that if x_0 is solution for (MP) and a constraint qualification is satisfied, then there exists y_0 such that (x_0, y_0) is solution for (WD).

Mangasarian in [7] points out that if in (MP), f is only pseudo-convex and g is quasiconvex, Wolfe duality does not hold necessarily for such functions. So in order to weaken the convexity requirements, Mond and Weir [8], proposed a different dual to (MP) as the following:
Find $\bar{x} \in X_0$ and $\bar{u} \in \mathbb{R}^m$ if they exist, such that

$$\begin{cases} f(\bar{x}) = \min\limits_{x \in X_0} f(x) \\ \bigtriangledown f(\bar{x}) + \bar{u} \bigtriangledown g(\bar{x}) = 0 \qquad\qquad (MWD) \\ \bar{u}g(\bar{x}) >= 0 \\ \bar{u} >= 0. \end{cases}$$

It is implicit in the above statement that f and g are differentiable at $\bar{x}$.

In two following theorems the relation between minimization problem and its Mond-Weir dual problem in η-convex case is investigated.

Theorem 4.1. *Let X_0 be open and x, $(\bar{x}, \bar{u})$ be feasible point of (MP) and (MWD) respectively. Suppose that f, g are differentiable at $\bar{x}$. If $f + \bar{u}g$ is η-convex at $\bar{x}$ such that η satisfies condition A, then*

$$f(\bar{x}) <= f(x).$$

Proof. For any $\lambda \in (0, 1]$ and from η-convexity of $f + \bar{u}g$ we have

$$f\big(\lambda x + (1-\lambda)\bar{x}\big) + \bar{u}g\big(\lambda x + (1-\lambda)\bar{x}\big) <=$$
$$f(\bar{x}) + \bar{u}g(\bar{x}) + \lambda\eta\big(f(\bar{x}) + \bar{u}g(\bar{x}), f(x) + \bar{u}g(x)\big).$$

So

$$\frac{f\big(\lambda x + (1-\lambda)\bar{x}\big) + \bar{u}g\big(\lambda x + (1-\lambda)\bar{x}\big) - f(\bar{x}) - \bar{u}g(\bar{x})}{\lambda} <=$$
$$\eta\big(f(\bar{x}) + \bar{u}g(\bar{x}), f(x) + \bar{u}g(x)\big).$$

Now Letting $\lambda \to 0^+$ we have

$$\bigtriangledown f(\bar{x}) + \bar{u} \bigtriangledown g(\bar{x})(x - \bar{x}) <= \eta\big(f(\bar{x}) + \bar{u}g(\bar{x}), f(x) + \bar{u}g(x)\big).$$

Since $\bar{x}$ satisfies conditions of (MWD),

$$\eta\big(f(\bar{x}) + \bar{u}g(\bar{x}), f(x) + \bar{u}g(x)\big) >= 0.$$

Condition A implies that

$$f(x) + \bar{u}g(x) >= f(\bar{x}) + \bar{u}g(\bar{x}).$$

From the fact that x and $(\bar{x}, \bar{u})$ satisfy conditions of (MP) and (MWD) respectively,

$$\begin{cases} g(x) <= 0 \\ \bar{u}g(\bar{x}) >= 0 \\ \bar{u} >= 0. \end{cases}$$

Therefore

$$\begin{cases} \bar{u}g(\bar{x}) >= 0 \\ \bar{u}g(x) <= 0. \end{cases}$$

Then

$$f(\bar{x}) <= f(\bar{x}) + \bar{u}g(\bar{x}) <= f(x) + \bar{u}g(x) <= f(x). \quad \square$$

Theorem 4.2. *Suppose that $\bar{x}$ is a solution of (MP) and all conditions of Theorem 3.6 hold. Then there exists $\bar{u} >= 0$ such that $(\bar{x}, \bar{u})$ is a feasible point of (MWD). Furthermore if the conditions of Theorem 4.1 hold, then $(\bar{x}, \bar{u})$ solves (WMD).*

Proof. It is straight forward. $\square$

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