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Classical Prime and 2-Absorbing L-Submodules

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Abstract. Let L be a complete lattice. Let R be a commutative ring, M an R-module and ν an L-submodule of M. ν is called a classical prime L-submodule of M if for any L-fuzzy points $a_r, b_s \in L^R$ and $x_t \in L^M$ ($a, b \in R$, $x \in M$ and $r, s, t \in L$), $a_r b_s x_t \in \nu$ implies that either $a_r x_t \in \nu$ or $b_s x_t \in \nu$. Assume that ν is an L-submodule of $mmu \in L(M)$. We say that ν is a 2-absorbing L-submodule of μ if for any L-fuzzy points $a_r, b_s \in L^R$ and $x_t \in L^M$ ($a, b \in R$, $x \in M$ and $r, s, t \in L$), $a_r b_s x_t \in \nu$ implies that $a_r b_s \mu \subseteq \nu$ or $a_r x_t \in \nu$ or $b_s x_t \in \nu$. In this case every prime L-submodule of M is a classical prime L-submodule, and every classical prime L-submodule is a 2-absorbing L-submodule. In this paper we give some basic results concerning these classes of L-submodules. Finally we topologize $L - Cl.Spec(M)$, the set of all classical prime L-submodules of M, with Zariski topology.

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1. Introduction

Throughout this paper R is a commutative ring with a nonzero identity, M is a unitary R-module and L stands for a complete lattice with least element 0 and greatest element 1. For every submodule N of M, we denote the annihilator of M/N by $(N :_R M)$, i.e. $(N :_R M) =$

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$\{r \in R | rM \subseteq N\}$. In his paper [3], Badawi introduced the notion of 2-absorbing ideals of a commutative ring, where a proper ideal A of R is said to be 2-absorbing provided that whenever $a, b, c \in R$ with $abc \in A$ then either $ab \in A$ or $ac \in A$ or $bc \in A$. In [15] this concept was generalized to submodules of M by the author and Soheilnia. Let N be a proper submodule of M. Then, N is said to be a 2-absorbing submodule of M provided that whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then either $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$.

Behboodi and Koohi introduced the notion of weakly prime submodules in [4], where a proper submodule N of M is said to be weakly prime if whenever $a, b \in R$ and $m \in M$ with $abm \in N$, then either $am \in N$ or $bm \in N$. Ebrahimi Atani and Farzalipour gave a different definition for weakly prime submodules in [8]. According to their definition, a proper submodule N of M is called weakly prime provided that for every $a \in R$ and $m \in M$ with $0 \neq am \in N$, then either $m \in N$ or $a \in (N :_R M)$. To avoid the ambiguity, Behboodi et al. renamed weakly prime submodules to classical prime submodules [6]. The set of all classical prime submodules of M is denoted by $Cl.Spec(M)$.

We recall that a proper submodule N of M is called a prime submodule of M if, for every $a \in R$ and $m \in M$, $am \in N$ implies that either $m \in N$ or $a \in (N :_R M)$. The notion of prime submodules was first introduced and studied in [7] and recently it has received a good deal of attention from several authors. We denote the set of all prime submodules of M by $Spec(M)$. Clearly every prime submodule is classical prime and every classical prime submodule is 2-absorbing.

Let R be a commutative ring and consider $Spec(R)$, the spectrum of all prime ideals of R. The Zariski topology on $Spec(R)$ is a useful implement in algebraic geometry. For each ideal I of R, the variety of I is the set $V(I) = \{P \in Spec(R) : I \subseteq P\}$. Then the set $\{V(I) : I \trianglerighteq R\}$ satisfies the axioms for the closed sets of a topology on $Spec(R)$, called the Zariski topology on $Spec(R)$ [2]. Let M be an R-module. In [11], the $Spec(M)$ topologized with the Zariski topology in a similar way to that of $Spec(R)$. For any submodule $N \leqslant M$, denote by $V(N)$ the variety of N, which is the set $V(N) = \{P \in Spec(M) : N \subseteq P\}$. Then the set $\zeta(M) = \{V(N) : N \leqslant M\}$ is not closed under finite unions.

The R-module M is called a Top-module provided that $\zeta(M)$ is closed under finite unions, whence $\zeta(M)$ constitute the closed sets in a Zariski topology on $Spec(M)$. Later Behboodi et. al. in [5] generalized the Zariski topology on $Cl.Spec(M)$. If, for every submodule $N \leqslant M$, we define the classical variety of N, denoted by $\mathbb{V}(N)$, to be the set of all $P \in Cl.Spec(M)$ with $N \subseteq P$, then if $\mathbb{C}(M) = \{\mathbb{V}(N) : N \leqslant M\}$ is closed under finite unions, M is called a classical Top-module. In this case the sets $\mathbb{V}(N)$ satisfy the axioms for the closed sets of a topology on $Cl.Spec(M)$, called the Zariski topology on $Cl.Spec(M)$.

Zadeh in [16] introduced the notion of a fuzzy subset μ of a non-empty set X as a function μ from X to $[0, 1]$. Goguen in [9] generalized the notion of a fuzzy subset of X to that of an L-fuzzy subset, namely a function from X to a lattice L. Later Rosenfeld considered the fuzzification of algebraic structures [14]. Liu [10], introduced and examined the notion of a fuzzy ideal of a ring. Since then several authors have obtained interesting results on L-fuzzy ideals of R and L-fuzzy modules. See [12] for a comprehensive survey of the literature on these developments. In [1], Ameri and Mahjoob introduced and studied $L - Spec(M)$, the set of all prime L-submodules of M, and topologized it in a similar way to that of $Spec(M)$.

In Sections 3 and 4, we introduce the concepts of 2-absorbing L-submodules and classical prime L-submodules of M. We denote by $L - Cl.Spec(M)$, the set of all classical prime L-submodules of M. In Section 5 we define the concept of L-classical Top-modules and show that an L-classical Top-module can be equipped with a Zariski topology.

2. Preliminaries

Given a nonempty set X, an L-subset μ is a function from X to L. The set of all L-subsets of X is called the L-power set of X and is denoted by L^X. In particular, when L is $[0, 1]$, the L-subsets of X are called fuzzy subsets and the set $[0, 1]^X$ is referred to as the fuzzy power set of X. For $\mu, \nu \in L^X$ we say $\mu \subseteq \nu$ if and only if $\mu(x) \leqslant \nu(x)$, for all $x \in X$. Also, $\mu \subset \nu$ if and only if $\subseteq \nu$ and $\mu \neq \nu$.

By an L-fuzzy point x_r of X, $x \in X$; $r \in L\backslash\{0\}$, we mean $x_r \in L^X$) defined by

$$x_r(y) = \begin{cases} r, & \text{if y=x;} \\ 0, & \text{otherwise.} \end{cases}$$

If x_r is an L-fuzzy point of X and $x_r \subseteq \mu \in L^X$, we write $x_r \in \mu$. For $A \subseteq X$ the characteristic function of A, $\chi_A \in L^X$, is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.1. *Let $\mu \in L^X$. For $t \in L$, define μ_t as follows:*

$$\mu_t = \{x \in X | \mu(x) \geqslant t\},$$

μ_t is called the t-cut (or t-level set) of μ.

We recall the two following basic definitions given in [12].

Definition 2.2. *Let $\xi \in L^R$. Then μ is called an L-ideal of R if for all $x; y \in R$,*

(i) $\mu(x - y) \geqslant \mu(x) \wedge \mu(y)$,
(ii) $\mu(xy) \geqslant \mu(x) \vee \mu(y)$.

Definition 2.3. *Let $\mu \in L^M$. Then μ is called an L-submodule of M if for all $x, y \in M$ and for all $r \in R$,*

(i) $\mu(x + y) \geqslant \mu(x) \wedge \mu(y)$,
(ii) $\mu(rx) \geqslant \mu(x)$,
(iii) $\mu(0_M) = 1$.

Let $L(M)$ denote the set of all L-submodules of M and $LI(R)$ the set of all L-ideals of R. We note that when $R = M$, then $\mu \in L(M)$ if and only if $\mu(0) = 1$ and $\mu \in LI(R)$.

Definition 2.4. *For every $\mu \in L(M)$, we define μ_* as follows:*

$$\mu_* = \{x \in X | \mu(x) = \mu(0)\}.$$

Definition 2.5. *For $\nu, \mu \in L(M)$, ν is called an L-submodule of μ if $\nu \subseteq \mu$.*

The following are two basic operations which will be used to define prime L-submodules, classical prime L-submodules and 2-absorbing L-submodules.

Definition 2.6. *Let $\xi \in L^R$ and $\mu \in L^M$. Define the composition $\xi o \mu$ and the product $\xi\mu$ respectively as follows: For all $w \in M$,*
$(\xi o \mu)(w) = sup\{\xi(r) \wedge \mu(x) | r \in R, x \in M, w = rx\}$,
$(\xi\mu)(w) = sup\{inf_{i=1}^{n}\{\xi(r_i) \wedge \mu(x_i)\} | r_i \in R, x_i \in M, n \in \mathbb{N}, w = \Sigma_{i=1}^{n} r_i x_i\}$,
where as usual the supremum of an empty set is taken to be 0.

Notice that $\xi o \mu$ is the case $n = 1$ in the definition of $\xi\mu$. Thus $\xi o \mu \subseteq \xi\mu$.

Definition 2.7. *Let $\{\mu_i\}_{i \in I}$ be a family of L-submodules of M. Then L-submodule $\sum_{i \in I} \mu_i$ of M is defined by*

$$(\sum_{i \in I} \mu_i)(x) = \bigvee\{\bigwedge_{i \in I} \mu_i(x_i) | x = \sum_{i \in I} x_i, x_i \in M \forall i \in I\},$$

for all $x \in M$.

Definition 2.8. *Let $\mu \in L^M$. Then the L-submodule of M generated by μ, denoted by $< \mu >$, is defined to be the intersection of all L-submodules of M containing μ, i.e.*

$$< \mu > = \bigcap\{\nu | \nu \in L(M), \mu \subseteq \nu\}.$$

Lemma 2.9. *For every L-fuzzy points $a_r \in L^R$ and $x_s \in L^N$ we have $< a_r >< x_s > = < a_r x_s >$.*

Proof. See [13, Lemma 3.4]. □

Definition 2.10. *For a non-constant $\xi \in LI(R)$, ξ is called an L-fuzzy prime ideal of R if for any L-fuzzy points $x_r, y_s \in L^R$, $x_r y_s \in \xi$ implies that either $x_r \in \xi$ or $y_s \in \xi$.*

Definition 2.11. *For $\mu, \nu \in L^M$ and $\xi \in L^R$, we define $(\mu : \nu)$ and $(\mu : \xi)$ by:*

$$(\mu : \nu) = \bigcup\{\eta \in L^R | \eta.\nu \subseteq \mu\},$$
$$(\mu : \xi) = \bigcup\{\lambda \in L^M | \xi.\lambda \subseteq \mu\}.$$

In the case where $\nu \in L^M$, $\mu \in L(M)$ and $\xi \in LI(R)$ we have:

$$(\mu : \nu) = \bigcup\{\eta \in LI(R) | \eta.\nu \subseteq \mu\},$$
$$(\mu : \xi) = \bigcup\{\lambda \in L(M) | \xi.\lambda \subseteq \mu\}.$$

In this case $(\mu : \nu) \in LI(R)$ and $(\mu : \xi) \in L(M)$.

Definition 2.12. *([1]) A non-constant L-submodule μ of M is said to be prime if for every $\xi \in LI(R)$ and $\nu \in L(M)$ such that $\xi.\nu \subseteq \mu$, then either $\nu \subseteq \mu$ or $\xi \subseteq (\mu : 1_R)$. The set of all prime L-submodules of M is denoted by $L - Spec(M)$.*

We recall from [1] that, for any L-submodule μ of M, $V^*(\mu)$, denotes the set of all prime L-submodule of M containing μ, i.e, $V^*(\mu) = \{P \in L - Spec(M) | \mu \subseteq P\}$. Thus if $\xi(M)$ denotes the collection of all subsets $V^*(\mu)$ of $L - Spec(M)$, then $\xi(M)$ contains the empty set, and $L - Spec(M)$ and it is closed under arbitrary intersections. If $\xi(M)$ is also closed under finite unions, i.e, for any L-submodules μ and ν of M, there exists an L-submodule θ of M, such that $V^*(\mu) \cup V^*(\nu) = V^*(\theta)$, then $\xi(M)$ satisfies the axioms of closed subsets of a topological space, which is called Zariski topology. An R-module M equipped with Zariski topology is called L-Top module. An L-submodule $\mu \in L(M)$ is called L-semiprime if $\mu = \bigcap_{i \in I} \mu_i$ such that μ_i is a prime L-submodule of M for all $i \in I$, and μ is called L-extraordinary if whenever $\mu_1, \mu_2 \in L(M)$ are semiprime L-submodules such that $\mu_1 \cap \mu_2 \subseteq \mu$, then either $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. It is proved in [1, Theorem 4.5] that an R-module M is an L-Top module if and only if every prime L-submodule of M is L-extraordinary if and only if $V^*(\mu_1) \cup V^*(\mu_2) = V^*(\mu_1 \cap \mu_2)$, for $\mu_1, \mu_2 \in L(M)$. For $\mu \in L(M)$, we define the radical of μ, denoted by $Rad(\mu)$, as the intersection of all prime L-submodules of M containing μ. In other words, $Rad(\mu) = \bigcap_{P \in V^*(\mu)} P$ and it is equal to 1_M if $V^*(\mu) = \emptyset$.

Definition 2.13. *Let $\alpha \in L \backslash \{1\}$. Then α is called a prime element of L if $x \wedge y \leqslant \alpha$ implies that $x \leqslant \alpha$ or $y \leqslant \alpha$ for all $x, y \in \alpha$.*

3. Classical Prime L-Submodules

In this section we introduce the notion of classical prime L-submodules of M which is a generalization of prime L-submodules of M. Then we provide some basic results on classical prime L-submodules.

Definition 3.1. *Let ν be an L-submodule of M. ν is called a classical prime L-submodule of M if for any L-fuzzy points $a_r, b_s \in L^R$) and $x_t \in L^M$ $(a, b \in R,\ x \in M$ and $r, s, t \in L)$, we have*

$$a_r b_s x_t \in \nu \text{ implies that either } a_r x_t \in \nu \text{ or } b_s x_t \in \nu.$$

Clearly, every prime L-submodule of M is a classical prime L-submodule.

Theorem 3.2. *Let μ be a classical prime L-submodule of M. Then, for every $t \in L$ with $\mu_t \neq M$, μ_t is a classical prime submodule of M.*

Proof. Assume that $a, b \in R$ and $m \in M$ are such that $abm \in \mu_t$. Then $\mu(abm) \geqslant t$. Then we have $a_t b_t m_t = (abm)_t \in \mu$. Since μ is a classical prime L-submodule of M, we get $(am)_t = a_t m_t \in \mu$ or $(bm)_t = b_t m_t \in \mu$. If $m_t \in \mu$ for some $m \in M$, then $\mu(m) \geqslant t$. So $m \in \mu_t$. Therefore, $am \in \mu_t$ or $bm \in \mu_t$. Hence μ_t is a classical prime submodule of M. □

Corollary 3.3. *If μ is a classical prime L-submodule of M, then μ_* is a classical prime submodule of M.*

Proof. Since μ is a non-constant L-fuzzy submodule of M, $\mu_* \neq M$. Now the result follows from Theorem 4.3. □

Theorem 3.4. *Let N be a classical prime submodule of M and α a prime element of L. If η is the L-subset of M defined by*

$$\eta(x) = \begin{cases} 1, & \text{if } x \in N; \\ \alpha, & \text{otherwise.} \end{cases} \tag{1}$$

for all $x \in M$, then η is a classical prime L-submodule of M.

Proof. Since N is a classical prime submodule of M, $N \neq M$. Therefore η is a non-constant L-fuzzy submodule of M. Suppose that $a_r, b_s \in L^R$ and $x_t \in L^M$ are L-fuzzy points such that $a_r b_s x_t \in \eta$. Then $r \wedge s \wedge t = (abx)_{r \wedge s \wedge t}(abx) = (a_r b_s x_t)(abx) \leqslant \eta(abx)$. If $a_r x_t \notin \eta$ and $b_s x_t \notin \eta$, then from $r \wedge t = (ax)_{r \wedge t}(ax) \nleq \eta(ax)$ we have $\eta(ax) = \alpha$ and so $ax \notin N$. Similarly, $s \wedge t = (bx)_{s \wedge t}(bx) \leqslant \eta(bx)$. So $\eta(bx) = \alpha$ and $bx \notin N$. So $r \wedge s \wedge t \nleq \alpha$ since α is assumed to be a prime element of L. Since N is a classical prime submodule of M, we have $abx \notin N$. Consequently, $\eta(abx) = \alpha$; so $r \wedge s \wedge t \leqslant \alpha$, which is a contradiction. □

Lemma 3.5. *(1) Let ν be an L-submodule of M. Then ν is a classical prime L-submodule of M if and only if for each L-fuzzy point $x_r \notin \nu$, $\nu : x_r$ is an L-fuzzy prime ideal of R.*

(2) Let $\{\eta_i\}_{i\in I}$ be a family of classical prime L-submodules of M such that for each $x_r \notin \cap_{i\in I}\eta_i$, $\{(\eta_i : x_r)\}_{i\in I}$ is a chain of L-fuzzy ideals of R. Then $\cap_{i\in I}\eta_i$ is a classical prime L-submodule of M.

(3) Let $\{\mu_i\}_{i\in I}$ be a family of prime L-submodules of M such that $\{(\mu_i : 1_M)\}_{i\in I}$ is a chain of L-ideals of R. Then $\cap_{i\in I}\mu_i$ is a classical prime L-submodule of M.

Proof. (1) It is obvious from the definition.

(2) Assume that $a_r, b_s \in L^R$ and $x_t \in L^M$ are L-fuzzy points such that $a_r b_s x_t \in \cap_{i\in I}\eta_i$, but $a_r x_t \notin \cap_{i\in I}\eta_i$ and $b_s x_t \notin \cap_{i\in I}\eta_i$. Hence $a_r x_t \notin \eta_k$ and $b_s x_t \notin \eta_l$ for some $k, l \in I$. In this case $a_r \notin (\eta_k : x_t)$ and $b_s \notin (\eta_l : x_t)$. Since $x_t \notin \cap_{i\in I}\eta_i$, we can assume that $(\eta_k : x_t) \subseteq (\eta_l : x_t)$. Therefore $a_r x_t \notin \eta_k$ and $b_s x_t \notin \eta_K$ while $a_r b_s x_t \in \eta_k$. This contradicts the assumption that η_k is a classical prime L-submodule of M.

(3) Assume that $a_r b_s x_t \in \bigcap_{i\in I}\mu_i$ for some L-fuzzy point $a_r, b_s \in L^R$ and $x_t \in L^M$. If $a_r x_t \notin \bigcap_{i\in I}\mu_i$ and $b_s x_t \notin \bigcap_{i\in I}\mu_i$, then $a_r x_t \notin \mu_k$ and $b_s x_t \notin \mu_l$ for some $k, l \in I$. In this case $a_r \notin (\mu_k : 1_M)$ and $b_s \in (\mu_l : 1_M)$. We can assume that $(\mu_k : 1_M) \subseteq (\mu_l : 1_M)$. By [1, Theorem 3.6], $(\mu_k : 1_M)$ is an L-fuzzy prime ideal of R. Therefore $a_r b_s \notin (\mu_k : 1_M)$. As μ_k is a prime L-submodule of M, it follows from $a_r b_s x_t \in \mu_k$ that $x_t \in \mu_k$, and hence $a_r x_t \in \mu_k$ which is a contradiction. □

4. 2-Absorbing L-Submodules

In this sections, we introduce the concepts of 2-absorbing L-submodules and strongly 2-absorbing L-submodules. We give some basic properties of these classes of L-submodules and then investigate the interplay between 2-absorbing submodules and 2-absorbing L-submodules.

Definition 4.1. *(1) Let ν be a non-constant L-submodule of μ. ν is called a 2-absorbing L-submodule of μ if for any L-fuzzy points $a_r, b_s \in L^R$ and $x_t \in L^M$ ($a, b \in R$, $x \in M$ and $r, s, t \in L$), $a_r b_s x_t \in \nu$ implies that $a_r b_s \mu \subseteq \nu$ or $a_r x_t \in \nu$ or $b_s x_t \in \nu$. ν is called a 2-absorbing L-submodule of M if it is a 2-absorbing L-submodule of 1_M.*

(2) Let η be an L-submodule of M. η is said to be an strongly 2-absorbing L-submodule of M if it is non-constant and whenever $\mu, \nu \in LI(R)$ and $\xi \in L(M)$ with $\mu\nu\xi \subseteq \eta$, then $\mu\nu \subseteq (\eta : 1_M)$ or $\mu\xi \subseteq \eta$ or $\nu\xi \subseteq \eta$.

Theorem 4.2. *(1) Every classical prime L-submodule of M is a 2-absorbing L-submodule.*

(2) Every prime L-submodule of M is an strongly 2-absorbing L-submodule.

(3) Every strongly 2-absorbing L-submodule of M is a 2-absorbing L-submodule.

Proof. (1) and (2) Immediate consequences of definition.

(3) Let η be an strongly 2-absorbing L-submodule of M. Assume that $a_r, b_s \in L^R$ and $x_t \in L^M$ be L-fuzzy points with $a_r b_s x_t \in \eta$. Then, by Lemma 2.9, we have $< a_r >< b_s >< x_t >=< a_r b_s x_t >\subseteq \eta$. Since η is an strongly 2-absorbing L-submodule , we have $< a_r b_s >=< a_r >< b_s >\subseteq (\eta : 1_M)$ or $< a_r x_t >=< a_r >< x_t >\subseteq \eta$ or $< b_s x_t >=< b_s >< x_t >\subseteq \eta$. Therefore $a_r b_s 1_M \subseteq \eta$ or $a_r x_t \in \eta$ or $b_s x_t \in \eta$, that is η is a 2-absorbing L-submodule of M. □

Example 4.3. By Theorem 4.2, every prime L-submodule is 2-absorbing, but the converse does not necessarily true. For example consider the case where $R = M = \mathbb{Z}$. Let p and q be a pair of distinct prime numbers, and set $A = pq\mathbb{Z}$. Clearly, A is a 2-absorbing ideal of $\mathbb{Z}$. Now define $\eta : \mathbb{Z} \to [0, 1]$ by

$$\eta(x) = \begin{cases} 1, & \text{if } pq|x; \\ 0, & \text{otherwise.} \end{cases}$$

Then η is a fuzzy 2-absorbing ideal of R. Moreover $\eta_0 = A$ is a 2-absorbing ideal of $\mathbb{Z}$ that is not a prime ideal. Hence η is not a fuzzy prime ideal of R.

Theorem 4.4. *If ν is a 2-absorbing L-submodule of μ, then ν_t is a*

2-absorbing submodule of μ_t for every $t \in L$ with $\nu_t \neq \mu_t$.

Proof. Let $abm \in \nu_t$ for some $a, b \in R$ and $m \in M$. In this case from $\nu(abm) \geqslant t$ we get $a_t b_t m_t = (abm)_t \in \nu$. As ν is a 2-absorbing L-submodule, we have $(ab)_t\mu = a_t b_t \mu \subseteq \nu$ or $(am)_t = a_t m_t \in \nu$ or $(bm)_t = b_t m_t \in \nu$. If $(ab)_t\mu \subseteq \nu$, then for every $w \in ab\mu_t$ we have $w = abz$ for some $z \in \mu_t$. Then from $\mu(z) \geqslant t$ we have

$$t = t \wedge \mu(z) \leqslant sup_{w=abx}\{t \wedge \mu(x)\} = (ab)_t\mu(w) \leqslant \nu(w).$$

Therefore

$$\nu(w) \geqslant t \Rightarrow w \in \nu_t \Rightarrow ab\mu_t \subseteq \nu_t \Rightarrow ab \in (\nu_t :_R \mu_t).$$

If $(am)_t \in \nu$, then $\nu(am) \geqslant t$. Hence $am \in \nu_t$. Similarly, if $(bm)_t \in \eta$ then $bm \in \nu_t$. This implies that ν_t is a 2-absorbing submodule of μ_t. □

Corollary 4.5. *If ν is an 2-absorbing L-submodule of M, then ν_* is a 2-absorbing submodule of M.*

Proof. The result follows from Theorem 4.3 since ν is L-fuzzy 2-absorbing; hence it is a non-constant L-fuzzy submodule of M and so $\nu_* \neq M$. □

Definition 4.6. *Let $\alpha \in L\backslash\{1\}$. Then α is called a 2-absorbing element of L if $x \wedge y \wedge z \leqslant \alpha$ implies that $x \wedge y \leqslant \alpha$ or $x \wedge z \leqslant \alpha$ or $y \wedge z \leqslant \alpha$ for all $x, y, z \in \alpha$.*

Theorem 4.7. *Assume that N is a 2-absorbing submodule of M and let α be a 2-absorbing element of L. If η is the L-subset of M defined by*

$$\eta(m) = \begin{cases} 1, & \text{if } x \in N; \\ \alpha, & \text{otherwise.} \end{cases}$$

for all $m \in M$, then η is a 2-absorbing L-submodule of M.

Proof. Assume that N is a 2-absorbing submodule of M. Then N is a proper submodule of M. Therefore η is a non-constant L-submodule of M. Suppose that $a_r, b_s \in L^R$ and $x_t \in L^M$ are L-fuzzy points such that $a_r b_s x_t \in \eta$ but $a_r x_t \notin \eta$ and $b_s x_t \notin \eta$. In this case $\eta(ax) = \alpha$ and $\eta(bx) = \alpha$. Therefore $ax \notin N$ and $bx \notin N$. Moreover from $a_r b_s x_t \in \eta$ we have

$$(abx)_{r\wedge s\wedge t}(abx) \leqslant \eta(abx) \Rightarrow r\wedge s\wedge t \leqslant \eta(abx).$$

If $\eta(abx) = 1$, then from $abx \in N$, $ax \notin N$ and $bx \notin N$ we get $ab \in (N :_R M)$ since N is a 2-absorbing submodule of M. Then $\eta(abm) = 1$ for every $m \in M$. Now we have $a_r b_s 1_M(abm) = r \wedge s \leqslant \eta(abm)$.

If $\eta(abx) = \alpha$, then from $r \wedge s \wedge t \leqslant \alpha$, $r \wedge t \nleqslant \alpha$ and $s \wedge t \nleqslant \alpha$ we get $r \wedge s \leqslant \alpha$ since α is a 2-absorbing element of L. In this case $a_r b_s 1_M(w) = r \wedge s \leqslant \alpha \leqslant \eta(w)$ for all $w \in M$.

Therefore $a_r b_s \in (\eta : 1_M)$, that is η is a 2-absorbing L-submodule of M. □

5. Classical $L - Top$-Modules

The set of all classical prime L-submodules of M is called the L-fuzzy classical prime spectrum of M and denoted by $L - Cl.Spec(M)$. In this section we introduce and study a topology on $L - Cl.Spec(M)$ which is analogous to that of $L - Spec(M)$, the spectrum of prime L-submodules of M. For every $\mu \in L^M$ let $\mathbb{V}^*(\mu)$, to be the set of all classical prime L-submodules P of M such that $\mu \subseteq P$. Then:

Proposition 5.1. *Let $\{\mu_i\}_i \in I$ be a family of L-submodules of M. Then*

(1) $\mathbb{V}^*(1_{\{0\}}) = L - Cl.Spec(M)$ and $\mathbb{V}^*(1_M) = \emptyset$;

(2) $\bigcap_{i\in I} \mathbb{V}^*(\mu_i) = \mathbb{V}^*(\sum_{i\in I} \mu_i)$;

(3) $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) \subseteq \mathbb{V}^*(\mu \cap \nu)$ for every $\mu, \nu \in L(M)$.

Proof. (1) is obvious. For (2), assume that $P \in \mathbb{V}^*(\sum_{i\in I} \mu_i)$. Then, $\mu_i \subseteq \sum_{i\in I} \mu_i \subseteq P$ for every $i \in I$. Hence $P \in \mathbb{V}^*(\mu_i)$ for every $i \in I$, and hence $\mathbb{V}^*(\sum_{i\in I} \mu_i) \subseteq \bigcap_{i\in I} \mathbb{V}^*(\mu_i)$. For the reverse containment, assume that $P \in \bigcap_{i\in I} \mathbb{V}^*(\mu_i)$. Then $\mu_i \subseteq P$ for every $i \in I$. Now, for every $m \in M$, we have

$$\begin{aligned}
&(\textstyle\sum_{i\in I} \mu_i)(m)\\
&= \bigvee\{\bigwedge_{i\in I} \mu_i(m_i) | \textstyle\sum_{i\in I} m_i = m, \text{ and } m_i \in M \text{ for every } i \in I\}\\
&\leqslant \bigvee\{\bigwedge_{i\in I} P(m_i) | \textstyle\sum_{i\in I} m_i = m, \text{ and } m_i \in M \text{ for every } i \in I\}\\
&\leqslant P(m).
\end{aligned}$$

It follows that $\sum_{i\in I}\mu_i \subseteq P$, that is $P \in \mathbb{V}^*(\sum_{i\in I}\mu_i)$. Therefore $\bigcap_{i\in I}\mathbb{V}^*(\mu_i) \subseteq \mathbb{V}^*(\sum_{i\in I}\mu_i)$. Hence we have the equality. (3) For every $P \in \mathbb{V}^*(\mu)\cup\mathbb{V}^*(\nu)$, either $\mu \subseteq P$ or $\nu \subseteq P$. Hence $\mu\cap\nu \subseteq P$. Therefore $P \in \mathbb{V}^*(\mu\cap v)$. $\square$

The inclusion in (3) in general is not an equality. In this section we study R-modules for which the last inclusion is an equality.

Definition 5.2. *Let M be a non-zero unitary R-module. M is called a L-classical Top-module (briefly $L - Cl.Top$ module) if $\mathbb{V}^*(\mu)\cup\mathbb{V}^*(\nu) = \mathbb{V}^*(\mu\cap\nu)$ for every $\mu,\nu \in L(M)$.*

For an $L - Cl.Top$ module, the set

$$L - \varrho^*(M) = \{\mathbb{V}^*(\mu) | \mu \in L(M)\},$$

satisfies the axioms for closed sets in a topology ς^* on $L - Cl.Spec(M)$. We call this topology the quasi-Zariski topology on $L - Cl.Spec(M)$.

Let μ be an L-submodule of M. We define the classical L-prime radical of μ, denoted by $Cl.Rad(\mu)$, to be the intersection of all classical prime L-submodules of M containing μ. In the other words, $Cl.Rad(\mu) = \bigcap_{P\in\mathbb{V}^*(\mu)} P$, and it is equal to 1_M if $\mathbb{V}^*(\mu) = \emptyset$.

Definition 5.3. *(1) An L-fuzzy submodule $\mu \in L(M)$ is called a classical semiprime L-submodule of M if μ is an intersection of classical prime L-submodules.*

(2) A classical prime L-submodule μ of M is called $L-Cl.$-extraordinary if for any two classical semiprime L-submodules λ_1 and λ_2 of M, $\lambda_1\cap\lambda_2 \subseteq \mu$ implies that $\lambda_1 \subseteq \mu$ or $\lambda_2 \subseteq \mu$.

We immediately have:

Lemma 5.4. *Let μ be a non-constant L-fuzzy submodule of M. Then*
(1) $Cl.Rad(\mu) \in L(M)$;
(2) $\mathbb{V}^*(\mu) = \mathbb{V}^*(Cl.Rad(\mu))$;
(3) $Cl.Rad(\mu)$ is a classical semiprime L-submodule of M.
(4) $\mu \subseteq Cl.Rad(\mu) \subseteq Rad(\mu)$;
(5) $Cl.Rad(\mu)\cap Cl.Rad(\nu) = Cl.Rad(\mu\cap\nu)$ for every $\mu,\nu \in L(M)$.

We provide a condition on L-fuzzy classical prime submodules under which the inclusion of (3) in Proposition 5.1 becomes an equality.

Proposition 5.5. *Let M be an R-module. The following statements are equivalent*

(i) M is an $L-Cl.Top$ module.

(ii) Every classical prime L-submodule if M is L-extraordinary.

(iii) $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) = \mathbb{V}^*(\mu \cap \nu)$ for every classical semiprime L-submodules $\mu, \nu \in L(M)$.

Proof. The result is clear when $Cl.Spec(M) = \emptyset$. So assume that $Cl.Spec(M) \neq \emptyset$.

$(i) \Rightarrow (ii)$ Let M be an $L-Cl.Top$-module. Assume that P is a classical prime L-submodule of M and that λ_1, λ_2 are classical semiprime L-submodules of M with $\lambda_1 \cap \lambda_2 \subseteq P$. By assumption, there exists $\mu \in L(M)$ with $\mathbb{V}^*(\lambda_1) \cup \mathbb{V}^*(\lambda_2) = \mathbb{V}^*(\mu)$. Since λ_1 is classical semiprime L-submodule, $\lambda_1 = \bigcap_{i\in I} P_i$ in which $\{P_i\}_{i\in I}$ is a collection of classical prime L-submodules of M. For every $i \in I$, we have

$$P_i \in \mathbb{V}^*(\lambda_1) \subseteq \mathbb{V}^*(\mu) \Rightarrow \mu \subseteq P_i \Rightarrow \mu \subseteq \bigcap_{i\in I} P_i = \lambda_1.$$

Similarly, $\mu \subseteq \lambda_2$. So $\mu \subseteq \lambda_1 \cap \lambda_2$. Now we have

$$\mathbb{V}^*(\lambda_1) \cup \mathbb{V}^*(\lambda_2) \subseteq \mathbb{V}^*(\lambda_1 \cap \lambda_2) \subseteq \mathbb{V}^*(\mu) = \mathbb{V}^*(\lambda_1) \cup \mathbb{V}^*(\lambda_2).$$

Consequently, $\mathbb{V}^*(\lambda_1) \cup \mathbb{V}^*(\lambda_2) = \mathbb{V}^*(\lambda_1 \cap \lambda_2)$. Now from $\lambda_1 \cap \lambda_2 \subseteq P$ we have $P \in \mathbb{V}^*(\lambda_1 \cap \lambda_2) = \mathbb{V}^*(\lambda_1) \cup \mathbb{V}^*(\lambda_2)$. Hence either $P \in \mathbb{V}^*(\lambda_1)$ or $P \in \mathbb{V}^*(\lambda_2)$, that is either $\lambda_1 \subseteq P$ or $\lambda_2 \subseteq P$. So P is $L-Cl.$-extraordinary.

$(ii) \Rightarrow (iii)$ Suppose that every classical prime L-submodule of M is $L-Cl.$-extraordinary. Assume that μ and ν are two classical semiprime L-submodules of M. By Proposition 5.1, $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) \subseteq \mathbb{V}^*(\mu \cap \nu)$. For the other containment, choose $P \in \mathbb{V}^*(\mu \cap \nu)$. Then $\mu \cap \nu \subseteq P$. By assumption, P is $L-Cl.$-extraordinary. So $\mu \subseteq P$ or $\nu \subseteq P$, that is either $P \in \mathbb{V}^*(\mu)$ or $P \in \mathbb{V}^*(\nu)$. Therefore $\mathbb{V}^*(\mu \cap \nu) \subseteq \mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu)$, and so $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) = \mathbb{V}^*(\mu \cap \nu)$.

$(iii) \Rightarrow (i)$ Let μ, ν be two L-submodules of M. We can assume that $\mathbb{V}^*(\mu)$ and $\mathbb{V}^*(\nu)$ are both nonempty, for otherwise $\mathbb{V}^*(\mu) \cap \mathbb{V}^*(\nu) = \mathbb{V}^*(\mu)$ or $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) = \mathbb{V}^*(\nu)$. We know that $Cl.Rad(\mu)$ and $Cl.Rad(\nu)$ are both classical semiprime L-submodules of M. Setting $\eta = Cl.Rad(\mu) \cap Cl.Rad(\nu)$ we have $\eta = Cl.Rad(\mu \cap \nu)$. Now

$$\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) = \mathbb{V}^*(Cl.Rad(\mu)) \cup \mathbb{V}^*(Cl.Rad(\nu)) =$$
$$\mathbb{V}^*(Cl.Rad(\mu) \cap Cl.Rad(\nu)) = \mathbb{V}^*(\eta) = \mathbb{V}^*(Cl.Rad(\mu \cap \nu)) = \mathbb{V}^*(\mu \cap \nu),$$

by (iii) and Lemma 5.4. Hence M is $L - Cl.Top$ module. □

Corollary 5.6. *Every $L - Cl.Top$ module is an $L - Top$ module.*

Proof. Assume that M is an $L - Cl.Top$ module. Let P be a prime L-submodule of M. As every prime L-submodule is a classical prime L-submodule, P is $L - Cl.$-extraordinary by Proposition 5.5. Hence it is L-extraordinary. Now the result follows from [1, Theorem 4.5]. □

6. Zariski-Like Topology on the Spectrum of L-Fuzzy Classical Prime Submodules

Now assume that $\mathbb{C}^*(M)$ denotes the collection of all subsets $\mathbb{V}^*(N)$ of $L - Cl.Spec(M)$. In this case
(i) $\emptyset \in \mathbb{C}^*(M)$, $L - Cl.Spec(M) \in \mathbb{C}^*(M)$,
(ii) $\mathbb{C}^*(M)$ is closed under arbitrary intersections, and
(iii) $\mathbb{C}^*(M)$ is not necessarily closed under finite unions.

From $(i) - (iii)$ above, we can see easily that there exists a topology, $\widetilde{\tau}^*$ say, on $L - Cl.Spec(M)$ having $\mathbb{C}^*(M)$ as the collection of closed sets if and only if $\mathbb{C}^*(M)$ is closed under finite union.

Definition 6.1. *An R-module M is is called a classical L-Top-module provided that $\mathbb{C}^*(M)$ is closed under finite unions, i.e., for very $\mu, \nu \in L(M)$, there exists $\xi \in L(M)$ such that $\mathbb{V}^*(\mu) \cup \mathbb{V}^*(\nu) = \mathbb{V}^*(\xi)$.*

Definition 6.2. *Let M be a non-zero unitary R-module. For every $\mu \in L(M)$ let $\mathbb{U}^*(\mu) = L - Cl.Spec(M) \setminus \mathbb{V}^*(\mu)$ and $\mathbb{B}^*(M) = \{\mathbb{U}^*(\mu) : \mu \in L(M)\}$. Then, we define $\mathbb{T}^*(M)$ to be the collection of all unions of finite intersections of elements of $\mathbb{B}^*(M)$. In fact, $\mathbb{T}^*(M)$ is the topology* *on* $L - Cl.Spec(M)$ *by the sub-basis* $\mathbb{B}^*(M)$*. We say that* $\mathbb{T}^*(M)$ *is the Zariski-like topology on* $L - Cl.Spec(M)$*.*

Let M be an R-module. Then the set

$\{\mathbb{U}^*(\mu_1) \cap \mathbb{U}^*(\mu_2) \cap ... \cap \mathbb{U}^*(\mu_n) : k \in \mathbb{N}$ and $\mu_i \in L(M)$ for every $1 \leqslant i \leqslant k\}$

is a basis for the Zariski-like topology on $L - Cl.Spec(M)$, and for a ring R, the Zariski-like topology of R as an R-module and the Zariski topology of $L - Spec(R)$ coincide.

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