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A Meshless Method for the Variable-Order Time Fractional Telegraph Equation

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Abstract. In this paper, the radial basis functions (RBFs) method is used for solving a class of variable-order time fractional telegraph equation (V-TFTE), which appears extensively in various fields of science and engineering. Fractional derivatives based on Caputo's fractional derivative as a function of the independent variable are defined of order $1 < \alpha(x,t) \leqslant 2$. The proposed method combines the radial basis functions and finite difference scheme to produce a semi-discrete algorithm. In the first stage the variable-order time-dependent derivative is discreticized, and then we approximate the solution by the radial basis functions. The aim of this paper is to show that the collocation method based on RBFs is suitable for the treatment of the variable-order fractional partial differential equations. The efficiency and accuracy of the proposed method are shown for some concrete examples. The results reveal that the proposed method is very efficient and accurate.

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1. Introduction

The telegraph equation is one of the most important equations of mathematical physics with applications in many diverse fields such as transmission and propagation of electrical signals [16, 20], vibrational systems [8], random walk theory [5], and mechanical systems [41], etc., and moreover, it is noteworthy that the heat diffusion and wave propagation equations are particular cases of the telegraph equation [38].

In recent years fractional calculus has been developed to describe some phenomena in physics and engineering. Also, fractional integral and derivative have been successfully to used describe many events in fluid mechanics, viscoelasticity, chemical physics,electricity, finance, control theory, biomedical engineering, heat conduction, diffusion problems and other sciences [22, 29, 31,34], Podlubny. Fractional partial differential equations (FPDEs), and particularly space and time-fractional equations, have been widely utilised to prove validity of these models and the existence of solution [24,44, 47, 48]. In addition, some reliable and powerful numerical and analytical methods has been focused on solving FPDEs in the last two decades. According to the mathematical literature, fractional partial differential equations have been progressed formulating in various problems in science and engineering, such as the Schroinger, diffusion and telegraph fractional equations [11, 15, 18, 23, 24, 25, 30, 32, 46].

Variable-order fractional calculus is an extension of traditional fractional calculus. Since fractional calculus allows integration and differentiation at any fractional order, the order of fractional integration and differentiation can be a function of space and/or time. Therefor, Samko and Ross [36] introduced the variable-order fractional differentiation operators and discussed some of their properties. Moreover, some of the Mapping features in Holder spaces and Riemann-Liouville fractional and Marchaud operators, have been generalized by Ross [35] and Samko [37]. Lorenzo and Hartley [27, 28] suggested that the order of Riemann-Liouville fractional derivative is allowed to variate as a function of time or space. Different definitions of fractional-order operators in different areas have been discussed [13, 19, 33, 39]. Relatively new research has been done to solve fractional differential equations of variable-orders and the numerical approximate solution of these equations is still in an early stage of development. Lin [26] worked on the stability and convergence of finite difference approximates of the solutions of variable-order nonlinear fractional diffusion equations. Zhuang [49] worked on the stability and convergence of the Euler approximation for variable-order fractional advection-diffusion equations with nonlinear source terms. Also a variable-order anomalous subdiffusion equation has been studied by Chen [10]. Some applications of variable-order fractional differential equations can be found in [3, 4, 12, 40], and the references therein.

The authors of [1] proposed a novel spectral method for solving the mobile-immobile advection-dispersion model with Coimbra variable time fractional derivative for different kinds of non-local conditions. Their proposed method is a fully collocation algorithm based on the shifted Jacobi-Gauss-Lobatto method in combination with the shifted Jacobi-Gauss-Radau method. As a novel work, Bhrawy and Zaky [7] proposed a numerical algorithm for V-FDEs with Dirichlet boundary conditions. Their proposed algorithm converts the variable-order fractional functional problems under study into systems of linear/nonlinear algebraic equations. Recently, Wei et.al provided an effective collocation method based on the RBFs to solve the variable order time fractional diffusion equation [43].

Our goal in this paper is to apply the method of radial basis functions for the solution of V-TFTE of the form:

$$\begin{aligned} {}_0^cD_t^{\alpha(x,t)}u(x,t) + 2\rho\, {}_0^cD_t^{\alpha(x,t)-1}u(x,t) + \sigma^2 u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ (x,t) \in [0,L] \times [0,T], \end{aligned} \tag{1}$$

subject to the following initial-boundary conditions:

$$\begin{aligned} u(x,0) &= g_1(x), \quad u_t(x,0) = g_2(x) \\ u(0,t) &= h_1(t), \quad u(L,t) = h_2(t), \end{aligned} \tag{2}$$

where ρ and σ are non-negative constants, $1 < \alpha(x,t) \leqslant 2$, $f(x,t)$, $g_1(x)$, $g_2(x)$, $h_1(t)$ and $h_2(t)$ are given functions, and ${}_0^cD_t^{\alpha(x,t)}$ denotes the variable-order fractional derivative in the Caputo sense of order $1 < \alpha(x,t) \leqslant 2$, as defined in [39] by:

$$ {}_0^cD_t^{\alpha(x,t)}u(x,t) = \frac{1}{\Gamma\left(2-\alpha(x,t)\right)} \int_0^t (t-\tau)^{1-\alpha(x,t)} \frac{\partial^2 u(x,\tau)}{\partial \tau^2} d\tau, \qquad t > 0. \tag{3} $$

and regarding $(0 < \alpha(x,t) - 1 \leqslant 1)$, we have:

$$ {}_0^cD_t^{\alpha(x,t)-1}u(x,t) = \frac{1}{\Gamma\left(2-\alpha(x,t)\right)} \int_0^t (t-\tau)^{1-\alpha(x,t)} \frac{\partial u(x,\tau)}{\partial \tau} d\tau, \qquad t > 0. \tag{4} $$

The proposed method uses a combination of the RBFs and finite difference methods to obtain a semi-discrete solution for the problem under consideration. In space domain, the RBFs approximation and in time domain, the finite difference technique are employed to obtain a good numerical solution for the problem. The efficiency and accuracy of the proposed method are demonstrated by some numerical examples. The obtained results show that the proposed method is very efficient and accurate for solving such kind of problems.

The rest of the paper is organized as follows: In Section 2, radial basis functions are introduced. In Section 3, the variable-order fractional derivative in the Caputo sense is expressed, and the proposed method are described. Convergence and stability of the proposed method are investigated in Section 4. Section 5 is devoted to numerical experiments. In Section 6, we conclude our results.

2. RBFs Method

Recently, RBFs have played important roles in various applications of numerical analysis. These functions, which can be used in interpolation of functions and solving partial differential equations [6, 9], have different types, such as Gaussian functions, inverse quadratic functions, multi quadratic functions and thin plate Spline functions (Table 1). In this paper, we use the multi quadratic functions (MQ functions) $\varphi(r) = \sqrt{r^2 + c^2}$, where $c \geqslant 0$ is the shape parameter. Moreover, we use Kansa approach [21] to approximate $u(x_i, t_n)$ as follows:

$$u^n(x_i) \simeq \sum_{j=1}^{N} \lambda_j^n \varphi_j(r_{ij}), \qquad i = 1, 2, \ldots, N. \tag{5}$$

where λ^n's are unknown constants at time level n, $r_{ij} = |x_i - x_j|$ and x_j's are centers which also coincide with the collocation points x_i. Eq. (5) can be rewritten in the matrix form as follows:

$$U^n \simeq \mathbf{A}\lambda^n. \tag{6}$$

Table 1: Some well-known radial basis functions

Gaussian functions:	$\exp(-c^2 r^2)$
Inverse quadratic functions:	$\dfrac{1}{c^2 + r^2}$
Multi quadratic functions:	$\sqrt{r^2 + c^2}$
Thin plate Spline functions:	$r^2 \ln(r)$

3. Description of the Proposed Method

In this section, we describe the process of solving the V-TFTE (1), subject to the initial-boundary conditions (2) as follows:

Consider N points $\{x_i = a + (i-1)h\}_{i=1}^{N}$ in the domain $[a, b]$, where $x_1 = a$, $x_N = b$ and $h = \frac{(b-a)}{(N-1)}$. Also, for time interval $[0, T]$ the grid points are $t^n = n\delta t$ $(n = 0, 1, 2, ..., M)$, where $M = \frac{T}{\delta t}$. In order to discrete the variable-order time fractional derivatives we use the finite difference technique and substitute t^{n+1} into Eqs. (3) and (4). Then, we have:

$$\begin{aligned}
&{}_0^cD_t^{\alpha(x,t^{n+1})}u(x,t^{n+1}) = \\
&\frac{1}{\Gamma\left(2-\alpha(x,t^{n+1})\right)}\int_0^{t^{n+1}}\frac{\partial^2 u(x,\tau)}{\partial\tau^2}\left(t^{n+1}-\tau\right)^{1-\alpha(x,t^{n+1})}d\tau = \\
&\frac{1}{\Gamma\left(2-\alpha(x,t^{n+1})\right)}\sum_{k=0}^{n}\int_{t^k}^{t^{k+1}}\frac{\partial^2 u(x,\tau)}{\partial\tau^2}\left(t^{n+1}-\tau\right)^{1-\alpha(x,t^{n+1})}d\tau. \qquad (7)
\end{aligned}$$

By approximating the first and second order derivatives by using the forward finite difference scheme, we have:

$$\frac{\partial u(x,\sigma)}{\partial t} = \frac{u(x,t^{n+1})-u(x,t^n)}{\delta t} + O(\delta t), \qquad (8)$$

$$\frac{\partial^2 u(x,\sigma)}{\partial t^2} = \frac{u(x,t^{n+1})-2u(x,t^n)+u(x,t^{n-1})}{\delta t^2} + O(\delta t^2), \qquad (9)$$

where $\sigma \in \left[t^n, t^{n+1}\right]$.
By substituting Eq. (9) into Eq. (7), we obtain:

$$\begin{aligned}
&{}_0^cD_t^{\alpha(x,t^{n+1})}u(x,t^{n+1}) = \frac{1}{\Gamma\left(2-\alpha(x,t^{n+1})\right)}\times \\
&\sum_{k=0}^{n}\left(\frac{u^{k+1}-2u^k+u^{k-1}}{\delta t^2}+O(\delta t^2)\right)\int_{t^k}^{t^{k+1}}\left(t^{n+1}-\tau\right)^{1-\alpha(x,t^{n+1})}d\tau, \qquad (10)
\end{aligned}$$

where $u^k = u(x,t^k)$, $k = 0,1,\ldots,M$. By considering $t^{n+1}-\tau = r$, the above integral can be simply computed as follows:

$$\begin{aligned}
&\int_{t^k}^{t^{k+1}}\left(t^{n+1}-\tau\right)^{1-\alpha(x,t^{n+1})}d\tau = \frac{-1}{2-\alpha(x,t^{n+1})}\,r^{2-\alpha(x,t^{n+1})}\Big|_{t^{n-k+1}}^{t^{n-k}} \\
&= \frac{(\delta t)^{2-\alpha(x,t^{n+1})}}{2-\alpha(x,t^{n+1})}\left[(n-k+1)^{2-\alpha(x,t^{n+1})}-(n-k)^{2-\alpha(x,t^{n+1})}\right]. \qquad (11)
\end{aligned}$$

Rearranging Eq. (10) and assuming $b_k(x,t^{n+1}) = (k+1)^{2-\alpha(x,t^{n+1})} -$

$(k)^{2-\alpha(x,t^{n+1})}$ yield:

$$
\begin{aligned}
&{}_0^cD_t^{\alpha(x,t^{n+1})}u(x,t^{n+1}) = \frac{(\delta t)^{-\alpha(x,t^{n+1})}}{\Gamma\left(3-\alpha(x,t^{n+1})\right)}\times \\
&\sum_{k=0}^{n} b_k(x,t^{n+1})\left(u^{n-k+1}-2u^{n-k}+u^{n-k-1}\right) = a(x,t^{n+1})\left(u^{n+1}-2u^{n}+u^{n-1}\right) \\
&+ a(x,t^{n+1})\left[\sum_{k=1}^{n} b_k(x,t^{n+1})\left(u^{n-k+1}-2u^{n-k}+u^{n-k-1}\right)+O(\delta t^{4-\alpha(x,t)})\right],
\end{aligned}
\tag{12}
$$

where $a(x,t^{n+1}) = \frac{(\delta t)^{-\alpha(x,t^{n+1})}}{\Gamma(3-\alpha(x,t^{n+1}))}$ and $n = 0, 1, \ldots, M$.
Furthermore, in a similar manner, we have:

$$
\begin{aligned}
&{}_0^cD_t^{\alpha(x,t^{n+1})-1}u(x,t^{n+1}) = \delta t\, a(x,t^{n+1})\times \\
&\left[u^{n+1}-u^{n}+\sum_{k=1}^{n} b_k(x,t^{n+1})\left(u^{n-k+1}-u^{n-k}\right)+O(\delta t^{3-\alpha(x,t)})\right].
\end{aligned}
\tag{13}
$$

It should be noted that using θ-weighted finite difference scheme, Eq. (1) in $t = t^{n+1}$ becomes:

$$
\begin{aligned}
&{}_0^cD_t^{\alpha(x,t^{n+1})}u(x,t^{n+1}) + 2\rho\,{}_0^cD_t^{\alpha(x,t^{n+1})-1}u(x,t^{n+1}) + \\
&\sigma^2\left(\theta\, u^{n+1}+(1-\theta)\,u^{n}\right) = \theta\, u_{xx}^{n+1} + (1-\theta)u_{xx}^{n+1} + f^{n+1},
\end{aligned}
\tag{14}
$$

where $0 \leqslant \theta \leqslant 1$ is a constant, $u_{xx}^{n} = \frac{\partial^2 u(x,t^n)}{\partial x^2}$ and $f^{n+1} = f(x,t^{n+1})$.
By substituting Eq. (12) and Eq. (13) into Eq. (14), we obtain:

$$
\begin{aligned}
&\left[a^{n+1}(2\rho\delta t+1)+\theta\sigma^2\right]u^{n+1} - \theta\, u_{xx}^{n+1} = \\
&\left[a^{n+1}(2\rho\delta t+2)-(1-\theta)\sigma^2\right]u^{n} + (1-\theta)u_{xx}^{n} - a^{n+1}u^{n-1} \\
&-a^{n+1}\sum_{k=1}^{n} b_k^{n+1}\left[(2\rho\delta t+1)u^{n-k+1}-(2\rho\delta t+2)u^{n-k}+u^{n-k-1}\right]+f^{n+1},
\end{aligned}
\tag{15}
$$

where $a^{n+1} = a(x, t^{n+1})$, $b_k^{n+1} = b_k(x, t^{n+1})$ and $n = 0, 1, \ldots, M$.
Note that the above recursive relation can be rewritten as:

$$\left[a^1(2\rho\delta t + 1) + \theta\sigma^2\right] u^1 - \theta\, u_{xx}^1 = \left[a^1(2\rho\delta t + 2) - (1-\theta)\sigma^2\right] u^0 + (1-\theta)u_{xx}^0 - a^1 u^{-1} + f^1 \tag{16}$$

and

$$\begin{aligned}\left[a^{n+1}(2\rho\delta t + 1) + \theta\sigma^2\right] u^{n+1} - \theta\, u_{xx}^{n+1} = \\ \left[a^{n+1}(2\rho\delta t + 2) - (1-\theta)\sigma^2\right] u^n + (1-\theta)u_{xx}^n - a^{n+1}u^{n-1} - \\ a^{n+1}\sum_{k=1}^{n} b_k^{n+1}\left[(2\rho\delta t + 1)u^{n-k+1} - (2\rho\delta t + 2)u^{n-k} + u^{n-k-1}\right] + f^{n+1},\end{aligned} \tag{17}$$

for $n \geqslant 1$.

Remark 3.1. *Although Eq. (15) is valid for any value of* $\theta \in [0, 1]$, *we use* $\theta = \frac{1}{2}$ *as the famous Crank-Nicholson scheme.*

Now we approximate $u^n(x)$ by RBFs method as follows. First we assume:

$$u^n(x) = \sum_{j=1}^{N} c_j^n \varphi_j(x), \tag{18}$$

where $\varphi_j(x),\ j = 1, 2...N$ are for the RBFs approximation and c_1^n, $c_2^n, ..., c_N^n$ are unknown coefficients, which should be computed.

Next, we consider N collocation points to compute the values of the coefficients c_k^n , $k = 1, 2, ..., N$ in the interpolant of $u^n(x)$, i.e.:

$$u^n(x_i) = \sum_{j=1}^{N} c_j^n \varphi_j(x_i) \qquad i = 1, 2, ..., N. \tag{19}$$

By writing Eq. (19) in a matrix form, we have:

$$[\mathbf{u}]^n = \mathbf{T}[\mathbf{c}]^n, \tag{20}$$

where $[\mathbf{u}]^n = [u_1^n\, u_2^n ... u_N^n]^T$, $[\mathbf{c}]^n = [c_1^n\, c_2^n ... c_N^n]^T$ and $\mathbf{T}$ is an $N \times N$ matrix given by

$$\mathbf{T} = [t_{ij}] = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{N1} \\ \vdots & \ddots & \vdots \\ \varphi_{1N} & \cdots & \varphi_{NN} \end{pmatrix}, \tag{21}$$

where $\varphi_{ji} = \varphi_j(x_i)$.

Assuming that there are $N-2$ internal points and 2 boundary points, the matrix $\mathbf{T}$ can be split into $\mathbf{T} = \mathbf{T_1} + \mathbf{T_2}$, where the entries of $\mathbf{T_1}$ and $\mathbf{T_2}$ are

$$\begin{aligned} \mathbf{T_1} &= \begin{cases} t_{ij} & 2 \leqslant i \leqslant N-1, 1 \leqslant j \leqslant N, \\ 0 & o.w., \end{cases} \\ \mathbf{T_2} &= \begin{cases} t_{ij} & i = 1, N, 1 \leqslant j \leqslant N, \\ 0 & o.w., \end{cases} \end{aligned} \tag{22}$$

Using Eq. (18), we discretize u_{xx} as follows:

$$u_{xx}^n(x) = \sum_{j=1}^{N} c_j^n \varphi_j^{''}(x), \tag{23}$$

By substituting the collocation points in Eq. (23), we obtain:

$$u_{xx}^n(x_i) = \sum_{j=1}^{N} c_j^n \varphi_j^{''}(x_i), \quad i = 2, 3, ..., N-1 \tag{24}$$

which can be rewritten as:

$$[u_{xx}]^n = \mathbf{D}[\mathbf{c}]^n, \tag{25}$$

where

$$\mathbf{D} = [d_{ij}] = \begin{pmatrix} 0 & \cdots & 0 \\ \varphi_{12}^{''} & \cdots & \varphi_{(N)2}^{''} \\ \vdots & \ddots & \vdots \\ \varphi_{1(N-1)}^{''} & \cdots & \varphi_{(N)(N-1)}^{''} \\ 0 & \cdots & 0 \end{pmatrix}. \tag{26}$$

It should be noted that the derivative operator is only applied on internal nodes. Now by substituting Eq. (20) and Eq. (25) into Eq. (15) together with boundary conditions in Eq. (2) and using the collocation points, we obtain the following matrix form:

$$\begin{gathered}\left[\left[\mathbf{a}^{n+1}(2\rho\delta t+1)+\theta\sigma^2\right]*\mathbf{T}_1-\theta\,\mathbf{D}+\mathbf{T}_2\right]\mathbf{c}^{n+1}=\\ \left[\left[\mathbf{a}^{n+1}(2\rho\delta t+2)-(1-\theta)\sigma^2\right]*\mathbf{T}_1+(1-\theta)\mathbf{D}\right]\mathbf{c}^{n}-\\ \mathbf{a}^{n+1}*\mathbf{T}_1\mathbf{c}^{n-1}+\mathbf{g}^{n+1}+\mathbf{f}^{n+1}+\mathbf{h}^{n+1},\end{gathered}\tag{27}$$

or briefly

$$\mathbf{B}\mathbf{c}^{n+1}=\mathbf{E}\mathbf{c}^{n}-\mathbf{a}^{n+1}*\mathbf{T}_1\mathbf{c}^{n-1}+\mathbf{g}^{n+1}+\mathbf{f}^{n+1}+\mathbf{h}^{n+1},\tag{28}$$

where

$$\begin{aligned}&\mathbf{B}=\left[\left[\mathbf{a}^{n+1}(2\rho\delta t+1)+\theta\sigma^2\right]*\mathbf{T}_1-\theta\,\mathbf{D}+\mathbf{T}_2\right],\\ &\mathbf{E}=\left[\left[\mathbf{a}^{n+1}(2\rho\delta t+2)-(1-\theta)\sigma^2\right]*\mathbf{T}_1+(1-\theta)\mathbf{D}\right],\\ &\mathbf{g}^{n+1}=-\mathbf{a}^{n+1}*\sum_{k=1}^{n}\mathbf{b}_k^{n+1}*\left[(2\rho\delta t+1)u_i^{n-k+1}-(2\rho\delta t+2)u_i^{n-k}+u_i^{n-k-1}\right],\\ &\mathbf{f}^{n+1}=[0,f_2^{n+1},...,f_{N-1}^{n+1},0]^T,\\ &\mathbf{h}^{n+1}=[h_1^{n+1},0,...,0,h_2^{n+1}]^T,\end{aligned}$$

and in which the operation $*$ is as defined in [14], and $\mathbf{g}^{n+1}$ is an $N\times 1$ column vector. Moreover, $\mathbf{u}^{n+1}$ can be computed by recurrence relation Eq. (28) for $n=1,\dots,M$. According to Eq. (28) for $n=1$, we have

$$\mathbf{B}\mathbf{c}^{2}=\mathbf{E}\mathbf{c}^{1}-\mathbf{a}^{2}*\mathbf{T}_1\mathbf{c}^{0}+\mathbf{g}^{2}+\mathbf{f}^{2}+\mathbf{h}^{2},\tag{29}$$

where $\mathbf{g}^2=\mathbf{a}^2*\left(\mathbf{b}_k^2*\left[\mathbf{u}^1-\mathbf{u}^0+\mathbf{u}^{-1}\right]\right)$.
Clearly $\mathbf{u}^0=u(\mathbf{x},0)=[(g_1)_1\ (g_1)_2...\ (g_1)_N]^T$, and then $\mathbf{c}^0=\mathbf{T}^{-1}\mathbf{u}^0$. Also, in order to compute $\mathbf{u}^1$ and $\mathbf{u}^{-1}$ we use numerical derivatives as follows

$$u_t(x,0)=\frac{u^1(x)-u^0(x)}{\delta t}=g_2(x),\qquad x\in[0,L].\tag{30}$$

So, we get

$$u^1(x)=u^0(x)+\delta t g_2(x),\tag{31}$$

and also, we have

$$u_t(x,0) = \frac{u^1(x) - u^{-1}(x)}{2\delta t}. \tag{32}$$

Now, Eqs. (31) and (32) gives $u^{-1}(x) = 2u^0(x) - u^1(x)$ and consequently $[\mathbf{c}]^{-1} = 2[\mathbf{c}]^0 - [\mathbf{c}]^1$.

4. Convergence and Stability

Suppose that the exact solution of Eq. (1) at time t^{n+1} is denoted by $\mathbf{U}_{ex}^{n+1}$. It is mentioned in [45, 42] that $|D^\alpha \mathbf{U}_{ex} - D^\alpha \mathbf{u}| \leqslant C_l h^{l-|\alpha|} |\mathbf{u}|_{N_\phi(\Omega)}$, where h is the distance between any two nodes in the bounded domain Ω, $N_\phi(\Omega)$ is a native space of RBFs ϕ, $l \in \mathbb{N}$ and $|\alpha| \leqslant l$.
From Eq. (28), we have

$$\mathbf{c}^{n+1} = \mathbf{B}^{-1}\mathbf{E}\mathbf{c}^n + \mathbf{B}^{-1}(-\mathbf{a}^{n+1} * \mathbf{T}_1)\mathbf{c}^{n-1} + \mathbf{B}^{-1}(\mathbf{g}^{n+1} + \mathbf{f}^{n+1} + \mathbf{h}^{n+1}), \tag{33}$$

and then

$$\begin{aligned} \mathbf{T}\mathbf{c}^{n+1} = \mathbf{T}\mathbf{B}^{-1}\mathbf{E}\mathbf{c}^n + \mathbf{T}\mathbf{B}^{-1}(-\mathbf{a}^{n+1} * \mathbf{T}_1)\mathbf{c}^{n-1} + \\ \mathbf{T}\mathbf{B}^{-1}(\mathbf{g}^{n+1} + \mathbf{f}^{n+1} + \mathbf{h}^{n+1}). \end{aligned} \tag{34}$$

Eq. (34) can also be written as follows

$$\begin{aligned} \mathbf{T}\mathbf{c}^{n+1} = \mathbf{T}\mathbf{B}^{-1}\mathbf{E}\mathbf{T}^{-1}\mathbf{T}\mathbf{c}^n + \mathbf{T}\mathbf{B}^{-1}(-\mathbf{a}^{n+1} * \mathbf{T}_1)\mathbf{T}^{-1}\mathbf{T}\mathbf{c}^{n-1} + \\ \mathbf{T}\mathbf{B}^{-1}(\mathbf{g}^{n+1} + \mathbf{f}^{n+1} + \mathbf{h}^{n+1}), \end{aligned} \tag{35}$$

or equivalently

$$\mathbf{u}^{n+1} = \mathbf{H}\mathbf{u}^n + \mathbf{K}\mathbf{u}^{n-1} + \mathbf{v}, \tag{36}$$

where

$$\begin{aligned} \mathbf{H} &= \mathbf{T}\mathbf{B}^{-1}\mathbf{E}\mathbf{T}^{-1}, \\ \mathbf{K} &= \mathbf{T}\mathbf{B}^{-1}(-\mathbf{a}^{n+1} * \mathbf{T}_1)\mathbf{T}^{-1}, \\ \mathbf{v} &= \mathbf{T}\mathbf{B}^{-1}(\mathbf{g}^{n+1} + \mathbf{f}^{n+1} + \mathbf{h}^{n+1}). \end{aligned}$$

By assuming that the recurrence scheme (36) is pth order accurate in space and considering Eqs. (12), (13) and (36), we obtain

$$\mathbf{U}_{ex}^{n+1} = \mathbf{H}\mathbf{U}_{ex}^{n} + \mathbf{K}\mathbf{U}_{ex}^{n-1} + \mathbf{v} + O(\delta t^{3-\alpha(x,t)} + h^p). \tag{37}$$

Subtracting Eqs. (36) from Eq. (37) and defining $\epsilon^n = \mathbf{U}_{ex}^n - \mathbf{u}^n$ yields

$$\epsilon^{n+1} = \mathbf{H}\epsilon^n + \mathbf{K}\epsilon^{n-1} + O(\delta t^{3-\alpha(x,t)} + h^p). \tag{38}$$

Hence, there exists a constant η such that

$$\|\epsilon^{n+1}\| \leqslant \|\mathbf{H}\|\|\epsilon^n\| + \|\mathbf{K}\|\|\epsilon^{n-1}\| + \eta O(\delta t^{3-\alpha(x,t)} + h^p). \tag{39}$$

Let $\bar{H}$ and $\bar{K}$ are upper bounds of $\|\mathbf{H}\|$ and $\|\mathbf{K}\|$, then we have

$$\|\epsilon^{n+1}\| \leqslant \bar{H}\|\epsilon^n\| + \bar{K}\|\epsilon^{n-1}\| + \eta O(\delta t^{3-\alpha(x,t)} + h^p). \tag{40}$$

Now we prove a theorem about an upper bound for $\|\epsilon^{n+1}\|$.

Theorem 4.1. *If $\alpha, \beta, \lambda \geqslant 0$, $0 \leqslant d_0 \leqslant c_0$, $0 \leqslant d_1 \leqslant c_1$ and the following recurrence equation and inequality relation are confirmed:*

$$d_n \leqslant \alpha d_{n-1} + \beta d_{n-2} + \lambda,$$
$$c_n = \alpha c_{n-1} + \beta c_{n-2} + \lambda,$$

then for $n \in \mathbb{N} - \{1\}$ we have

$$d_n \leqslant c_n.$$

Proof. We use mathematical induction:
As the base of induction suppose:

$$\begin{aligned} &d_2 \leqslant \alpha d_1 + \beta d_0 + \lambda \leqslant \alpha c_1 + \beta c_0 + \lambda = c_2,\\ &d_3 \leqslant c_3,\\ &\quad\vdots\\ &d_n \leqslant c_n. \end{aligned}$$

and we must prove the induction step:

$$d_{n+1} \leqslant c_{n+1}.$$

But we have

$$d_{n+1} \leqslant \alpha d_n + \beta d_{n-1} + \lambda \leqslant \alpha c_n + \beta c_{n-1} + \lambda = c_{n+1}. \quad \square$$

Because $\bar{H}$ and $\bar{K}$ are nonnegative we can consider the following recurrence equation corresponding to the recurrence inequality relation Eq. (40):

$$e_{n+1} = \bar{H}e_n + \bar{K}e_{n-1} + \eta O(\delta t^{3-\alpha(x,t)} + h^p). \tag{41}$$

Assuming $e_0 = \|\epsilon^0\| = 0$ and from Eq. (31) $e_1 = \|\epsilon^1\| = O(\delta t)$. Now putting $e_n = \gamma y^n$, for $\gamma, y \neq 0, n \in \mathbb{N} \cup \{0\}$. the difference equation Eq. (41) has the characteristic equation:

$$y^2 - \bar{H}y - \bar{K} = 0,$$

with the characteristic roots

$$y_1, y_2 = \frac{\bar{H} \pm \sqrt{\bar{H}^2 + 4\bar{K}}}{2},$$

and according to theorem (4.1)

$$\|\epsilon^{n+1}\| \leqslant e_{n+1} \leqslant \gamma_1|y_1|^n + \gamma_2|y_2|^n + \frac{\eta}{1 - \bar{H} - \bar{K}} O(\delta t^{3-\alpha(x,t)} + h^p), \tag{42}$$

where the stability condition $|y_1|, |y_2| < 1$ results in $\bar{H} + \bar{K} < 1$ [17, 2]. In Eq. (42), the unknown coefficients γ_1 and γ_2 must be calculated by using initial conditions $\|\epsilon^0\| = 0$ and $\|\epsilon^1\| = O(\delta t)$. Thus, we obtain

$$\|\epsilon^{n+1}\| \leqslant \bar{C}_1 O(\delta t) + \bar{C}_2 O(\delta t^{3-\alpha(x,t)} + h^p), \tag{43}$$

which guarantees the stability and convergence of our method.

5. Numerical Results

In this section, some numerical examples are illustrated to show the applicability and efficiency of the proposed method. To measure the accuracy of the proposed method, L_∞, L_2 errors and Root-Mean-Square

(RMS) of errors are used with the following definitions:

$$L_\infty = \max_{1 \leqslant i \leqslant N} |u(i) - u_{exact}(i)|,$$

$$L_2 = \sqrt{\sum_{i=1}^{N} |u(i) - u_{exact}(i)|^2},$$

$$RMS = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{N} |u(i) - u_{exact}(i)|^2\right)}.$$

The computations associated with the examples were performed using MATLAB R2014 software on a Personal Computer with intel core i3 3.60 GHz and RAM 8.00.

Example 5.1. Consider the V-TFTE (1) on the domain $[0, 2\pi]$ with $\rho = 1$, $\sigma = 1$ and $\alpha(x,t) = 1.8 - 0.1\cos(xt)\sin(x)$. The right hand side $f(x,t)$ is chosen such that the exact solution is $u(x,t) = t^3 \sin^2(x)$. The L_∞, L_2 errors and RMS errors for $N = 10,\ 20$ with some different values of δt are reported in Table 2. Note that for constant-orders $\alpha(x,t) = \alpha$, this problem has been solved in [18]. The space-time graphs of the absolute error and the approximate solution for $N = 20$ and $\delta t = 0.01$ are shown in Fig. 1. Also, the behavior of the approximate solution and absolute error for $N = 20$ and $\delta t = 0.001$ are illustrated in Fig. 2. From Figs. 1 and 2, one can see that by decreasing δt the numerical results improve.

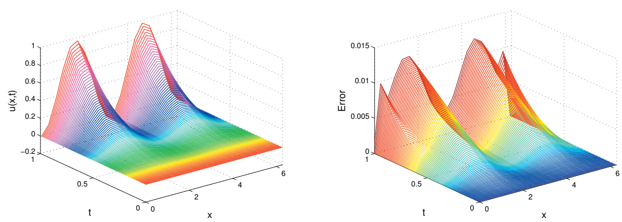

Figure 1. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 1.0$ and $\delta t = 0.01$ for Example 5.1.

Example 5.2. Consider the V-TFTE (1) on the domain $[0, 1]$ with $\rho = \frac{1}{2}$, $\sigma = 1$ and $\alpha(x,t) = 1 + 2^{-xt}$. The right hand side $f(x,t)$ which is compatible with the exact solution $u(x,t) = (t + x)\exp(-x^2)$. The L_∞, L_2 errors and RMS errors for $N = 10,\ 20$ with some different values of δt are reported in Table 3. Note that for constant-orders $\alpha(x;t) = \alpha$, this problem has been solved in [18]. The space-time graphs of the absolute error and the approximate solution for $N = 20$ and $\delta t = 0.01$ are shown in Fig. 3. Also, the behavior of the approximate solution and absolute error for $N = 20$ and $\delta t = 0.001$ are illustrated in Fig. 4. From Figs. 3 and 4, one can see that by decreasing δt the numerical results improve.

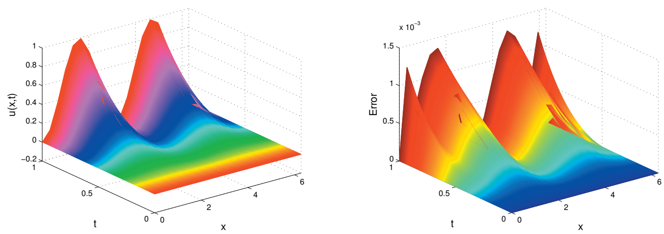

Figure 2. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 0.5$ and $\delta t = 0.01$ for Example 5.1.

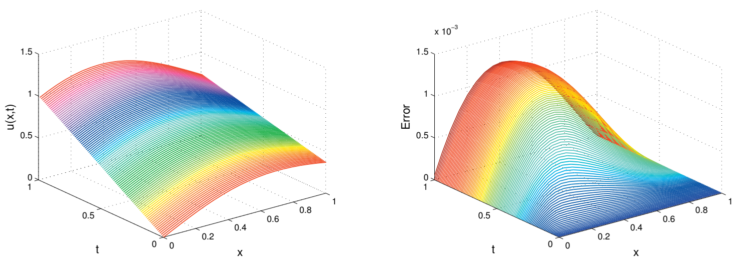

Figure 3. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 0.4$ and $\delta t = 0.001$ for Example 5.2.

Example 5.3. Consider the V-TFTE (1) on the domain $[0, 1]$ with $\rho = 1$, $\sigma = 1$ and $\alpha(x,t) = 1.2 + 0.6|x - t|$. The right hand side $f(x,t)$ is chosen such that the exact solution is $u(x,t) = x^2 t^3$. The L_∞, L_2 errors and RMS errors for $N = 10,\ 20$ with some different values of δt are reported in Table 4. The space-time graphs of the absolute error and the approximate solution for $N = 20$ and $\delta t = 0.01$ are shown in Fig.5. Also, the behavior of the approximate solution and absolute error for $N = 20$

and $\delta t = 0.001$ are illustrated in Fig. 6. From Figs. 5 and 6, one can see that by decreasing δt the numerical results improve.

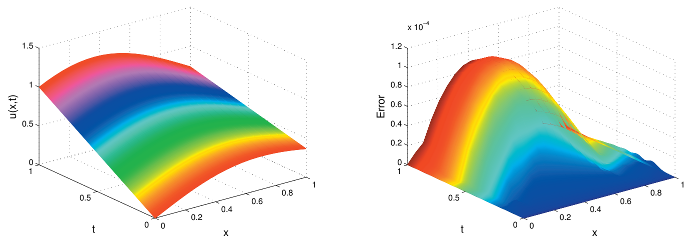

Figure 4. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 0.5$ and $\delta t = 0.01$ for Example 5.2.

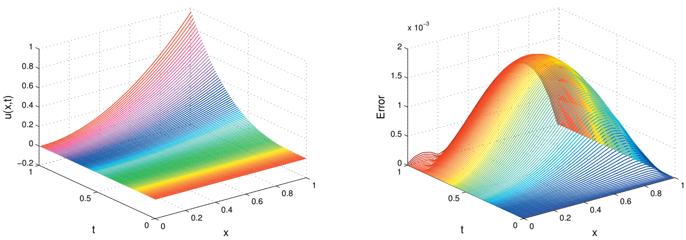

Figure 5. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 0.5$ and $\delta t = 0.001$ for Example 5.3.

Table 2: The errors for Example 5.1.

	$N = 10$					$N = 20$				
δt	L_∞-error	L_2-error	RMS	c	CPU time(s)	L_∞-error	L_2-error	RMS	c	CPU time(s)
0.050	7.5052×10^{-2}	1.5090×10^{-1}	4.7720×10^{-3}	1.0	00.10	7.0518×10^{-2}	1.7444×10^{-1}	3.9006×10^{-2}	0.1	00.10
0.010	1.7279×10^{-2}	3.3501×10^{-2}	1.0594×10^{-3}	1.0	00.35	1.2346×10^{-2}	3.8355×10^{-2}	8.5752×10^{-3}	0.3	00.43
0.005	8.8509×10^{-3}	1.7132×10^{-2}	5.4175×10^{-3}	2.0	01.16	7.2919×10^{-3}	2.0141×10^{-2}	4.5037×10^{-3}	1.0	01.37
0.001	3.8768×10^{-3}	5.7807×10^{-3}	1.8280×10^{-3}	2.5	25.09	1.3597×10^{-3}	4.0130×10^{-3}	8.9733×10^{-4}	1.5	30.56

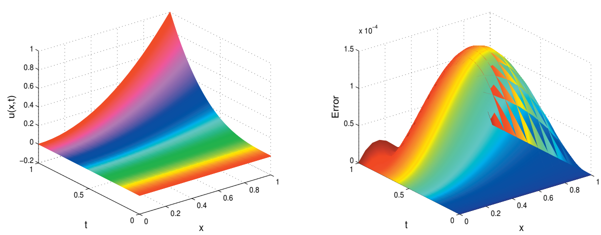

Figure 6. The graphs of the numerical solution (left side) and absolute error (right side) with $N = 20$, $c = 0.5$ and $\delta t = 0.001$ for Example 5.3.

Table 3: The errors for Example 5.2.

δt	$N=10$					$N=20$				
	L_∞-error	L_2-error	RMS	c	CPU time(s)	L_∞	L_2-error	RMS	c	CPU time(s)
0.050	5.8763×10^{-3}	1.2521×10^{-2}	3.9596×10^{-3}	1.0	00.08	5.8778×10^{-3}	1.8144×10^{-2}	4.0571×10^{-3}	0.5	00.10
0.010	1.1880×10^{-3}	2.5303×10^{-3}	8.0014×10^{-4}	1.0	00.36	1.1757×10^{-3}	3.6257×10^{-3}	8.1072×10^{-4}	0.5	00.45
0.005	5.9903×10^{-4}	1.2758×10^{-3}	4.0344×10^{-4}	1.0	01.20	5.8820×10^{-4}	1.8205×10^{-3}	4.0309×10^{-4}	0.5	01.51
0.001	1.2756×10^{-4}	2.7222×10^{-4}	8.6084×10^{-5}	1.0	25.06	8.8966×10^{-5}	2.5456×10^{-4}	5.6920×10^{-5}	0.4	30.59

Table 4: The errors for Example 5.3.

δt	$N=10$					$N=20$				
	L_∞-error	L_2-error	RMS	c	CPU time(s)	L_∞	L_2-error	RMS	c	CPU time(s)
0.050	7.6521×10^{-3}	1.4544×10^{-2}	4.5992×10^{-3}	1.0	00.08	7.7379×10^{-3}	2.1114×10^{-2}	4.7209×10^{-3}	0.5	00.11
0.010	1.2287×10^{-3}	2.3038×10^{-3}	7.2852×10^{-4}	1.0	00.35	1.2440×10^{-3}	3.3038×10^{-3}	7.3815×10^{-4}	0.5	00.39
0.005	5.7283×10^{-4}	1.0577×10^{-3}	3.3448×10^{-4}	1.0	01.11	5.6696×10^{-4}	1.4880×10^{-3}	3.3273×10^{-4}	0.5	01.32
0.001	1.1231×10^{-4}	2.0326×10^{-4}	6.4276×10^{-5}	1.0	24.03	9.3729×10^{-5}	2.4279×10^{-4}	5.4289×10^{-5}	0.5	28.19

6. Conclusion

In this paper, we studied and discussed on the V-TFTE. We proposed a hybrid numerical scheme based on the radial basis functions (RBFs) and finite difference methods in combination with collocation method. A time stepping approach was used to deal with the fractional time derivatives. The proposed method is very efficient and convenient for solving such initial-boundary value problems. The implementation of the proposed method is simple for solution of the problems under consideration. The main characteristic behind this approach is that it reduces such problems to those of solving a linear system of algebraic equations that leads to obtain a good approximate solution for the problem under study. The obtained results through the figures and tables illustrate that the proposed method computes numerical results with a good accuracy.

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