Journal of Mathematical Extension
Vol. 13, No. 2, (2019), 137-156
ISSN: 1735-8299
URL: http://www.ijmex.com

Classification of Translation Surfaces of Type 1 in Semi-Isotropic Space

M. K. Karacan
Usak University

A. Çakmak*
Bitlis Eren University

N. Yüksel
Erciyes University

S. Kızıltuğ
Erzincan University

Abstract. In this paper, we classify translation surfaces of Type1 in the three dimensional semi- isotropic space $\mathbb{SI}^3$ under the condition $\Delta^J x_i = \lambda_i x_i$, where Δ^J denotes the Laplacian of the surface with respect to the first, the second and the third fundamental forms. We also give explicit forms of these surfaces.

AMS Subject Classification: 53A35; 53B30
Keywords and Phrases: Semi-isotropic space, translation surfaces, Laplace operator

1. Introduction

A surface that arises when a curve $\alpha(u)$ is translated over another curve $\beta(v)$, is called a translation surface. A translation surface can be de-

Received: December 2017; Accepted: March 2018
*Corresponding author

fined as the sum of the two generating curves $\alpha(u)$ and $\beta(v)$. Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in a Euclidean 3-space $\mathbb{E}^3$ formed by translating two curves lying in orthogonal planes is the graph of a function $z(u,v) = f(u)+g(v)$, where $f(u)$ and $g(v)$ are smooth functions on some interval of $\mathbb{R}$ ([1,11]).

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(u,v) = f(u) + g(v)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(u,v) = \frac{1}{a}\log\left|\frac{\cos(au)}{\cos(av)}\right| = \frac{1}{a}\log\left|\cos(au)\right| - \frac{1}{a}\log\left|\cos(av)\right|,$$

where a is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces ([19]).

Translation surfaces have been investigated from various viewpoints by many differential geometers. Liu described translation surfaces having constant Gaussian and mean curvature in the Euclidean and Minkowski space ([14]). Goemans proved classification theorems of Weingarten translation surfaces ([11]). Baba-Hamed, Bekkar and Zoubir studied coordinate finite type translation surfaces in a 3-dimensional Minkowski space ([5]). Yoon classified coordinate finite type translation surfaces in a 3-dimensional Galilean space ([19]). Bekkar and Senoussi studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{\mathbf{III}}\mathbf{r}_i = \mu_i\mathbf{r}_i,$$

([6]). Cakmak, Karacan, Kiziltug and Yoon studied the translation surfaces in the 3-dimensional Galilean space under the condition

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i\mathbf{x}_i,$$

([10]). Sipus described translation surfaces in a simply isotropic space having constant isotropic Gaussian or mean curvature ([17]). Aydin studied the translation surfaces generated by a space curve and a planar curve in the isotropic 3-space $\mathbb{I}_3$ ([4]). Bukcu, Karacan and Yoon classified translation surfaces of Type 1 and Type 2 that satisfy the condition

$$\Delta^{\mathbf{I,II,III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

in the 3-dimensional simply isotropic space ([8,9,13]). Aydin defined semi-isotropic space $\mathbb{SI}^3$ which is Lorentz-Minkowski version of the isotropic space ([2,3]).

In this work, we describe the translation surfaces of Type 1 that satisfy the conditions $\Delta^{\mathbf{I,II,III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \lambda_i \in \mathbb{R}$.

2. Preliminaries

The semi-isotropic space $\mathbb{SI}^3$ is an a affine 3-space $\mathbb{R}^3$ endowed with the (semi-) norm defined as

$$\|u\| = \begin{cases} \sqrt{(u_1)^2 + (u_2)^2}\ , & \text{if } u_1 \neq 0 \text{ or } u_2 \neq 0 \\ u_3\ , \quad \text{if } u_1 = u_2 = 0, & u = (u_1, u_2, u_3) \in \mathbb{SI}^3. \end{cases}$$

The group of motions of $\mathbb{SI}^3$ is based a six-parameter group G_6 of affine transformations $(x, y, z) \to (x', y', z')$,

$$\begin{cases} x' = a + x\cosh\varphi + y\sinh\varphi \\ y' = b + x\sinh\varphi + y\cosh\varphi \\ \quad z' = c + dx + ey + z, \end{cases} \tag{1}$$

where $a, b, c, d, \varphi \in \mathbb{R}$. We call such transformations semi-isotropic congruence transformations or $(s-i)$-motions. Note that $(s-i)$-motions are the composition an affine shear transformation in z-direction and a Lorentzian motion in $xy-$plane. By a Lorentzian motion, we mean a translation and the following Lorentzian rotation

$$\begin{pmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{pmatrix},$$

which is one of the four kind isometries of the Lorentz-Minkowski plane $\mathbb{R}^2_1$. In the sequel, many of metric properties in semi-isotropic geometry (invariants under (1)) are Lorentzian invariants in their projections onto $\mathbb{R}^2_1$.

The semi-isotropic scalar product between two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in \mathbb{SI}^3$ is given by

$$\langle u, v\rangle = \begin{cases} u_1v_1 - u_2v_2 \ , & \text{if at least one of } u_i \text{ or } v_i \text{ is non zero, } i = 1, 2 \\ u_3v_3 \ , & \text{if } u_i = v_i = 0, \ i = 1, 2. \end{cases}$$

The vector product in the sense of semi-isotropic space is

$$u \times v = \begin{vmatrix} e_1 & -e_2 & 0 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

for $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$. It can be easily check that

$$\langle u \times v, w\rangle = \det (u, v, \widetilde{w}),$$

where $\widetilde{w}$ denotes the canonical projection of w onto $\mathbb{R}^2_1$. We call the vectors of the form $u = (0, 0, u_3)$ in $\mathbb{SI}^3$ isotropic vectors and ones of the form $u = (\widetilde{u} \neq 0, u_3)$ in $\mathbb{SI}^3$ non-isotropic vectors. A vector $u \in \mathbb{SI}^3$ is called spacelike, timelike and null (or lightlike) respectively if $\langle u, u\rangle > 0$ or $u = 0$, $\langle u, u\rangle < 0$ and $\langle u, u\rangle = 0$ $(u \neq 0)$, respectively. We remark that only non-isotropic vectors have causal character which is the property to be spacelike, timelike or null.

The set of all null vectors of $\mathbb{SI}^3$ is called null-cone, i.e.,

$$C : \left\{(x, y, z) \in \mathbb{SI}^3 | \ x^2 - y^2 = 0\right\} - \left\{0 \in \mathbb{SI}^3\right\}.$$

Timelike-cone is the set of all timelike vectors of $\mathbb{SI}^3$,

$$T : \left\{(x, y, z) \in \mathbb{SI}^3 | \ x^2 - y^2 < 0\right\}.$$

The semi-isotropic angle of two timelike non-isotropic vectors $u, v \in \mathbb{SI}^3$ lying in the same timelike-cone is defined as the Lorentzian angle between their projections onto $\mathbb{R}^2_1$, i.e.,

$$\langle u, v\rangle = - \|u\| \, \|v\| \cosh \varphi.$$

For a spacelike plane Γ determined by the non-isotropic vectors u, v the induced metric on Γ is positive definite and hence the angle between u and v is the usual Euclidean angle between $\widetilde{u}$ and $\widetilde{v}$.

Note that all isotropic vectors are orthonogal to non-isotropic ones. Further, two non-isotropic vectors u, v in $\mathbb{SI}^3$ are orthonogal if $\langle u, v\rangle = 0$. Let $\mathbf{M}$ be a surface immersed in $\mathbb{SI}^3$ without isotropic tangent planes. Then we call such a surface admissible. Denote g the metric on $\mathbf{M}$ induced from $\mathbb{SI}^3$. The surface $\mathbf{M}$ is said to be spacelike (resp. timelike, null) if g is positive definite (resp.a metric with index 1, degenerate).

Throughout this paper we consider only spacelike and timelike admissible surfaces in $\mathbb{SI}^3$. Assume that $\mathbf{M}$ has a local parameterization in $\mathbb{SI}^3$ as follows

$$\mathbf{x} : D \subseteq \mathbb{R}^2 \to \mathbb{SI}^3 : \quad (u_1, u_2) \to (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

If (g_{ij}) is the matrical expression of g with respect to the basis $\{\mathbf{x}_{u_1}, \mathbf{x}_{u_2}\}$, then we have

$$g_{ij} = \left\langle \mathbf{x}_{u_i}, \mathbf{x}_{u_j} \right\rangle, \quad \mathbf{x}_{u_i} = \frac{\partial \mathbf{x}}{\partial u_i}, \ i, j = 1, 2.$$

The metric g is positive definite if and only if $\det(g_{ij}) > 0$. If the surface $\mathbf{M}$ is timelike then $\det\,(g_{ij}) < 0$. If $\mathbf{M}$ is a graph surface in $\mathbb{SI}^3$ of the form

$$\mathbf{x}(u_1, u_2) = (u_1, u_2, z(u_1, u_2)) \tag{2}$$

then the metric on M induced from $\mathbb{SI}^3$ is $g = du_1^2 - du_2^2$ and it always becomes a at timelike surface. So, its Laplacian turns to

$$\Delta := \frac{\partial^2}{\partial u_1^2} - \frac{\partial^2}{\partial u_2^2}.$$

The unit normal vector field of $\mathbf{M}$ is the isotropic vector $(0, 0, 1)$ since it is perpendicular to all non-isotropic vectors. The coefficients of the second fundamental form are

$$h_{ij} := \frac{\det\left(\mathbf{x}_{u_1}, \mathbf{x}_{u_2}, \mathbf{x}_{u_i u_j}\right)}{\sqrt{\det(g_{ij})}}, \ \mathbf{x}_{u_i u_j} = \frac{\partial^2 \mathbf{x}}{\partial u_i \partial u_j}, \ i, j = 1, 2.$$

For the surfaces of the form (2), these coefficients are

$$h_{ij} := z_{u_i u_j} = \frac{\partial^2 z}{\partial u_i \partial u_j}, \; i,j = 1,2.$$

Thus the semi-relative curvature and the semi-isotropic mean curvature of $\mathbf{M}$ are defined by

$$\mathbf{K} = -\epsilon \frac{\det(h_{ij})}{\det(g_{ij})}$$

and

$$\mathbf{H} = -\epsilon \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\det(g_{ij})},$$

where $\epsilon = sgn(\det(g_{ij}))$,respectively. We call a surface semi-isotropic flat or $(s-i)$-flat (resp. semi-isotropic minimal or $(s-i)$-minimal) in $\mathbb{SI}^3$ if $\mathbf{K}$ (resp. $\mathbf{H}$) vanishes ([2, 3]).

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the first, the second and the third fundamental forms on $\mathbf{M}$ are defined by ([5, 6, 7, 9, 12, 13, 16])

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}} \begin{bmatrix} \frac{\partial}{\partial u}\left(\frac{g_{22}\mathbf{x}_u - g_{12}\mathbf{x}_v}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{g_{12}\mathbf{x}_u - g_{11}\mathbf{x}_v}{\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right) \end{bmatrix}, \tag{3}$$

$$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{\left|h_{11}h_{22} - h_{12}^2\right|}} \begin{bmatrix} \frac{\partial}{\partial u}\left(\frac{h_{22}\mathbf{x}_u - h_{12}\mathbf{x}_v}{\sqrt{\left|h_{11}h_{22} - h_{12}^2\right|}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{h_{12}\mathbf{x}_u - h_{11}\mathbf{x}_v}{\sqrt{\left|h_{11}h_{22} - h_{12}^2\right|}}\right) \end{bmatrix}, \tag{4}$$

and

$$\Delta^{\mathbf{III}}\mathbf{x} = -\frac{1}{h_{11}h_{22} - h_{12}^2\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}} \begin{bmatrix} \frac{\partial}{\partial u}\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{\left(h_{11}h_{22} - h_{12}^2\right)\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right) \\ -\frac{\partial}{\partial v}\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{\left(h_{11}h_{22} - h_{12}^2\right)\sqrt{\left|g_{11}g_{22} - g_{12}^2\right|}}\right) \end{bmatrix}, \tag{5}$$

where

$$
\begin{aligned}
X &= g_{11}h_{12}^2 - 2g_{12}h_{11}h_{12} + g_{22}h_{11}^2, \\
Y &= g_{11}h_{12}h_{22} - g_{12}h_{11}h_{22} + g_{22}h_{11}h_{12} - g_{12}h_{12}^2, \\
Z &= g_{22}h_{12}^2 - 2g_{12}h_{22}h_{12} + g_{11}h_{22}^2.
\end{aligned}
$$

2.1 Translation surfaces in $\mathbb{SI}^3$

In order to describe the semi-isotropic analogues of translation surfaces of constant curvatures, we consider translation surfaces obtained by translating two planar curves. The local surface parametrization is given by

$$\mathbf{x}(u,v) = \alpha(u) + \beta(v). \tag{6}$$

Therefore, the obtained translation surfaces allow the following parametrizations:

Type 1: The surface **M** is parametrized by

$$\mathbf{x}(u,v) = (u, v, f(u) + g(v)), \tag{7}$$

and the translated curves are $\alpha(u) = (u, 0, f(u))$, $\beta(v) = (0, v, g(v))$.

Type 2: The surface **M** is parametrized by

$$\mathbf{x}(u,v) = (u, f(u) + g(v), v), \tag{8}$$

and the translated curves are $\alpha(u) = (u, f(u), 0)$, $\beta(v) = (0, g(v), v)$. In order to obtain admissible surfaces, $g'(v) \neq 0$ is assumed (i.e. $g(v) \neq$const.).

Type 3: The surface **M** is parametrized by

$$\mathbf{x}(u,v) = \frac{1}{2}\left(f(u) + g(v), u - v + \pi, u + v\right), \tag{9}$$

and the translated curves are

$$\alpha(u) = \frac{1}{2}\left(f(u), u + \frac{\pi}{2}, u - \frac{\pi}{2}\right), \beta(v) = \left(g(v), \frac{\pi}{2} - v, \frac{\pi}{2} + v\right).$$

In order to obtain admissible surfaces, $f'(u) + g'(v) \neq 0$ is assumed (i.e. $f'(u) \neq -g'(v) = a =$constant.) ([17]).

In this paper, we will investigate the translation surface of Type 1 in the three dimensional semi-isotropic space.

3. Translation Surfaces of Type 1 Satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify translation surface in $\mathbb{SI}^3$ satisfying the equation

$$\Delta^{\mathbf{I}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{10}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{I}}\mathbf{x} = \left(\Delta^{\mathbf{I}}\mathbf{x}_1, \Delta^{\mathbf{I}}\mathbf{x}_2, \Delta^{\mathbf{I}}\mathbf{x}_3\right), \tag{11}$$

where

$$\mathbf{x}_1 = u,\ \mathbf{x}_2 = v,\ \mathbf{x}_3 = f(u) + g(v).$$

For the translation surface given by (7), the coefficients of the first and second fundamental forms are

$$g_{11} = 1, g_{12} = 0, g_{22} = -1, \tag{12}$$

$$h_{11} = f'', h_{12} = 0, h_{22} = g'', \tag{13}$$

respectively. Since $g_{11}g_{22} - g_{12}^2 < 0$, translation surface of Type 1 is timelike. The semi relative Gaussian curvature $\mathbf{K}$ and the semi- isotropic mean curvature $\mathbf{H}$ are

$$\mathbf{K} = -f''(u)g''(v), \quad \mathbf{H} = \frac{f''(u) - g''(v)}{2}, \tag{14}$$

respectively. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (3), (11) and (12) turns out to be

$$\Delta^{\mathbf{I}}\mathbf{x}_i = \left(0, 0, f''(u) - g''(v)\right). \tag{15}$$

Suppose that $\mathbf{M}$ satisfies (10). Then from (15), we have

$$\left(f''(u) - g''(v)\right) = \lambda\left(f(u) + g(v)\right), \tag{16}$$

where $\lambda \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 1-type. We discuss two cases according to constant λ. First of all, we assume that $\mathbf{M}$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x}_i = 0$. We call a surface satisfying that condition a harmonic surface or semi-isotropic minimal. In this case, we get from (16)

$$f''(u) - g''(v) = 0. \tag{17}$$

Here u and v are independent variables, so each side of (17) must equal to a constant, call it p. Hence, the two equations

$$f'' = p = g''. \tag{18}$$

Thus we get

$$\begin{aligned} f(u) &= p\frac{u^2}{2} + c_1 u + c_2, \\ g(v) &= p\frac{v^2}{2} + c_3 v + c_4. \end{aligned} \tag{19}$$

where $p, c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \left(p\frac{u^2}{2} + c_1 u + c_2\right) + \left(p\frac{v^2}{2} + c_3 v + c_4\right)\right). \tag{20}$$

In particular, if $p = 0$, we have

$$\begin{aligned} f(u) &= c_1 u + c_2, \\ g(v) &= c_3 u + c_4, \end{aligned} \tag{21}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = (u, v, (c_1 u + c_2) + (c_3 v + c_4)). \tag{22}$$

Theorem 3.1. *Let* $\mathbf{M}$ *be a translation surface given by (7) in* $\mathbb{SI}^3$*. If* $\mathbf{M}$ *is harmonic or semi-isotropic minimal, then it is congruent to an open part of the surface (20) or (22).*

If $\lambda \neq 0$, from (16), we have

$$f''(u) - \lambda f(u) = g''(v) + \lambda g(v). \tag{23}$$

Here u and v are independent variables, so each side of (23) is equal to a constant, call it p. Hence, we have the two equations

$$f''(u) - \lambda f(u) = p = g''(v) + \lambda g(v). \tag{24}$$

These equations are second order linear differential equations with constant coefficients. We discuss two cases according to constant λ.

Case 1: $\lambda > 0$, from (24), we obtain

$$\begin{aligned} f''(u) - \lambda f(u) &= p, \\ g''(v) + \lambda g(v) &= p, \end{aligned} \tag{25}$$

and

$$\begin{aligned} f(u) &= -\frac{p}{\lambda} + c_1 e^{u\sqrt{\lambda}} + c_2 e^{-u\sqrt{\lambda}}, \\ g(v) &= \frac{p}{\lambda} + c_3 \cos v\sqrt{\lambda} + c_4 \sin v\sqrt{\lambda}, \end{aligned} \tag{26}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u, \\ v, \\ \left(-\frac{p}{\lambda} + c_1 e^{u\sqrt{\lambda}} + c_2 e^{-u\sqrt{\lambda}}\right) \\ + \left(\frac{p}{\lambda} + c_3 \cos v\sqrt{\lambda} + c_4 \sin v\sqrt{\lambda}\right) \end{pmatrix}. \tag{27}$$

In particular, if $p = 0$, we have

$$\begin{aligned} f(u) &= c_1 e^{u\sqrt{\lambda}} + c_2 e^{-u\sqrt{\lambda}}, \\ g(v) &= c_3 \cos v\sqrt{\lambda} + c_4 \sin v\sqrt{\lambda}, \end{aligned} \tag{28}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u, \\ v, \\ \left(c_1 e^{u\sqrt{\lambda}} + c_2 e^{-u\sqrt{\lambda}}\right) \\ + \left(c_3 \cos v\sqrt{\lambda} + c_4 \sin v\sqrt{\lambda}\right) \end{pmatrix}. \tag{29}$$

Case 2: $\lambda < 0$, from (23), we obtain

$$\begin{aligned} f''(u) + \lambda f(u) &= p, \\ g''(v) - \lambda g(v) &= p, \end{aligned} \tag{30}$$

and

$$\begin{aligned} f(u) &= \frac{p}{\lambda} + c_1 \cos u\sqrt{\lambda} + c_2 \sin u\sqrt{\lambda}, \\ g(v) &= -\frac{p}{\lambda} + c_3 e^{v\sqrt{-\lambda}} + c_4 e^{-v\sqrt{-\lambda}}, \end{aligned} \tag{31}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u, \\ v, \\ \left(\frac{p}{\lambda} + c_1 \cos u\sqrt{\lambda} + c_2 \sin u\sqrt{\lambda}\right) \\ + \left(-\frac{p}{\lambda} + c_3 e^{v\sqrt{-\lambda}} + c_4 e^{-v\sqrt{-\lambda}}\right) \end{pmatrix}. \tag{32}$$

In particular, if $p = 0$, we have

$$\begin{aligned} f(u) &= c_1 \cos u\sqrt{\lambda} + c_2 \sin u\sqrt{\lambda}, \\ g(v) &= c_3 e^{v\sqrt{-\lambda}} + c_4 e^{-v\sqrt{-\lambda}}, \end{aligned} \tag{33}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u, \\ v, \\ \left(c_1 \cos u\sqrt{\lambda} + c_2 \sin u\sqrt{\lambda}\right) \\ + \left(c_3 e^{v\sqrt{-\lambda}} + c_4 e^{-v\sqrt{-\lambda}}\right) \end{pmatrix}. \tag{34}$$

Theorem 3.2. *Let* $\mathbf{M}$ *be a non harmonic translation surface given by (7) in the three dimensional semi-isotropic space* $\mathbb{SI}^3$*. If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{I}}\mathbf{x}_i=\lambda_i\mathbf{x}_i$*, where* $\lambda_i\in\mathbb{R}$*,* $i=1,2,3$*, then it is congruent to an open part of the surfaces (27), (29), (32) or (34).*

4. Translation Surfaces of Type 1 Satisfying $\Delta^{\mathbf{II}}\mathbf{x}_i= \lambda_i\mathbf{x}_i$

In this section, we classify translation surfaces with non-degenerate second fundamental form in $\mathbb{SI}^3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i= \lambda_i\mathbf{x}_i, \tag{35}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\Delta^{\mathbf{II}}\mathbf{x}_1, \Delta^{\mathbf{II}}\mathbf{x}_2, \Delta^{\mathbf{II}}\mathbf{x}_3\right). \tag{36}$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4), (14) and (36) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\frac{f'''}{2f''^2}, \frac{g'''}{2g''^2}, -2 + \frac{ff'''}{2f''^2} + \frac{g'g'''}{2g''^2}\right). \tag{37}$$

The equation (35) by means of (37) gives rise to the following system of ordinary differential equations

$$\frac{f'''}{2f''^2} = \lambda_1 u, \tag{38}$$

$$\frac{g'''}{2g''^2} = \lambda_2 v, \tag{39}$$

$$-2 + f'\frac{f'''}{2f''^2} + g'\frac{g'''}{2g''^2} = \lambda_3 \left(f(u) + g(v)\right), \tag{40}$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3- types. Combining equations (38), (39) and (40), we have

$$\lambda_1 u f' - \lambda_3 f - 2 = -\lambda_2 v g' + \lambda_3 g. \tag{41}$$

Here u and v are independent variables, so each side of (41) is equal to a constant, call it p. Hence, we have the two equations

$$\lambda_1 u f' - \lambda_3 f - 2 = p = -\lambda_2 v g' + \lambda_3 g. \tag{42}$$

Thus we get

$$\begin{aligned} f(u) &= -\frac{2+p}{\lambda_3} + c_1 u^{\frac{\lambda_3}{\lambda_1}}, \\ g(v) &= \frac{p}{\lambda_3} + c_2 v^{\frac{\lambda_3}{\lambda_2}}, \end{aligned} \tag{43}$$

where for some constants $c_i \neq 0$ and $\lambda_i \neq 0$. In particular, if $p = 0$, then we have

$$\begin{aligned} f(u) &= -\frac{2}{\lambda_3} + c_1 u^{\frac{\lambda_2}{\lambda_1}}, \\ g(v) &= c_2 v^{-\frac{\lambda_3}{\lambda_2}}. \end{aligned} \tag{44}$$

We discuss seven cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1: Let $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$, from (42), we obtain

$$-\lambda_3 f - 2 = p = -\lambda_2 v g' + \lambda_3 g. \tag{45}$$

This differential equations admit the solutions

$$\begin{aligned} f(u) &= -\frac{2+p}{\lambda_3}, \\ g(v) &= c_1 v^{\frac{\lambda_3}{\lambda_2}} + \frac{p}{\lambda_3}, \end{aligned} \tag{46}$$

where $p, c_1 \neq 0 \in \mathbb{R}$.

Case 2: Let $\lambda_1 = 0, \lambda_2 = 0,\ \lambda_3 \neq 0$, from (42), we obtain

$$-\lambda_3 f - 2 = p = \lambda_3 g. \tag{47}$$

We can get easily

$$\begin{aligned} f(u) &= -\frac{2+p}{\lambda_3}, \\ g(v) &= \frac{p}{\lambda_3}, \end{aligned} \tag{48}$$

where $p \in \mathbb{R}$.

Case 3: Let $\lambda_1 = 0, \lambda_2 \neq 0,\ \lambda_3 = 0$, from (42), we obtain

$$-2 = -\lambda_2 v g'. \tag{49}$$

We can get easily

$$g(v) = c_1 + \frac{2 \log v}{\lambda_2}, \tag{50}$$

where $c_1 \in \mathbb{R}$. Here, the function $f(u)$ independent of selection of the function $g(v)$. We can choose the function $f(u)$ as below

$$f(u) = c_2 u^2 + c_3 u + c_4. \tag{51}$$

Case 4: Let $\lambda_1 \neq 0, \lambda_2 = 0,\ \lambda_3 = 0$, from (42), we obtain

$$\lambda_1 u f' - 2 = 0. \tag{52}$$

Also, the general solution of (52) can be given by

$$f(u)=c_1+\frac{2\log u}{\lambda_1}, \tag{53}$$

where $c_1\in\mathbb{R}$. Here, the function $g(v)$ independent of selection of the function $f(u)$. We can choose the function $g(v)$ as below

$$g(v)=c_2v^2+c_3v+c_4. \tag{54}$$

Case 5: Let $\lambda_1\neq 0,\lambda_2\neq 0,\ \lambda_3=0$, from (42), we obtain

$$\lambda_1uf'-2=p=-\lambda_2vg'. \tag{55}$$

Hence, the general solutions of (55) are given by

$$\begin{aligned} f(u)&=c_1+\frac{(2+p)\log u}{\lambda_1},\\ g(v)&=c_2-\frac{p\log v}{\lambda_2}, \end{aligned} \tag{56}$$

where $c_1,c_2,p\in\mathbb{R}$.

Case 6: Let $\lambda_1\neq 0,\lambda_2=0,\ \lambda_3\neq 0$, from (42), we obtain

$$\lambda_1uf'-\lambda_3f-2=p=\lambda_3g. \tag{57}$$

and its general solutions are

$$\begin{aligned} f(u)&=-\frac{2+p}{\lambda_3}+c_1u^{\frac{\lambda_3}{\lambda_1}},\\ g(v)&=\frac{p}{\lambda_3}, \end{aligned} \tag{58}$$

where $c_1,p\in\mathbb{R}$.

Case 7: Let $\lambda_1=\lambda_2=\lambda_3=0$, from (42), we obtain $-2=0$. We obtain a contradiction.

The solutions (48) and (58) give a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form. The solutions (43), (44), (46) do not satisfy (38) and (39) simultaneously. The solutions (50), (51), (53), (54) and (56) satisfy (38) and (39) simultaneously.

Definition 4.1. *A surface in the three dimensional semi-isotropic space* $\mathbb{SI}^3$ *is said to be* $\mathbf{II}-$*harmonic if it satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}=\mathbf{0}$.

Corollary 4.2. *There is no* $\mathbf{II}-$*harmonic translation surface of Type 1 given by (7) in the three dimensional semi- isotropic space* $\mathbb{SI}^3$.

Theorem 4.3. *Let* $\mathbf{M}$ *be a non* $\mathbf{II}$*-harmonic translation surface of Type 1 with non-degenerate second fundamental form given by (7) in the three dimensional semi-isotropic space* $\mathbb{SI}^3$*. If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}_i{=}\lambda_i\mathbf{x}_i$, *where* $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$, *then it is congruent to an open part of the following surfaces:*

$$\mathbf{x}(u,v)=\left(u,v,\left(c_2u^2+c_3u+c_4\right)+\left(c_1+\frac{p\log v}{\lambda_2}\right)\right),$$

$$\mathbf{x}(u,v)=\left(u,v,\left(c_1+\frac{2\log u}{\lambda_1}\right)+\left(c_2v^2+c_3u+c_4\right)\right),$$

or

$$\mathbf{x}(u,v)=\left(u,v,\left(c_1+\frac{(2+p)\log u}{\lambda_1}\right)+\left(c_2-\frac{p\log v}{\lambda_2}\right)\right).$$

5. Translation Surfaces of Type 1 Satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i{=}\ \lambda_i\mathbf{x}_i$

In this section, we classify translation surface of Type 1 with non-degenerate second fundamental form in $\mathbb{SI}^3$ satisfying the equation

$$\Delta^{\mathbf{III}}\mathbf{x}_i{=}\ \lambda_i\mathbf{x}_i, \tag{59}$$

where $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$ and

$$\Delta^{\mathbf{III}}\mathbf{x}=\left(\Delta^{\mathbf{III}}\mathbf{x}_1,\Delta^{\mathbf{III}}\mathbf{x}_2,\Delta^{\mathbf{III}}\mathbf{x}_3\right). \tag{60}$$

Using (60), the Laplacian of **M** can be expressed as follows

$$\Delta^{\mathbf{III}}\mathbf{x} = \left(\frac{-f'''}{f''^3}, \frac{+g'''}{g''^3}, \frac{-f''^3 g''^2 + f''^2 g''^3 - f' g''^3 f''' + g' f''^3 g'''}{f''^3 g''^3} \right). \tag{61}$$

By using (59) and (61), we have the following equations

$$-\left(\frac{f'''}{f''^3}\right) = \lambda_1 u, \tag{62}$$

$$\left(+\frac{g'''}{g''^3}\right) = \lambda_2 v, \tag{63}$$

$$\frac{-f''^3 g''^2 + f''^2 g''^3 - f' g''^3 f''' + g' f''^3 g'''}{f''^3 g''^3} = \lambda_3 \left(f(u) + g(v)\right), \tag{64}$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. This means that **M** is at most of 3- types. Combining equations (62), (63) and (64), we have

$$f'\lambda_1 u + \frac{1}{f''} - \lambda_3 f = -g'\lambda_2 v + \frac{1}{g''} + \lambda_3 g. \tag{65}$$

Here u and v are independent variables, so each side of (65) is equal to constant, call it p. Hence, we have

$$f'\lambda_1 u + \frac{1}{f''} - \lambda_3 f = p = -g'\lambda_2 v + \frac{1}{g''} + \lambda_3 g. \tag{66}$$

If we choose $p = 0$, then we get

$$f'\lambda_1 u + \frac{1}{f''} - \lambda_3 f = 0 = -g'\lambda_2 v + \frac{1}{g''} + \lambda_3 g, \tag{67}$$

where $c_i, \lambda_i \in \mathbb{R}$.
We discuss only one case according to constants $\lambda_1, \lambda_2, \lambda_3$. Because, there are no any suitable solutions for the functions $f(u)$ and $g(v)$ satisfying the equation $\Delta^{\mathbf{III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ in the other cases.

Case 1: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (62), (63) and (68), we obtain

$$\begin{aligned} f'''(u) &= 0, \\ g'''(v) &= 0, \\ -\frac{1}{f''} + \frac{1}{g''} &= 0. \end{aligned} \tag{68}$$

Their common solutions are given by

$$\begin{aligned} f(u) &= \frac{u^2}{2p} + c_1 u + c_2 \\ g(v) &= \frac{v^2}{2p} + c_3 v + c_4, \end{aligned} \tag{69}$$

where $c_i, p \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \left(\frac{u^2}{2p} + c_1 u + c_2\right) + \left(\frac{v^2}{2p} + c_3 v + c_4\right)\right). \tag{70}$$

Definition 5.1. *A surface in the three dimensional semi-isotropic space $\mathbb{SI}^3$ is said to be* **III**-*harmonic if it satisfies the condition* $\Delta^{\mathbf{III}}\mathbf{x} = \mathbf{0}$.

Theorem 5.2. *Let* $\mathbf{M}$ *be a translation surface of Type 1 with non-degenerate second fundamental form given by (7) in the three dimensional semi-isotropic space* $\mathbb{SI}^3$. *The surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i$=0, *then it is congruent to an open part of the surface (74).*

Theorem 5.3. *(Classification)Let* $\mathbf{M}$ *be a translation surface of Type 1 with non-degenerate second fundamental form given by (7) in the three dimensional semi-isotropic space* $\mathbb{SI}^3$. *There is no surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i=\lambda_i\mathbf{x}_i$, *where* $\lambda_i\in\mathbb{R}$.

Acknowledgements
The authors would like to thank the anonymous referees for their valuable comments and suggestions.

References

[1] K. Arslan, B. Bayram, B. Bulca, and G. Ozturk, On translation surfaces in 4-dimensional Euclidean space, *Acta Et Commentationes Universitatis Tartuensis De Mathematica*, 20 (2) (2016), 123-133.

[2] M. E. Aydin, Surfaces of constant curvature in a semi-isotropic space, https://arxiv.org/pdf/1609.02437v1.pdf.

[3] M. E. Aydin, Certain classifications on surfaces of revolution in a semi-isotropic space, https://arxiv.org/pdf/1609.07270.pdf.

[4] M. E. Aydin, A generalization of translation surfaces with constant curvature in the isotropic space, *Journal of Geometry*, 107 (3) (2016), 603–615.

[5] Ch. Baba-Hamed, M. Bekkar, and H. Zoubir, Translation surfaces in the three-dimensional Lorentz-Minkowski space satisfying $\Delta r_i = \lambda_i r_i$, *Int. Journal of Math. Analysis*, 4 (17) (2010), 797-808.

[6] M. Bekkar and B. Senoussi, Translation surfaces in the 3-dimensional space satisfying $\Delta^{III} r_i = \mu_i r_i$, *J. Geom.*, 103 (2012), 367–374.

[7] M. Bekkar and H. Zoubir, Surfaces of revolution in the 3-Dimensional Lorentz-Minkowski Space Satisfying $\Delta x^i = \lambda^i x^i$, *Int. J. Contemp. Math. Sciences*, 3 (24) (2008), 1173-1185.

[8] B. Bukcu, D. W. Yoon, and M. K. Karacan, Translation surfaces of type 2 in the three dimensional simply isotropic space $\mathbb{I}_3^1$, *Bull. Korean Math. Soc.*, https://doi.org/10.4134/BKMS.b160377.

[9] B. Bukcu, D. W. Yoon, and M. K. Karacan, Translation surfaces in the three dimensional simply isotropic space $\mathbb{I}_3^1$ satisfying $\Delta^{III} x_i = \lambda_i x_i$, *Konuralp Journal of Mathematics*, 4 (1) (2016), 275-281.

[10] A. Cakmak, M. K. Karacan, S. Kiziltug, and D. W. Yoon, Translation surfaces in the 3-dimensional Galilean space satisfying $\Delta^{II} x_i = \lambda_i x_i$, *Bull. Korean Math. Soc.*, https://doi.org/10.4134/BKMS.b160442.

[11] W. Goemans, *Surfaces in Three-Dimensional Euclidean and Minkowski Space, in Particular a Study of Weingarten Surfaces*, PhD. Dissertation, September 2010.

[12] G. Kaimakamis, B. Papantoniou, and K. Petoumenos, Surfaces of revolution in the 3-dimensional Lorentz-Minkowski space satisfying $\Delta^{III} r = Ar$, *Bull.Greek Math. Soc.*, 50 (2005), 75-90.

[13] M. K. Karacan, D. W. Yoon, and B. Bukcu, Translation surfaces in the three dimensional simply isotropic space $\mathbb{I}_3^1$, *Int. J. Geom. Methods Mod. Phys.*, 13, (2016), (9 pages). DOI:http://dx.doi.org/10.1142/S0219887816500882.

[14] H. Liu, Translation surfaces with constant mean curvature in 3-dimensioanl spaces,*J. Geom.*, 64 (1999), 141–149.

[15] H. Sachs, *Isotrope Geometrie des Raumes*, Vieweg Verlag, Braunschweig, 1990.

[16] B. Senoussi and M. Bekkar, Helicoidal surfaces with $\Delta^J r = Ar$ in 3-dimensional Euclidean space, *Stud. Univ. Babes-Bolyai Math.*, 60 (3) (2015), 437-448.

[17] Z. M. Sipus, Translation Surfaces of constant curvatures in a simply isotropic space, *Period Math. Hung.,* 68 (2014), 160–175.

[18] K. Strubecker, *Differentialgeometrie des Isotropen Raumes III*, Flachentheorie, Math. Zeitsch, 48 (1942), 369-427.

[19] D. W. Yoon, Some Classification of Translation surfaces in Galilean 3-space, *Int. Journal of Math. Analysis*, 6 (28) (2012), 1355-1361.

[20] D. W. Yoon, C. W. Lee, and M. K.Karacan, Some translation surfaces in the 3-dimensional Heisenberg Group, *Bull. Korean Math. Soc.*, 50 (4) (2013), 1329–1343.

Murat Kemal Karacan
Associate Professor of Mathematics
Department of Mathematics
Faculty of Arts and Sciences
Usak University, 1 Eylul Campus, 64200
Usak, Turkey
E-mail: murat.karacan@usak.edu.tr

Ali Çakmak
Assistant Professor of Mathematics
Department of Mathematics
Faculty of Arts and Sciences
Bitlis Eren University, 13000
Bitlis, Turkey
E-mail: acakmak@beu.edu.tr

Nural Yüksel
Assistant Professor of Mathematics
Department of Mathematics
Faculty of Sciences
Erciyes University, 38039
Kayseri, Turkey
E-mail: yukseln@erciyes.edu.tr

Sezai Kiziltug
Associate Professor of Mathematics
Department of Mathematics
Faculty of Arts and Sciences
Erzincan University, 24000
Erzincan, Turkey
E-mail: skiziltug@erzincan.edu.tr