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The Dual Notion of Fuzzy Prime Submodules

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Abstract. Let R be a commutative ring and let M be an R-module. In this paper, we introduce the dual notion of fuzzy prime (that is, fuzzy second) submodules of M and explore some of the basic properties of this class of submodules. We say a non-zero fuzzy submodule μ of M is fuzzy second if for each $r \in R$, we have $1_r.\mu = \mu$ or $1_r.\mu = 1_\theta$. It is shown that the fuzzy second submodules is a proper subclass of the fuzzy coprimary submodules.

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1. Introduction

Throughout this paper, R is a commutative ring with a non-zero identity, M is an R-module, and θ is the zero element of M.

In 1965, Zadeh [10] introduced a fuzzy subset μ of X as a map from X to the unit interval $I := [0 : 1]$. Since then, the theory of fuzzy set has advanced in many disciplines. Application of this theory can be found,

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for example in artificial intelligence, computer science, medicine, control engineering, decision theory, expert system, logic, etc. The notion of fuzzy sets was applied in algebra as one as one of the first branches from among various branches of pure mathematics. In 1971, Rosenfeld [15] introduced the fuzzy subgroupoid and fuzzy subgroups. The concept of fuzzy modules and fuzzy submodules was first introduced by Negoita and Ralescu in 1975 [5]. Consequently, fuzzy finitely generated submodules, fuzzy quotient modules [12], radical of fuzzy submodules, and primary fuzzy submodules [13], [15] were investigated.

A non-constant fuzzy submodule μ is said to be prime if for fuzzy ideal ζ and fuzzy submodule ν such that $\zeta.\nu \subseteq \mu$, then either $\nu \subseteq \mu$ or $\zeta \subseteq (\mu : 1_M)$ [3].

The concept of second submodules, as a dual of prime submodules, was first introduced by Yassemi [4]. A non-zero submodule N of M is said to be *second* if for each $a \in R$, the endomorphism of M given by multiplication by a is either surjective or zero. This implies that $Ann_R(N)$ is a prime ideal of R [4].

A non-zero fuzzy submodule μ of M is called a *coprimary (or secondary) fuzzy submodule* if for each $r \in R$ either $1_r.\mu = \mu$ or there exists $n \in \mathbb{N}$ such that $1_{r^n}.\mu = 1_\theta$ [1].

In the last two decades, there is a considerable amount of researches concerning the fuzzy prime submodules of modules and this notion attracted attention by a number of authors. It is natural to ask the following question: To what extend does the dual of results hold for fuzzy second submodules of an R-module M? The main purpose of this paper is to provide some information in this case.

We say a non-zero fuzzy submodule μ of M is *fuzzy second* if for each $r \in R$, $1_r.\mu = \mu$ or $1_r.\mu = 1_\theta$.

In Section 3 of this paper, among the other results, we investigate the relationship between fuzzy second and fuzzy coprimary submodules (see Theorem 3.8). Moreover, we give an example which illustrates fuzzy second submodules is a proper subclass of fuzzy coprimary submodules (see Example 3.7). In Theorem 3.11, we show that the sum (resp. quotient) of two fuzzy submodules is a fuzzy second submodule. Theorem 3.15 says that if λ and μ are two fuzzy submodules of M and 1_M is a fuzzy second with $\mu + \lambda \supseteq 1_M$, then either $\lambda \supseteq 1_M$ or $(1_\theta : 1_M) \supseteq (1_\theta : \mu)$. In Theorem 3.18, we provide some useful characterization for fuzzy second submodules. Also in Theorem 3.20., we show that if $(1_\theta : \mu)$ is a maximal fuzzy ideal, then μ is a fuzzy second submodule of M. Furthermore, we introduce the notion of minimal fuzzy submodules and prove that for a minimal fuzzy submodule μ of M, $(1_\theta : \mu)$ is a maximal fuzzy ideal (see Theorem 3.24). Finally, we show that every minimal fuzzy submodule is a fuzzy second submodule (see Corollary 3.25).

2. Preliminaries

In this section, we recall some basic definitions and remarks which are needed in the sequel.

A fuzzy subset μ of a non-empty set X is defined as a map from X to the unit interval $I := [0, 1]$. We denote the set of all fuzzy subsets of X by $F(X)$. Let $\mu, \lambda \in F(X)$. Then the inclusion $\mu \subseteq \lambda$ (resp. $\mu \subset \lambda$) is denoted by $\mu(x) \leqslant \lambda(x)$ (resp. $\mu(x) < \lambda(x)$) for all $x \in X$.

We write $\wedge$ and $\vee$ for infimum and supremum, respectively.
Let $\mu, \lambda \in F(X)$. Then $\mu \cap \lambda$ and $\mu \cup \lambda$ are defined as follows:

$$(\mu \cap \lambda)(x) = \mu(x) \wedge \lambda(x),$$
$$(\mu \cup \lambda)(x) = \mu(x) \vee \lambda(x), \text{ for all } x \in X.$$

Let $\mu \in F(X)$. Then μ has sup property if every subset of $\mu(x)$ has a maximal element.

Let $\mu \in F(X)$ and $t \in I$. Then the set $\mu_t = \{x \in X | \mu(x) \geqslant t\}$ is called the *t-level subset* of X with respect to μ and $\mu^* = \{x \in X, \mu(x) > \mu(0)\}$ is called *the support* of μ. Also if $\mu \in F(X)$, then $\mu_* = \{x \in X,\ \mu(x) = \mu(0)\}$.

Definition 2.1. *(See [2].) Let $Y \subseteq X$ and $t \in I$. Then*

$$t_Y(x) = \begin{cases} t & \text{if } x \in Y\ ; \\ 0 & \text{Otherwise.} \end{cases}$$

In particular, if $Y = \{y\}$*, then* t_y *is often refereed to a fuzzy singleton point (or fuzzy point). Moreover,* 1_Y *is refereed as the characteristic function of* Y*.*

If t_y *is a fuzzy singleton point and* $t_y \subseteq \mu \in F(X)$*, we write* $t_y \in \mu$*.*

Definition 2.2. *(See [4].)*

• *Let* f *be a mapping from* X *into* Y *such that* $\mu \in F(X)$ *and* $\nu \in F(Y)$*. Then* $f(\mu) \in F(Y)$ *and* $f^{-1}(\nu) \in F(X)$*, defined by* $\forall y \in Y$*,*

$$f(\mu(y)) = \begin{cases} \vee\{\mu(x) \mid x \in X,\, f(x) = y\} & \text{if } f^{-1}(y) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

and $\forall x \in X$*,*

$$f^{-1}(\nu)(x) = \nu(f(x)).$$

• *A fuzzy subset* ξ *of a ring* R *is called a fuzzy ideal of* R *if it satisfies the following properties:*

(i) $\xi(x-y) \geqslant \xi(x) \wedge \xi(y))$, for all $x, y \in R$; and

(ii) $\xi(xy) \geqslant \xi(x) \vee \xi(y))$, for all $x, y \in R$.

• *A fuzzy subset* $\mu \in F(M)$ *is called a fuzzy submodule if*

(i) $\mu(\theta) = 1$,

(ii) $\mu(rx) \geqslant \mu(x)$, for all $r \in R$ and $x \in M$,

(iii) $\mu(x+y) \geqslant \mu(x) \wedge \mu(y)$, for all $x, y \in M$.

In the following, we denote the set of fuzzy submodules (resp. fuzzy ideals) of M *(resp., 'of* R*) by* $FS(M)$ *(resp.* $FI(R)$*). The zero fuzzy submodule of* M *(resp. fuzzy ideal of* R*) is* 1_θ *(resp.* 1_0*).*

• *If* $\lambda \in FS(M)$*, then* $\lambda_* = \{x \in M \mid \lambda(x) = 1\}$ *is a submodule of* M*.*

• *Let* $\lambda \in FI(R)$*. Then* $\Re(\lambda) \in FI(R)$*, defined by* $\Re(\lambda)(x) = \vee_{n\in\mathbb{N}}\lambda(x^n)$ $\forall x \in R$*, is called the* $\Re-radical$ *of* λ*.*

• *Let* $\xi \in FI(R)$ *and* $\mu, \nu \in FS(M)$*. Define* $\mu+\nu, \xi.\mu \in F(M)$ *as follows*

$$(\mu+\nu)(x) = \vee\{\mu(y)\wedge\nu(z)\,|\,y,z\in M, y+z=x\},$$
$$(\xi.\mu)(x) = \vee\{\xi(r)\wedge\mu(y)\,|\,r\in R,\ y\in M,\ ry=x\},\ \text{for all } x\in M.$$

• *For $\mu,\nu \in FS(M)$ and $\zeta \in LI(R)$, define $(\mu:\nu)\in FI(R)$ and $(\mu:\zeta)\in FS(M)$ as follows:*

$$(\mu:\nu) = \cup\{\eta\,|\,\eta\in FI(R),\ \eta.\nu\subseteq\mu\},$$
$$(\mu:\zeta) = \cup\{\xi\,|\,\xi\in F(M),\ \xi.\zeta\subseteq\mu\}.$$

• *Let $\mu\in LI(R)$. Then μ is called a maximal fuzzy ideal of R if μ is a maximal element in the set of all non-constant fuzzy ideal of R under ponitwise partial ordering.*

Definition 2.3. *[1] A non-constant fuzzy submodule μ of M is said to be prime fuzzy (or fuzzy prime) if for $\zeta\in FI(R)$ and $\nu\in FS(M)$ such that $\zeta.\nu\subseteq\mu$, then either $\nu\subseteq\mu$ or $\zeta\subseteq(\mu:1_M)$. If $M=R$, then μ is said to be a fuzzy prime ideal. In this case, we have $(\mu:1_M)=(\mu:1_R)=\mu$.*

Remark 2.4. *([4, Theorem 4.5.2].) Let $\mu,\nu\in FS(M)$ and $\zeta\in FI(R)$. Then*
[(i)] $(\mu:\nu)=\cup\{t_r\,|\,r\in R\,, t_r.\nu\subseteq\mu\}$;
[(ii)] $(\mu:\zeta)=\cup\{s_x\,|\,x\in M\,, s_x.\zeta\subseteq\mu\}$.

Remark 2.5. *([1, Theorem 3.5].) Let $\mu\in FS(M)$. Then μ is a fuzzy prime submodule of M if and only if it satisfies the following conditions:*
(a) μ_* is a prime submodule of M,
(b) $(\mu:1_M)(1)$ is a prime element in I,
(c) $t_r.s_x\in\mu$, $r\in R$, $x\in M$, and $t,s\in I\Rightarrow$ either $t_r\in(\mu:1_M)$ or $s_x\in\mu$.

Remark 2.6. *([1, Theorem 3.6].) Let μ be a fuzzy prime submodule of M. Then $(\mu:1_M)$ is a fuzzy prime ideal.*

Remark 2.7. *([12, Corollary 2.1].) Let $1_\theta\neq\mu\in FS(M)$. Then for each $a\in R$ and each $t\in(0,1]$, $t_a\in(1_\theta:\mu)$. That is $t_a.\mu\subseteq 1_\theta$ if and only if $1_a.\mu\subseteq 1_\theta$. Consequently, $Im(1_\theta:\mu)=\{0,1\}$.*

Remark 2.8. *([12, Remark 2.1].) Let $\zeta\in FI(R)$ and let $\nu,\lambda\in FS(M)$ such that $\nu\subseteq\lambda$. Then* *(a) If $\zeta.\lambda \subseteq \nu$, then $\zeta.\frac{\lambda}{\nu} \subseteq 1_\theta$.*

(b) If $\zeta.\frac{\lambda}{\nu} \subseteq 1_\theta$, then $\zeta.\lambda \subseteq 1_{\nu^*}$.

Remark 2.9. *([1, Lemma 2.2].) Let $\nu, \lambda \in FS(M)$. Then $(\nu : \lambda)_* \subseteq (\nu_* :_R \lambda_*)$. In particular, $(1_\theta : \lambda)_* \subseteq (0 :_R \lambda_*)$.*

3. Main Result

In this section, we introduce the fuzzy second submodules and investigate some basic properties of this class of submodules.

Definition 3.1. *Let $\mu \in FS(M)$. We say that μ is a fuzzy second submodule of M if $\mu \neq 1_\theta$ and for each $r \in R$, either $1_r.\mu = \mu$ or $1_r.\mu = 1_\theta$. (For every $r \in R$, 1_r is a fuzzy point of R by Definition 2.1, where $t = 1$.)*

Theorem 3.2. *Let μ be a fuzzy second submodule of M. Then $(1_\theta : \mu)$ is a fuzzy prime ideal.*

Proof. By Remark 2.7, $(1_\theta : \mu)$ is a proper fuzzy ideal of R. Now let $(t \wedge s)_{ab} = t_a.s_b \in (1_\theta : \mu)$, where $t, s \in I$ and $a, b \in R$. If $t = 0$ or $s = 0$, then $t_a \in (1_\theta : \mu)$ or $s_b \in (1_\theta : \mu)$. So we assume that $t \neq 0 \neq s$ and $s_b \notin (1_\theta : \mu)$. Then $s_b.\mu \nsubseteq 1_\theta$ so that $1_b.\mu \nsubseteq 1_\theta$ by Remark 2.7. Therefore, $(1_\theta : \mu)$ is a fuzzy prime ideal. □

Remark 3.3. *Let μ be a fuzzy second submodule. Then by Theorem 3.2, $\xi := (1_\theta : \mu)$ is a fuzzy prime ideal of R. In this case, we say μ is ξ -fuzzy second.*

Definition 3.4. *Let $\mu \in FS(M)$. Then we define $W(\mu)$ as follows*

$$W(\mu) = \cup\{1_r |\, r \in R,\ 1_r.\mu \neq \mu\}.$$

Proposition 3.5. *The following assertions are equivalent;*

[(a)] μ is a ξ-fuzzy second submodule of M.

[(b)] $(1_\theta : \mu) = W(\mu) = \xi$.

Proof. "$(a) \Longrightarrow (b)$" Since $\mu \neq 1_\theta$, the claim follows easily from the fact that $(1_\theta : \mu) = \cup\{1_r \,|\, r \in R,\ 1_r.\mu \subseteq 1_\theta\}$ by Remark 2.7.

"$(b) \Longrightarrow (a)$" is clear. □

Proposition 3.6. *(a) Let P be prime ideal of R. Then 1_N is a 1_P-second fuzzy submodule of M if and only if N is a P-second submodule of M. (We recall that, for every submodule N of M, 1_N is the characterization function of N by Definition 2.1).*

(b) Let μ be a ξ-fuzzy second submodule of M. If $\vee\{\mu(x) \mid x \notin \mu_*\} < 1$, then μ_* is a ξ_*-second submodule of M.

Proof. (a) Let 1_N be 1_P-fuzzy second. Assume that $r \in R$ and $rN \neq 0$. Then $1_r.1_N = 1_rN \neq 1_\theta$. Since 1_N is fuzzy second, we have $1_rN = 1_N$. This implies that $rN = N$ so that N is a second submodule. Clearly, N is a P-second submodule of M. Conversely, let N be a P-second submodule of M. Suppose that $r \in R$ and $1_r.1_N \neq 1_\theta$. Then $1_rN \neq 1_\theta$ and hence $rN \neq \theta$. Since N is a second submodule, we have $rN = N$. Thus $1_r.1_N = 1_rN = 1_N$. Further, we see that $1_P = (1_\theta : 1_N)$. Hence 1_N is a 1_P-fuzzy second submodule of M.

(b) Let $r \in R$ and let $r\mu_* \neq 0$. We show that $r\mu_* = \mu_*$. Clearly, $r\mu_* \subseteq \mu_*$. So let $x \in \mu_*$. We show that $x = ry$ for some $y \in \mu_*$. Since $r\mu_* \neq 0$, $r \notin (1_\theta : \mu)_*$ by Remark 2.9. Thus $1_r.\mu \nsubseteq 1_\theta$. This implies that $1_r.\mu = \mu$ because μ is fuzzy second. Hence $(1_r.\mu)(x) = 1 \bigwedge(\bigvee_{y\in M,\, x=ry} \mu(y)) = \bigvee_{y\in M,\, x=ry} \mu(y) = 1$. So by hypothesis, there exists $y \in \mu_*$ with $x = ry$, as desired. Thus μ_* is a second submodule of M. Now we show that $(1_\theta : \mu)_* = (0 :_R \mu_*)$. By Remark 2.9, it is enough to show that $(0 :_R \mu_*) \subseteq (1_\theta : \mu)_*$. To see this, let $r \in (1_\theta :_R \mu_*) \setminus (1_\theta : \mu)_*$. Then we have $1_r.\mu = \mu$ because μ is fuzzy second and $1_r.\mu \nsubseteq 1_\theta$. Choose $\theta \neq x \in \mu_*$. Then $1 = \mu(x) = (1_r.\mu)(x) = \bigvee_{y\in M,\, x=ry} \mu(y) = 1$. From this, we conclude that there exists $y \in \mu_*$ such that $x = ry = 0$, a contradiction. $\square$

We recall that a non-zero fuzzy submodule μ of M is called a coprimary (or secondary) fuzzy submodule if for each $r \in R$, either $1_r.\mu = \mu$ or there exists $n \in \mathbb{N}$ such that $1_{r^n}.\mu = 1_\theta$ (or equivalently $r \in \Re(1_\theta : \mu)$, where $\Re(1_\theta : \mu)$ is the $\Re$-radical of fuzzy ideal $(1_\theta : \mu)$ [12]). Clearly, every fuzzy second submodule is a fuzzy coprimary submodule but the following example shows that the converse is not true in general.

Example 3.7. Let $M = \mathbb{Z}_{p^\infty}$. Let $n > 1$ and set $N :=< \frac{1}{p^n} + \mathbb{Z} >$. Then N is a coprimary $\mathbb{Z}$-module which is not a second submodule. Hence 1_N

is a fuzzy coprimary submodule which is not a fuzzy second submodule by Proposition 3.6 (a).

Theorem 3.8. *Let $1_\theta \neq \mu$ be a fuzzy coprimary submodule of M. Then μ is a fuzzy second submodule if and only if $(1_\theta : \mu)$ is a fuzzy prime ideal of R.*

Proof. The necessity follows from Theorem 3.2. Conversely, we assume that $(1_\theta : \mu)$ is a fuzzy prime ideal. Suppose that $1_r.\mu \neq \mu$. Since μ is a fuzzy coprimary submodule, there exists $n \in \mathbb{N}$ such that $1_{r^n} \in (1_\theta : \mu)$. This implies that $1_r \in (1_\theta : \mu)$ so that $1_r.\mu = 1_\theta$, as desired. □

Corollary 3.9. *Let $1_\theta \neq \lambda$ be a ξ-fuzzy coprimary submodule of M and let μ be a ξ-fuzzy second submodule of M which contains λ. Then λ is a ξ-fuzzy second submodule.*

Proof. Let $\lambda \subseteq \mu$ and μ be a fuzzy second submodule. Then $(1_\theta : \mu) \subseteq (1_\theta : \lambda) \subseteq \Re(1_\theta : \lambda) = \xi$. Hence by Theorem 3.8, λ is a ξ-fuzzy second submodule. □

We shall call a fuzzy submodule μ of M a minimal ξ-fuzzy coprimary (resp. ξ-fuzzy second) submodule of M if μ is a ξ-fuzzy coprimary (resp. ξ-fuzzy second) submodule of M. (We recall that $\xi = (1_\theta : \mu)$).

Proposition 3.10. *Let μ be a minimal ξ-fuzzy second submodule of M. Then μ is a minimal ξ-fuzzy coprimary submodule of M.*

Proof. Let ν be a ξ-fuzzy coprimary submodule of μ. Since μ is a ξ-fuzzy second, ν is a ξ-fuzzy second submodule by Corollary 3.9 and hence $\nu = \mu$ by minimality of μ. □

Theorem 3.11. *Let μ and η be two ξ-fuzzy submodules of M. Then we have the following*

- *(a) $\mu + \eta$ is a ξ-fuzzy second submodule.*
- *(b) Let $\nu \in FS(M)$ be such that $\nu \subset \mu$ and $\nu^* \subset \mu^*$, where ν^* (resp. μ^*) is support of ν (resp. μ). Then the fuzzy quotient submodule $\frac{\mu}{\nu}$ is ξ-fuzzy second.*

Proof. (a) It is clear that $\mu + \eta$ is a non-zero fuzzy submodule. If $1_r.\mu = \mu$ and $1_r.\eta = \eta$, then $1_r.(\mu + \eta) = \mu + \eta$. If $1_r.\mu = 1_\theta$ and $1_r.\eta = 1_\theta$, then clearly we have $1_r.(\mu+\eta) = 1_\theta$. It is easy to see that $(1_\theta : \mu+\eta) = \xi$. Hence $\mu+\eta$ is a ξ-fuzzy second submodule.

(b) Clearly, $\frac{\mu}{\nu} \neq 1_\theta$. Let $r \in R$ and $1_r.\mu = \mu$. We claim that $1_r.\frac{\mu}{\nu} = \frac{\mu}{\nu}$. To see this, suppose that $x \in \mu^*$. Then

$$\begin{aligned}(1_r.\mu)(x+\nu^*) &= \bigvee_{x-ry\in\nu^*} \tfrac{\mu}{\nu}(y+\nu^*)\\ &= \bigvee_{x-ry\in\nu^*} \bigvee_{e\in\nu^*} \mu(y+e)\\ &= \bigvee_{e\in\nu^*} \bigvee_{x-ry\in\nu^*} \mu(y+e)\\ &= \bigvee_{e\in\nu^*, \acute{e}\in\nu^*} \bigvee_{x+\acute{e}=r(y+e)} \mu(y+e)\\ &= \bigvee_{e\in\nu^*} (\bigvee_{\acute{e}\in\nu^*} (1_r.\mu)(x+\acute{e}))\\ &= \bigvee_{e\in\nu^*} (\bigvee_{\acute{e}\in\nu^*} \mu(x+\acute{e}))\\ &= \bigvee_{\acute{e}\in\nu^*} \tfrac{\mu}{\nu}(x+\nu^*) = \tfrac{\mu}{\nu}(x+\nu^*)\end{aligned}$$

Now assume that $1_r.\mu = 1_\theta$. Hence

$$(1_\theta : \tfrac{\mu}{\nu})(r) = \cup\{\zeta(r) \mid \zeta \in FI(R), \zeta.\tfrac{\mu}{\nu} \subseteq 1_\theta\} \geqslant$$

$$\cup\{\zeta(r) \mid \zeta \in FI(R), \zeta.\mu \subseteq 1_\theta\} = \cup\{t_r \mid r \in R, t \in [0,1], t_r.\tfrac{\mu}{\nu}\} \geqslant 1_r(r) = 1.$$

(Note that by Remark 2.8 (a), if $\zeta.\mu \subseteq 1_\theta$, then we have $\zeta.\frac{\mu}{\nu} \subseteq 1_\theta$). This implies that $1_r \in (1_\theta : \frac{\mu}{\nu})$. Hence $\frac{\mu}{\nu}$ is a ξ-fuzzy second submodule. □

For a multiplicative closed subset S of R and for a fuzzy submodule μ of M, we put $S(\mu) := \cap_{s\in S}(1_s.\mu)$. Clearly, $S(\mu) \in FS(M)$ and $S(\mu) \subseteq \mu$.

Theorem 3.12. *Let $f : M \to S^{-1}M$ be the natural homomorphism $x \to x/1$ for all $x \in M$. If μ is ξ-fuzzy second submodule of M, then the following are hold.*
(i) If $S \cap \xi_* \neq \emptyset$, then $S^{-1}\mu = 1_\theta$.
(ii) If $S\cap\xi_* = \emptyset$, then for each $s \in S$, we have $1_s.\mu = \mu$ and so $S(\mu) = \mu$.
(iii) If μ has the sup property, then either $S^{-1}\mu = 1_\theta$ or $S^{-1}\xi$-fuzzy second submodule with $(1_\theta : S^{-1}\mu) = S^{-1}\xi$.

Proof. Use the technique of [12, Theorem 3.3]. □

Remark 3.13. *In [12], Theorem 3.2 says that if $f : M \to N$ is a monomorphism of non-zero submodules such that $N/Im(f)$ is a torsion free R-module and if μ is a ξ-fuzzy coprimary submodule of M, then so* *is $f(\mu)$. But the given proof is valid just for fuzzy second submodules not for fuzzy coprimary submodules. However, their proof will be completed if we replace the claim "if $r \in R$ and $1_r.\mu = 1_\theta$, then $1_r.f(\mu) = 1_\theta$" by "if $r \in R$ and for some n, $1_{r^n}.\mu = 1_\theta$, then $1_{r^n}.f(\mu) = 1_\theta$". We think there has been a misprint in writing. By the above arguments, we have the following theorem.*

Theorem 3.14. *Let $f : M \to N$ be a monomorphism of non-zero R-modules such that $N/Im(f)$ is a torsion free R-module. If μ is a ξ-fuzzy second submodule of M, then so is $f(\mu)$.*

Theorem 3.15. *Let λ, μ be two fuzzy submodules of M and 1_M be fuzzy second with $\mu + \lambda \supseteq 1_M$. Then either $\lambda \supseteq 1_M$ or $(1_\theta : 1_M) \supseteq (1_\theta : \mu)$.*

Proof. Suppose that $(1_\theta : \mu) \nsubseteq (1_\theta : 1_M)$. Then there exists $t_r \in (1_\theta : \mu)$ such that $t_r \notin (1_\theta : 1_M)$. Thus by Remark 2.7, we have $1_r.\mu \subseteq 1_\theta$ and $1_r.1_M \nsubseteq 1_\theta$. Hence $1_r.1_M = 1_M$. By assumption, we have $1_r.\mu + 1_r.\lambda = 1_\theta + 1_r.\lambda \supseteq 1_M$. Now let $\theta \neq x \in M$. Then $1 = 1_M(x) \leqslant (1_\theta + 1_r.\lambda)(x) \leqslant 1_\theta(\theta) \wedge 1_r.\lambda(1) \leqslant \lambda(x)$. Therefore, $\lambda \supseteq 1_M$, as desired. $\square$

Corollary 3.16. *Let ζ be a fuzzy ideal of R and let μ be a fuzzy submodule of M. If 1_M is a fuzzy second submodule of M with $\mu + (1_\theta : \zeta) \supseteq 1_M$, then either $(1_\theta : \zeta) \supseteq 1_M$ or $\mu \supseteq 1_M$.*

Let μ be a fuzzy prime submodule of M (resp. a fuzzy prime ideal of R) and let $\xi = (\mu : 1_M)$ (resp. $\xi = (\mu : 1_R)$). Then μ is called a ξ-fuzzy prime submodule (resp. a ξ-fuzzy prime ideal).

Theorem 3.17. *Let $1_M \neq \lambda \in FS(M)$ and let $\zeta \in FI(R)$. Then ζ is a fuzzy prime ideal of R and λ is a ζ-fuzzy prime submodule if and only if it satisfies the following conditions:*

(a) $\forall t, s \in I, r \in R, x \in M$, if $t_r.s_x \in \lambda$, and $s_x \notin \lambda$, then $t_r \in \zeta$.

(b) $\zeta \subseteq (\lambda : 1_M)$.

Proof. The necessity is clear. Conversely, let (a), (b) hold. Firstly, we show that $\lambda_* \neq M$ is a fuzzy prime submodule of M. To see this, let $rx \in \lambda_*$ and $r \notin (\lambda_* :_R M)$, where $r \in R, x \in M$. Thus $1_r.1_x \in \lambda$ by Remark 2.9. Hence $1_r \notin \zeta$ and $1_x \in \lambda$ by (a). So $x \in \lambda_*$, as desired. Moreover, we see that $(\lambda : 1_M)(1) \neq 1$ is a prime element in I. Finally, let $t_r.s_x \in \lambda$

and $s_x \notin \lambda$, where $t, s \in I$, $r \in R$, and $x \in M$. Then $t_r \in (\lambda : 1_M)$ by (b). Therefore, λ is a fuzzy prime submodule of M by Remark 2.5. Next we show that ζ is a fuzzy prime ideal. By (b), $\zeta \subseteq (\lambda : 1_M)$. Now let $t_r \in (\lambda : 1_M)$. Then $t_r.1_M \subseteq \lambda$. Since $\lambda \neq 1_M$, there exists $x \in M$ such that $1_x \notin \lambda$. Hence $1_r.1_x \in \lambda$ so that $1_r \in \zeta$ by (a). Thus $\zeta = (\lambda : 1_M)$ is a fuzzy prime ideal by Remark 2.6. □

Theorem 3.18. *Let $1_\theta \neq \mu \in FS(M)$ and $\xi \in FI(R)$. Then ξ is a fuzzy prime ideal of R and μ is ξ-fuzzy second if and only if it satisfies the following conditions:*

(a) $r \in R$ and $1_r.\mu \neq \mu$ implies that $1_r \in \xi$.

(b) $\xi \subseteq (1_\theta : \mu)$.

Proof. The necessity is clear. Now we assume that (a), (b) hold and let $r \in R$ with $t_r \notin (1_\theta : \mu)$. This implies that $1_r \notin \xi$ by (b) and $1_r.\mu = \mu$ by (a). So μ is a fuzzy second submodule. Next we show that $\xi = (1_\theta : \mu)$. To see this, let $t_r \in (1_\theta : \mu)$, where $r \in R$ and $t \in I$. Then by Remark 2.7, $1_r \in (1_\theta : \mu)$ and hence $1_r.\mu \neq \mu$. Thus we have $1_r \in \xi$ by (a) and so $t_r \in \xi$ by Remark 2.7. Hence the equality holds by (b) and ξ is a fuzzy prime ideal. □

Theorem 3.19. *Let $1_\theta \neq \mu \in FS(M)$ be ξ-fuzzy second, where $\xi = (1_\theta : \mu)$. Let $\nu \in FS(M)$ be such that $\nu \subset \mu$ and $\nu^* \subset \mu^*$. Then $(1_{\nu^*} : 1_{\mu^*})$ is a ξ-fuzzy prime ideal of R. Moreover, $(\nu : \mu)$ is a fuzzy prime ideal.*

Proof. Let $t_a.s_b \in (1_{\nu^*} : 1_{\mu^*})$ and $t_a \notin (1_\theta : \mu)$ for some $t, s \in I$ and $a, b \in R$. Then $t_a.s_b.1_{\mu^*} \subseteq 1_{\nu^*}$ and we have $1_a.\mu = \mu$ by Remark 2.7. Hence $t_a.s_b.\frac{1_{\mu^*}}{1_{\nu^*}} \subseteq 1_\theta$ and $1_a.1_{\mu^*} = 1_{\mu^*}$ by Remark 2.8 (a). Thus $(t \wedge s)_{ab} = t_a.s_b \in (1_\theta : \frac{1_{\mu^*}}{1_{\nu^*}})$ and so $1_{ab} \in (1_\theta : \frac{1_{\mu^*}}{1_{\nu^*}})$ by Remark 2.7. Hence $1_a.1_b.\frac{1_{\mu^*}}{1_{\nu^*}} \subseteq 1_\theta$. Now we have $1_a.1_b.1_{\mu^*} \subseteq 1_{\nu^*}$ by Remark 2.8 (b). It follows that $1_b.1_{\mu^*} \subseteq 1_{\nu^*}$. Hence $s_b \in (1_{\nu^*} : 1_{\mu^*})$. One can see that $(1_\theta : \mu) \subseteq ((1_{\nu^*} : 1_{\mu^*}) : 1_R)$. So by Theorem 3.17, $(1_{\nu^*} : 1_{\mu^*})$ is a ξ-fuzzy prime ideal. Moreover, it is easy to see that $(\nu : \mu) \subseteq (1_{\nu^*} : 1_{\mu^*})$. Hence $(1_{\nu^*} : 1_{\mu^*}) = \xi = (1_\theta : \mu) \subseteq (\nu : \mu) \subseteq (1_{\nu^*} : 1_{\mu^*})$. Therefore, $(\nu : \mu)$ is a fuzzy prime ideal. □

Theorem 3.20. *Let μ be a fuzzy submodule of M. If $(1_\theta : \mu)$ is a* *maximal fuzzy ideal of R, then μ is a fuzzy second submodule of M.*

Proof. Clearly, $\mu \neq 1_\theta$. Now let $r \in R$ and $1_r.\mu \neq 1_\theta$. We show that $1_r.\mu = \mu$. Since $1_r \notin (1_\theta : \mu)$, we have $1_{<r>} + (1_\theta : \mu) = 1_R$ by maximality of $(1_\theta : \mu)$. Thus there exist $t_s \in (1_\theta : \mu)$ and $\acute{s} \in R$ such that $1_{\acute{s}r} + t_s = 1_1$. From this, we conclude that $1_r.\mu = \mu$, as needed. □

Corollary 3.21. *Suppose that μ is a non-zero fuzzy submodule of M which is contained in a fuzzy submodule λ such that $(1_\theta : \lambda)$ is a maximal fuzzy ideal. Then μ is a fuzzy second submodule of M.*

Corollary 3.22. *Let ξ be a maximal fuzzy ideal of R such that $1_\theta \neq (1_\theta : \xi)$. Then $(1_\theta : \xi)$ is a fuzzy second submodule of M.*

Proof. Use Theorem 3.20 with the fact that $\xi \subseteq (1_\theta : (1_\theta : \xi))$. □

Definition 3.23. *Let μ be a fuzzy submodule of M. Then μ is called a minimal fuzzy submodule of M if μ is a minimal element in the set of all non-zero fuzzy submodule of M under pointwise partial ordering.*

Theorem 3.24. *Let μ be a minimal fuzzy submodule of M. Then the fuzzy ideal $(1_\theta : \mu)$ is a maximal fuzzy ideal of R.*

Proof. Clearly, $(1_\theta : \mu)$ is a proper fuzzy ideal of R. Now suppose that η is a fuzzy ideal such that $(1_\theta : \mu) \subset \eta$. Then there exists $t_r \in \eta$ such that $t_r \notin (1_\theta : \mu)$. Thus $1_\theta \neq t_r.\mu \subseteq \mu$ and so $t_r.\mu = \mu$. Now $(1_1 - t_r)\mu = 1_1.\mu - t_r.\mu = \mu - \mu = 1_\theta$ and so $1_1 - t_r \in (1_\theta : \mu) \subset \eta$. Thus $1_1 \in \eta$ which means that $\eta = 1_R$, a contradiction. Hence $(1_\theta : \mu)$ is a maximal fuzzy ideal of R. □

Corollary 3.25. *Every minimal fuzzy submodule of M is a fuzzy second submodule.*

Proof. This follows from Theorem 3.24 and Corollary 3.22. □

4. Conclusion

In this paper, the concept of fuzzy second submodules have been introduced which dualize the notion of fuzzy prime submodules. Among other results, it is shown that fuzzy second submodules is a proper subclass of fuzzy coprimary submodules. Further, the relationship between fuzzy second and fuzzy prime submodules are investigated. Also it has been proved that the sum (resp. quotient) of two fuzzy submodules is a fuzzy second submodule. Moreover, if λ and μ are two fuzzy submodules of M and 1_M is a fuzzy second with $\mu + \lambda \supseteq 1_M$, then either $\lambda \supseteq 1_M$ or $(1_\theta : 1_M) \supseteq (1_\theta : \mu)$. The notion of minimal fuzzy submodule has been introduced and after expressing some characteristics of maximal and minimal of fuzzy ideals, we concluded that every minimal fuzzy submodule is a fuzzy second submodules.

Our next aim is to investigate further properties of fuzzy second submodules and explore their relationship with other fuzzy modules.

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