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Bi-Concave Functions Involving a Differential Operator

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Abstract. The purpose of the present paper is to introduce a class $\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}C_0(\beta)$ of bi-concave functions defined by a differential operator. We find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Several consequences of these results are also pointed out in the form of corollaries.

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1. Introduction

Let A indicate an analytic function family, which is $z \in C$ (C=complex numbers) normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\Delta = \{z : z \in \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Further, by S we shall denote the class of all functions in A which are univalent in Δ.

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It is well known that every function $f \in S$ has an inverse f^{-1}, satisfying $f^{-1}(f(z)) = z,\ (z \in \Delta)$ and $f\left(f^{-1}(w)\right) = w,\ \left(|w| < r_0(f); r_0(f) \geqslant \frac{1}{4}\right)$, where

$$f^{-1}(w) = w \ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots,$$

(for details, see Duren [16]). A function $f \in A$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ stand for the class of bi-univalent functions defined in the unit disk Δ. For a brief history of functions in the class, see [27] Srivastava 2010 (see also [11, 12, 20, 22]). More recently, Srivastava *et al.* [27], Altınkaya and Yalcın [3] made an effort to introduce various subclasses of the bi-univalent function class Σ and found non-sharp coefficient estimates on the initial coefficients $|a_2|$ and $|a_3|$ (see also [7], [17]). But determination of the bounds for the coeffcients

$$|a_n|, \quad n \in \mathbb{N} \setminus \{1, 2\}; \quad \mathbb{N} = \{1, 2, 3, ...\},$$

is still an open problem. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions (see, for Example [4, 18, 29]).

The study of operators plays an important role in the Geometric Function Theory and its related fields. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better (see, for example [2, 13, 19] and [21]). Recently, Darus and İbrahim [15] introduced a differential operator

$$\boldsymbol{D}_{\lambda,\delta}^{k,\alpha} : A \to A,$$

by

$$\boldsymbol{D}_{\lambda,\delta}^{k,\alpha} f(z) = z + \sum_{n=2}^{\infty} [n^\alpha + (n-1)n^\alpha \lambda]^k \begin{pmatrix} n+\delta-1 \\ \delta \end{pmatrix} a_n z^n,$$

where $z \in \Delta$ and $k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda, \delta \geqslant 0.$

It should be remarked that the operator $\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}$ is a generalization of many other linear operators studied by earlier researchers. Namely:

- for $\alpha = 1$, $\lambda = 0$; $\delta = 0$ or $\alpha = \ \delta = 0$; $\lambda = 1$, the operator $\boldsymbol{D}_{0,0}^{k,1} \equiv \boldsymbol{D}_{1,0}^{k,0} \equiv \boldsymbol{D}^{k}$ is the popular Salagean operator [26],
- for $\alpha = 0$, $\delta = 0$, the operator $\boldsymbol{D}_{\lambda,0}^{k,0} \equiv \boldsymbol{D}_{\lambda}^{k}$ has been studied by Al-Oboudi (see [1]),
- for $\alpha = 0$, the operator $\boldsymbol{D}_{\lambda,\delta}^{k,0} \equiv \boldsymbol{D}_{\lambda,\delta}^{k}$ has been studied by Darus and İbrahim (see [15]),
- for $k = 0$, the operator $\boldsymbol{D}_{\lambda,\delta}^{k,\alpha} \equiv \boldsymbol{D}^{\delta}$ has been studied by Ruscheweyh (see [24]).

2. Preliminaries

Conformal maps of the unit disk onto convex domains is a classical topic. Recently Avkhadiev and Wirths [8] discovered that conformal maps onto concave domains (the complements of convex closed sets) have some novel properties.

A function $f : \Delta \to \mathbb{C}$ is said to belong to the family $C_0(\beta)$ if f satisfies the following conditions:

- f is analytic in Δ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
- f maps Δ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
- The opening angle of $f(\Delta)$ at ∞ is less than or equal to $\pi\beta$, $\beta \in (1, 2]$.

The class $C_0(\beta)$ is referred to as the class of concave univalent functions and for a detailed discussion about concave functions, we refer to Avkhadiev et al.[9], Cruz and Pommerenke [14] and references there in.

In particular, the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 0, \qquad (z \in \Delta),$$

is used - sometimes also as a definition - for concave functions $f \in C_{0_O}$ (see e.g. [23] and others).

Bhowmik et al. [10] showed that an analytic function f maps Δ onto a concave domain of angle $\pi\beta$, if and only if $\Re(P_f(z)) > 0$, where

$$P_f(z) = \frac{2}{\beta - 1}\left[\frac{\beta+1}{2}\frac{1+z}{1-z} - 1 - z\frac{f''(z)}{f'(z)}\right].$$

There has been a number of investigations on basic subclasses of concave univalent functions (see, for example [5, 6] and [25]).

Let us recall now the following definition required in sequel.

Definition 2.1. *(see [28]) Let the functions $h, p : \Delta \to \mathbb{C}$ be so constrained that*

$$\min\{\Re(h(z)), \Re(p(z))\} > 0,$$

and

$$h(0) = p(0) = 1.$$

Motivated by each of the above definitions, we now define a new subclass of bi-concave analytic functions involving $\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}$ differential operator.

Definition 2.2. *A function $f \in \Sigma$ given by (1) is said to be in the class*

$$\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}C_0(\beta) \qquad (k, \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ \lambda, \delta \geqslant 0;\ \beta \in (1, 2];\ z, w \in \Delta)$$

if the following conditions are satisfied:

$$\frac{2}{\beta-1}\left[\frac{\beta+1}{2}\frac{1+z}{1-z} - 1 - z\frac{\left[\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}f(z)\right]''}{\left[\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}f(z)\right]'}\right] \in h(\Delta), \tag{2}$$

and

$$\frac{2}{\beta-1}\left[\frac{\beta+1}{2}\frac{1-w}{1+w} - 1 - w\frac{\left[\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}g(w)\right]''}{\left[\boldsymbol{D}_{\lambda,\delta}^{k,\alpha}g(w)\right]'}\right] \in p(\Delta), \tag{3}$$

where $g(w) = f^{-1}(w)$.

3. Main Results and Their Consequences

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}C_0(\beta)$.

Theorem 3.1. *Let f given by (1) be in the class $\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}C_0(\beta)$. Then*

$$|a_2| \leqslant \min\left\{\sqrt{\frac{(\beta+1)^2}{4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2}+\frac{(\beta-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2}+\frac{(\beta^2-1)\left(|h'(0)|+|p'(0)|\right)}{8[2^\alpha(1+\lambda)]^{k}(1+\delta)^2}},\right.$$
$$\left.\sqrt{\frac{(\beta-1)\left(|h''(0)|+|p''(0)|\right)}{8\left|4[2^\alpha(1+\lambda)]^{k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}+\frac{(\beta+1)}{\left|4[2^\alpha(1+\lambda)]^{k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}}\right\} \tag{4}$$

and

$$|a_3| \leqslant \min$$
$$\left\{\frac{8(\beta+1)^2+(\beta-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2}+\frac{\left(\beta^2-1\right)(|h'(0)|+|p'(0)|)}{8[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2}+\frac{(\beta-1)(|h''(0)|+|p''(0)|)}{24[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)},\right.$$
$$\frac{(\beta-1)(|h''(0)|+|p''(0)|)}{8\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}$$
$$+\frac{\beta+1}{\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}$$
$$\left.+\frac{(\beta-1)(|h''(0)|+|p''(0)|)}{24[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)}\right\}. \tag{5}$$

Proof. Let $f \in \boldsymbol{D}^{k,\alpha}_{\lambda,\delta}C_0(\beta)$ and g be the analytic extension of f^{-1} to Δ. It follows from (2) and (3) that

$$\frac{2}{\beta-1}\left[\frac{\beta+1}{2}\frac{1+z}{1-z}-1-z\frac{\left[\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}f(z)\right]''}{\left[\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}f(z)\right]'}\right]=h\left(z\right), \tag{6}$$

and

$$\frac{2}{\beta-1}\left[\frac{\beta+1}{2}\frac{1-w}{1+w}-1-w\frac{\left[\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}g(w)\right]''}{\left[\boldsymbol{D}^{k,\alpha}_{\lambda,\delta}g(w)\right]'}\right]=p\left(w\right), \tag{7}$$

where $h(z)$ and $p(w)$ satisfy the conditions of Definiton 2.1. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expansions:

$$h(z) = 1 + h_1 z + h_2 z^2 + \cdots ,$$

and

$$p(w) = 1 + p_1 w + p_2 w^2 + \cdots ,$$

respectively. Now, equating the coefficients in (6) and (7), we get

$$\frac{2\left[(\beta+1) - 2\left[2^{\alpha}(1+\lambda)\right]^{k}(1+\delta)a_2\right]}{\beta-1} = h_1, \tag{8}$$

$$\frac{2\left[(\beta+1) - 3\left[3^{\alpha}(1+2\lambda)\right]^{k}(1+\delta)(2+\delta)a_3 + 4\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2 a_2^2\right]}{\beta-1} = h_2 \tag{9}$$

and

$$-\frac{2\left[(\beta+1) - 2\left[2^{\alpha}(1+\lambda)\right]^{k}(1+\delta)a_2\right]}{\beta-1} = p_1, \tag{10}$$

$$\frac{2\left[(\beta+1) - 3\left[3^{\alpha}(1+2\lambda)\right]^{k}(1+\delta)(2+\delta)(2a_2^2 - a_3) + 4\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2 a_2^2\right]}{\beta-1} = p_2. \tag{11}$$

From (8) and (10), we find that

$$h_1 = -p_1. \tag{12}$$

Also, from (8), we can write

$$a_2 = \frac{\beta+1}{2\left[2^{\alpha}(1+\lambda)\right]^{k}(1+\delta)} - \frac{h_1(\beta-1)}{4\left[2^{\alpha}(1+\lambda)\right]^{k}(1+\delta)}. \tag{13}$$

Next, by using (8), (10), (12) and (13), we get

$$a_2^2 = \frac{(\beta+1)^2}{4\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2} + \frac{(\beta-1)^2\left(h_1^2+p_1^2\right)}{32\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2} - \frac{\left(\beta^2-1\right)(h_1-p_1)}{8\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2}. \tag{14}$$

By adding (9) to (11), we get

$$a_2^2 = \frac{(\beta-1)(h_2+p_2)}{4\left[4\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2 - 3\left[3^{\alpha}(1+2\lambda)\right]^{k}(1+\delta)(2+\delta)\right]} - \frac{\beta+1}{4\left[2^{\alpha}(1+\lambda)\right]^{2k}(1+\delta)^2 - 3\left[3^{\alpha}(1+2\lambda)\right]^{k}(1+\delta)(2+\delta)}. \tag{15}$$

Therefore, we find from the equations (14) and (15) that

$$|a_2|^2 \leqslant \frac{(\beta+1)^2}{4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} + \frac{(\beta-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} + \frac{\left(\beta^2-1\right)(|h'(0)|+|p'(0)|)}{8[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2}$$

and

$$|a_2|^2 \leqslant \frac{(\beta-1)(|h''(0)|+|p''(0)|)}{8\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|} + \frac{(\beta+1)}{\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}.$$

Similarly, subtracting (11) from (9), we have

$$a_3 = a_2^2 - \frac{(\beta-1)(h_2-p_2)}{12\left[3^\alpha(1+2\lambda)\right]^k(1+\delta)(2+\delta)}. \tag{16}$$

Then, upon substituting the value of a_2^2 from (14) and (15) into (16), it follows that

$$a_3 = \frac{(\beta+1)^2}{4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} + \frac{(\beta-1)^2\left(h_1^2+p_1^2\right)}{32[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} - \frac{\left(\beta^2-1\right)(h_1-p_1)}{8[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} - \frac{(\beta-1)(h_2-p_2)}{12[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)},$$

and

$$\begin{aligned} a_3 \;=\;& \frac{(\beta-1)(h_2+p_2)}{4\left[4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right]} \\ & - \frac{\beta+1}{4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)} - \frac{(\beta-1)(h_2-p_2)}{12[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)}. \end{aligned}$$

Consequently, we have

$$\begin{aligned} |a_3| \;\leqslant\;& \frac{8(\beta+1)^2+(\beta-1)^2\left(|h'(0)|^2+|p'(0)|^2\right)}{32[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} \\ & + \frac{\left(\beta^2-1\right)(|h'(0)|+|p'(0)|)}{8[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2} + \frac{(\beta-1)(|h''(0)|+|p''(0)|)}{24[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)} \end{aligned}$$

and

$$\begin{aligned} |a_3| \;\leqslant\;& \frac{(\beta-1)(|h''(0)|+|p''(0)|)}{8\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|} \\ & + \frac{\beta+1}{\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|} + \frac{(\beta-1)(|h''(0)|+|p''(0)|)}{24[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)}. \end{aligned}$$

This completes the proof of Theorem 3.1. □

It is easily seen that, by specializing the functions $h(z)$ and $p(z)$ involved in Theorem 3.1, several coefficient estimates can be obtained as special cases.

Corollary 3.2. *If we set*

$$h(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}=1+2\gamma z+2\gamma^2z^2+\dots \qquad (0<\gamma\leqslant 1),$$

and

$$p(z)=\left(\frac{1-z}{1+z}\right)^{\gamma}=1-2\gamma z+2\gamma^2z^2+\dots \qquad (0<\gamma\leqslant 1),$$

then inequalities (4) and (5) become

$$|a_2|\leqslant\min\left\{\frac{(\beta+1)+(\beta-1)\gamma}{2[2^{\alpha}(1+\lambda)]^{k}(1+\delta)},\sqrt{\frac{(\beta+1)+(\beta-1)\gamma^2}{\left|4[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2-3[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)\right|}}\right\},$$

and

$$\begin{aligned}|a_3| \;\leqslant\; \min\Bigg\{&\left(\frac{(\beta+1)+(\beta-1)\gamma}{2[2^{\alpha}(1+\lambda)]^{k}(1+\delta)}\right)^2+\frac{(\beta-1)\gamma^2}{3[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)},\\ &\frac{(\beta+1)+(\beta-1)\gamma^2}{\left|4[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2-3[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)\right|}+\frac{(\beta-1)\gamma^2}{3[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)}\Bigg\}.\end{aligned}$$

Corollary 3.3. *If we let*

$$h(z)=\frac{1+(1-2\eta)z}{1-z}=1+2(1-\eta)z+2(1-\eta)z^2+\cdots \qquad (0\leqslant\eta<1),$$

and

$$p(z)=\frac{1-(1-2\eta)z}{1+z}=1-2(1-\eta)z+2(1-\eta)z^2+\cdots \qquad (0\leqslant\eta<1),$$

then inequalities (4) and (5) become

$$|a_2|\leqslant\min\left\{\frac{(\beta+1)+(\beta-1)(1-\eta)}{2[2^{\alpha}(1+\lambda)]^{k}(1+\delta)},\sqrt{\frac{(\beta+1)+(\beta-1)(1-\eta)}{\left|4[2^{\alpha}(1+\lambda)]^{2k}(1+\delta)^2-3[3^{\alpha}(1+2\lambda)]^{k}(1+\delta)(2+\delta)\right|}}\right\},$$

and

$$|a_3| \leqslant \min\left\{\left(\frac{(\beta+1)+(\beta-1)(1-\eta)}{2[2^\alpha(1+\lambda)]^k(1+\delta)}\right)^2+\frac{(\beta-1)(1-\eta)}{3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)},\right.$$
$$\left.\frac{(\beta+1)+(\beta-1)(1-\eta)}{\left|4[2^\alpha(1+\lambda)]^{2k}(1+\delta)^2-3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)\right|}+\frac{(\beta-1)(1-\eta)}{3[3^\alpha(1+2\lambda)]^k(1+\delta)(2+\delta)}\right\}.$$

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