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On Matricial Ranges of Some Matrices

M. Dehghani*
University of Kashan

M. Kian
University of Bojnord

Abstract. The matricial ranges of 2×2 complex matrices are revisited. Moreover, using the standard block-matrix techniques, we describe matricial ranges of some special non-quadratic higher order matrices. Finally, we obtain the matricial ranges of some specific 3×3 matrices. Various examples are given as well.

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1. Introduction

One of the most well-known concept in study of operator algebras is the notion of numerical range of an operator. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $\mathbb{B}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the identity operator I. When $\mathscr{H}$ has finite dimension n, we identify $\mathbb{B}(\mathscr{H})$ with the algebra $\mathbb{M}_n := \mathbb{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices and I_n denotes the $n \times n$ identity matrix. The numerical range of an operator $T \in \mathbb{B}(\mathscr{H})$ is defined by

$$W(T) = \{\langle Tx, x\rangle;\ x \in \mathscr{H},\ \|x\| = 1\}$$

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*Corresponding author

and the numerical radius of T is defined by $\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. This set is a powerful tool which gives many information about T, particularly about its eigenvalues and eigenspaces. The numerical range has a unique nature in numerical analysis and differential equations. It has many desirable properties, which probably the most famous of them is the Toeplitz-Hausdorff result. It asserts that $W(T)$ is convex for every $T \in \mathbb{B}(\mathscr{H})$; see, e.g., [8, Theorem 1.1-2].

As a noncommutative extension of the numerical range, the matricial ranges of an operator $T \in \mathbb{B}(\mathscr{H})$ is defined by Arveson in [1] as the set of all $n \times n$ matrices of the form $\Phi(T)$ where Φ ranges over all unital completely positive linear mapppings of $C^*(T)$ into $\mathbb{M}_n$ in which $C^*(T)$ is the unital C^*-algebra generated by T and n is a positive integer. This definition is extensively connected with a noncommutative type of convexity, called C^*-convexity. A set $\mathcal{K} \subset \mathbb{B}(\mathscr{H})$ is called C^*-convex, if $X_1, \ldots, X_m \in \mathcal{K}$ and $A_1, \ldots, A_m \in \mathbb{B}(\mathscr{H})$ with $\sum_{j=1}^m A_j^* A_j = I$ imply that $\sum_{j=1}^m A_j^* X_j A_j \in \mathcal{K}$ [13]. Indeed, this is a noncommutative generalization of linear convexity. It is evident that the C^*-convexity of a set implies its convexity in the usual sense. But the converse is not true in general.

Matricial ranges are closely connected with C^*-convex sets. In fact, the matrix ranges turns out to be the compact C^*-convex sets. However, except in some special cases, it is not routine to obtain the matricial ranges of an operator. The reader is referred to [1, 5, 6, 13, 14, 15, 17] and the references therein for more information about C^*-convexity and matricial ranges.

The main purpose of this paper is to describe the matricial ranges of some matrices. In Section 2, we provide preliminaries concerning the matricial ranges. In Section 3, the matricial ranges of all 2×2 matrices are revisited and then the matricial ranges of some specific non-quadratic higher order matrices have been described, accoding to the matricial ranges of 2×2 matrices. Section 4 is devoted to describe the matricial ranges of some special 3×3 matrices. Various examples are given as well.

2. Matricial Ranges

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with units $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$, respectively. A mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is said to be positive if $\Phi(A) \geqslant 0$ in $\mathcal{B}$ whenever $A \geqslant 0$ in $\mathcal{A}$. It is called unital if $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. Let $\mathcal{M}_n(\mathcal{A})$ be the C^*-algebra of all $n \times n$ matrices with entries in $\mathcal{A}$. A linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called completely positive if the associated mappings

$$\Phi_{(n)} : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}), \quad \Phi_{(n)}([A_{ij}]) = [\Phi(A_{ij})], \quad (n \geqslant 1),$$

are all positive [2].

We denote by $\mathcal{CP}(C^*(T), \mathbb{M}_n)$ the space of all unital completely positive linear mappings from $C^*(T)$ into $\mathbb{M}_n$. For a positive integer n, the nth matricial range of $T \in \mathbb{B}(\mathscr{H})$ [1] is defined by

$$W^n(T) = \{\Phi(T); \quad \Phi \in \mathcal{CP}(C^*(T), \mathbb{M}_n)\},$$

and the sequence $\{W^1(T), W^2(T), \ldots\}$ is called the matrix range of T.

Let us collect some basic properties of this set. The set $W^n(T)$ is compact and clearly it is contained in the ball of radius $\|T\|$. Moreover, as a noncommutative version of Toeplitz-Hausdorff theorem, it is known that $W^n(T)$ is C^*-convex [13, Example 4].

Also, a subset $\mathcal{K}$ of $\mathbb{M}_n$ is compact and C^*-convex if and only if there exists a bounded operator T with $W^n(T) = \mathcal{K}$. More precisely, the C^*-convex set generated by $T \in \mathbb{M}_n$ is the matricial range $W^n(T)$ of T [13, Corollary 20], that is, if $T \in \mathbb{M}_n$, then

$$W^n(T) = \{\sum_{i=1}^{\infty} V_i^* T V_i \ : \ \sum_{i=1}^{\infty} V_i^* V_i = I_n \text{ (in norm)}\}.$$

Evidently, $W^1(T)$ is the closure of the numerical range of T. Indeed, since $\mathcal{CP}(C^*(T), \mathbb{M}_1 = \mathbb{C}) = \mathcal{S}(C^*(T))$ is the state space of $C^*(T)$, it is known that

$$\{\varphi(T) \ ; \ \varphi \in \mathcal{S}(C^*(T))\} = \overline{W(T)}.$$

Therefore $W^1(T) = \overline{W(T)}$; see, e.g., [2, 18].

The next lemma follows from the definition of the matricial ranges. We omit the proof.

Lemma 2.1. *Let $T \in \mathbb{B}(\mathscr{H})$ and let n be a positive integer. Then*

(i) $W^n(T^*) = W^n(T)$.

(ii) $W^n(U^*TU) = W^n(T)$ *for each unitay* $U \in \mathbb{B}(\mathscr{H})$.

(iii) $W^n(\alpha I) = \{\alpha I_n\}$ *and* $W^n(\alpha T + \beta I) = \alpha W^n(T) + \beta I_n$ *for all* $\alpha, \beta \in \mathbb{C}$.

The following known result shows that if T is a normal operator, then $W^n(T)$ is the closed matrix valued convex hull of the spectrum of T.

Proposition 2.2. *[2, Proposition 2.4.1] If $T \in \mathbb{B}(\mathscr{H})$ is a normal operator, then $W^n(T)$ is the closure in $\mathbb{M}_n$ of the set of operators of the form $\lambda_1 H_1 + \ldots + \lambda_r H_r$, where $r \geqslant 1$, $\lambda_i \in \sigma(T)$ and $\{H_i\}$ is a set of positive elements of $\mathbb{M}_n$ having sum I_n. In other words,*

$$W^n(T) = \mathrm{cl}\left\{\sum_{k=1}^{r} \lambda_k H_k; r \geqslant 1, \lambda_k \in \sigma(T), 0 \leqslant H_k \in \mathbb{M}_n, \sum_{k=1}^{r} H_k = I_n\right\}.$$

Proposition 2.2 and the spectral theorem give the next result.

Corollary 2.3. *[2, P. 302] If $T \in \mathbb{B}(\mathscr{H})$ is a unitary operator, then*

$$W^n(T) = \{B \in \mathbb{M}_n \ ; \ \|B\| \leqslant 1\}.$$

Moreover, if $T \in \mathbb{B}(\mathscr{H})$ is a self-adjoint operator with $\sigma(T) \subseteq [m, M]$ for some real numbers m, M, then

$$W^n(T) = \{B \in \mathbb{M}_n \ ; \ B^* = B, \ m \leqslant B \leqslant M\}.$$

Lemma 2.4. *[14, Theorem 3.2] Let $r > 0$, $T \in \mathbb{B}(\mathscr{H})$ and $z_0 \in W^1(T)$. If $\{z \in \mathbb{C} : |z - z_0| \leqslant r\} \subseteq W^1(T)$, then*

$$\{B \in \mathbb{M}_n; \ \|z_0 I_n - B\| \leqslant r\} \subseteq W^n(T), \quad (n \geqslant 1).$$

If $W^1(T)$ does not contain a disc, then it is a point or a segment. In either case, T is normal and has the form $T = z_1 I + z_2 S$ where S has the numerical range $[-1, 1]$. Hence

$$W^n(T) = \{B \in \mathbb{M}_n\,;\ B = z_1 I_n + z_2 Q,\ Q \in \mathbb{M}_n,\ Q = Q^*,\ \|Q\| \leqslant 1\}.$$

Example 2.5. [7, Proposition 5.2] Let $T \in \mathbb{B}(\mathscr{H})$ be a contraction with

$\mathbb{T} = \{z \in \mathbb{C} : \|z\| = 1\} \subseteq \sigma(T)$ and let n be a positive integer. Then

$$W^n(T) = \{B \in \mathbb{M}_n\,;\ B^*B \leqslant I_n\}.$$

Example 2.6. Let $\mathscr{H}$ be a separable Hilbert space with the standard orthonormal basis $\{e_j\}_{j=1}^{\infty}$. If $T \in \mathbb{B}(\mathscr{H})$ be the unilateral shift operator defined by $Te_j = e_{j+1}$ $(j \geqslant 1)$, then

$$W^n(T) = \{B \in \mathbb{M}_n\ ;\ B^*B \leqslant I_n\}.$$

Indeed, T is a contraction with $\mathbb{T} \subseteq \sigma(T) = \{z \in \mathbb{C} : |z| \leqslant 1\}$. It is known that T^* is the backward shift operator, i.e., $T^*e_1 = 0$ and $T^*e_j = e_{j-1}$ $(j \geqslant 2)$. Then

$$W^n(T^*) = W^n(T) = \{B \in \mathbb{M}_n\ ;\ B^*B \leqslant I_n\}.$$

We also recall the famous Stinespring's theorem which characterizing completely positive maps of C^*-algebras.

Theorem 2.7. *(Stinespring)[16](also, see [2, Theorem 0.4]) Let $\mathcal{A}$ be a unital C^*-algebra and $\Phi : \mathcal{A} \to \mathbb{B}(\mathscr{H})$ be a unital completely positive mapping. Then there exists a Hilbert space $\mathscr{K}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathbb{B}(\mathscr{K})$ and an isometry $V : \mathscr{H} \to \mathscr{K}$ with $\|\Phi(1_{\mathcal{A}})\| = \|V\|^2 = 1$ such that*

$$\Phi(A) = V^*\pi(A)V, \qquad \text{for all } A \in \mathcal{A}.$$

With the exception of certain special elements, which have been reviewed above, it is almost impossible to provide an explicit description of the matricial ranges of operators, in general. In the present paper, we review the matricial ranges of 2×2 matrices based on the shape of their numerical ranges and then we obtain the matricial ranges of some special types of matrices.

3. Matricial Ranges of 2×2 Matrices

An operator $T \in \mathbb{B}(\mathscr{H})$ is called quadratic if there are $\alpha_0, \alpha_1 \in \mathbb{C}$ such that $T^2+\alpha_1 T+\alpha_0 I = 0$. The matricial ranges of (non-normal) quadratic operators have been determined by Tso and Wu in [17] as follows.

Theorem 3.1. *[17, Theorem 3.1] Let $T \in \mathbb{B}(\mathscr{H})$ be a quadratic operator and let n be a positive integer. Then $W^1(T)$ is an elliptical disc and*

$$W^n(T) = \{B \in \mathbb{M}_n \,:\, W(B) \subseteq W^1(T)\}.$$

We notice that every 2×2 matrix is a two dimensional quadratic operator. Therefore the matricial ranges of a 2×2 matrix can be characterized by Theorem 3.1. However, we use the Schur decomposition theorem and the elliptical range theorem to obtain an explicit formula for the matricial ranges of 2×2 matrices. First we recall some results which we will use in the sequel.

Recall that the operator $A \in \mathbb{B}(\mathscr{H})$ has a dilation $B \in \mathbb{B}(\mathscr{K})$ if there is an isometry $V : \mathscr{H} \to \mathscr{K}$ such that $A = V^*BV$.

Lemma 3.2. *[17, Lemma 3.4] Let $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The operator $B \in \mathbb{B}(\mathscr{H})$ has a dilation of the form $T \otimes I = T \oplus T \oplus \dots$ if and only if $\omega(B) \leqslant \frac{1}{2}$.*

The following result can be found in [2, P. 302]. We give a simple proof of it for the sake of completeness.

Lemma 3.3. *If T is a two dimensional operator having a matrix representation*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then

$$W^n(T) = \{B \in \mathbb{M}_n \,;\, \omega(B) \leqslant \frac{1}{2}\}, \quad (n \geqslant 1).$$

Proof. Let $B \in \mathbb{M}_n$ with $\omega(B) \leqslant \frac{1}{2}$. By Lemma 3.2, B can be delated to $T \oplus T \oplus \dots$, i.e., there is an isometry $V : \mathbb{C}^n \to \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \dots$ such that $B = V^*(T \oplus T \oplus \ldots)V$. Notice that T is an irreducible matrix so that $C^*(T) = \mathbb{M}_2$. Define $\Phi : C^*(T) = \mathbb{M}_2 \to \mathbb{M}_n$ by $\Phi(X) = V^*(X \oplus X \oplus \ldots)V$, $(X \in \mathbb{M}_2)$. Then Φ is a unital completely positive mapping and $B = \Phi(T) \in W^n(T)$.

Now, assume that $\Phi : C^*(T) = \mathbb{M}_2 \to \mathbb{M}_n$ be a unital completely positive mapping and $B = \Phi(T) \in W^n(T)$. The Stinespring Theorem 2.7 implies that $\Phi(T) = V^*\pi(T)V$ for some isometry V and some unital $*$-homomorphism π. Then we have

$$\omega(B) = \omega(\Phi(T)) = \omega(V^*\pi(T)V) \leqslant \omega(\pi(T)) \leqslant \omega(T) = \frac{1}{2}. \quad \square$$

Theorem 3.4. *(Schur Decomposition Theorem) [9, Theorem 2.3.1] For every $A \in \mathbb{M}_q$ there exists a unitary matrix $U \in \mathbb{M}_q$ such that*

$$U^*AU = S,$$

is upper triangular and the diagonal elements of S are the eigenvalues of A.

Theorem 3.5. *(Elliptical Range Theorem) [12] Let $A \in \mathbb{M}_2$ with eigenvalues λ_1 and λ_2. Then $W(A)$ is an elliptical disc with foci λ_1, λ_2 and minor axis with length $\{\mathrm{Tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2\}^{\frac{1}{2}}$. Convesely, let $\mathcal{E}$ be the closed elliptical disk in $\mathbb{C}$ with foci λ_1 and λ_2, and let d be the length of the minor axis. Then $\mathcal{E} = W(A)$ with $A = \begin{bmatrix} \lambda_1 & d \\ 0 & \lambda_2 \end{bmatrix}$.*

Lemma 3.6. *Let $A \in \mathbb{M}_2$ such that $W(A) = \{z \in \mathbb{C} \ ; \ |z| \leqslant r\}$ for some $r > 0$. Then*

$$W^n(A) = \{B \in \mathbb{M}_n \ ; \ \omega(B) \leqslant r\} \quad (n \geqslant 1).$$

More generally, if $0 \neq z_0 \in \mathbb{C}$ and $W(A) = \{z \in \mathbb{C} \ ; \ |z - z_0| \leqslant r\}$, then

$$W^n(A) = \{B + z_0 I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant r\} \quad (n \geqslant 1).$$

Proof. Let $r > 0$ and $W(A) = \{z \in \mathbb{C} \ ; \ |z| \leqslant r\}$. By the elliptical range theorem, there is a unitary matrix $U \in \mathbb{M}_2$ such that

$A = U^* \begin{bmatrix} 0 & 2r \\ 0 & 0 \end{bmatrix} U$. Hence Lemma 2.1 and Lemma 3.3 imply that

$$W^n(A) = W^n(\begin{bmatrix} 0 & 2r \\ 0 & 0 \end{bmatrix}) = 2rW^n(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \{B \in \mathbb{M}_n \ ; \ \omega(B) \leqslant r\}.$$

In the general case, assume that $W(A) = \{z \in \mathbb{C} \ ; \ |z - z_0| \leqslant r\}$ for some $0 \neq z_0 \in \mathbb{C}$. Since

$$W(A - z_0 I_2) = W(A) - z_0 = \{z \in \mathbb{C} \ ; |z| \leqslant r\}$$

we have from the first part that

$$W^n(A - z_0 I_2) = \{B \in \mathbb{M}_n \ ; \ \omega(B) \leqslant r\}.$$

Therefore

$$W^n(A) = W^n(A - z_0 I_2 + z_0 I_2) = W^n(A - z_0 I_2) + z_0 I_n$$

from which we get the desired result. $\square$

Let $A \in \mathbb{M}_2$ and let

$$U^* A U = \begin{bmatrix} \lambda_1 & a \\ 0 & \lambda_2 \end{bmatrix}, \tag{1}$$

be the Schur decomposition of A, where λ_1 and λ_2 are the eigenvalues of A. Since $W^n(A)$ is invariant under unitary transformations, it suffices to consider only upper triangular matrices of the form (1).

In the next theorem we review the matricial ranges of 2×2 matrices.

Theorem 3.7. *Let $A \in \mathbb{M}_2$ has a decomposition of the form* (1) *and let n be a positive integer. Then*

$$W^n(A) = \left\{ e^{i\theta} f^{-1}(B) + \frac{\lambda_1 + \lambda_2}{2} I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{1}{2} \right\}$$

in which $\frac{\lambda_1 - \lambda_2}{2} = re^{i\theta}$ and $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(z = x + iy) = \frac{1}{\sqrt{4r^2 + |a|^2}} x + i \frac{1}{|a|} y.$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, then

$$W^n(A) = \left\{ B + \lambda I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{|a|}{2} \right\}.$$

Proof. First, assume that $\lambda_1 = \lambda_2 = \lambda$. Then

$$U^*AU - \lambda I_2 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}.$$

It follows from Lemma 2.1 that

$$W^n(U^*(A - \lambda I_2)U) = aW^n(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}).$$

Hence

$$W^n(A) - \lambda I_n = W^n(U^*(A - \lambda I_2)U) = a \left\{ B \in \mathbb{M}_n \, ; \, \omega(B) \leqslant \frac{1}{2} \right\}.$$

Therefore

$$\begin{aligned} W^n(A) &= \left\{ B \in \mathbb{M}_n \, ; \, \omega(B) \leqslant \frac{|a|}{2} \right\} + \lambda I_n \\ &= \left\{ B + \lambda I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{|a|}{2} \right\}. \end{aligned}$$

Now, suppose that $\lambda_1 \neq \lambda_2$. If $a = 0$, then

$$U^*AU = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

is normal. It yields that

$$W^n(A) = W^n(U^*AU) = W^n(\mathrm{diag}(\lambda_1, \lambda_2)).$$

Proposition 2.2 then implies that $W^n(A)$ is the closure of C^*-convex hull of the set $\{\lambda_1, \lambda_2\}$, i.e.,

$$W^n(A) = \mathrm{cl}\left\{ \lambda_1 H_1 + \lambda_2 H_2 \ ; \ 0 \leqslant H_1, H_2 \in \mathbb{M}_n, \ H_1 + H_2 = I_n \right\},$$

see also [11, Theorem 2.7].

In the case of $a \neq 0$, we have

$$e^{-i\theta}\left(U^*\left(A-\frac{\lambda_1+\lambda_2}{2}I_2\right)U\right)=\begin{bmatrix} r & ae^{-i\theta} \\ 0 & -r \end{bmatrix},$$

where $\frac{\lambda_1-\lambda_2}{2}=re^{i\theta}$, $r>0$ and $\theta\in[0,2\pi)$. Then

$$W^n(A)=e^{i\theta}W^n(\begin{bmatrix} r & ae^{-i\theta} \\ 0 & -r \end{bmatrix})+\frac{\lambda_1+\lambda_2}{2}I_n.$$

Let $C=\begin{bmatrix} r & ae^{-i\theta} \\ 0 & -r \end{bmatrix}$. It is known that [8, Example 3]

$$W(C)=\left\{(x,y)\ ;\ \frac{x^2}{r^2+\frac{|a|^2}{4}}+\frac{y^2}{\frac{|a|^2}{4}}\leqslant 1\right\} \quad \text{and} \quad \omega(C)=\frac{\sqrt{4r^2+|a|^2}}{2}.$$

For all nonzero $\alpha,\beta\in\mathbb{R}$, the function $f:\mathbb{C}\to\mathbb{C}$ defined by $f(x+iy)=\alpha x+i\beta y$ is an invertible affine transform. If $A=\mathrm{Re}(A)+i\mathrm{Im}(A)$ is the cartesian decomposition of $A\in\mathbb{M}_2$, then $f(A)$ is defined by $f(A)=\alpha\mathrm{Re}(A)+i\beta\mathrm{Im}(A)$.

Now, consider the affine transform $f:\mathbb{C}\to\mathbb{C}$ by

$$f(z=x+iy)=\frac{1}{\sqrt{4r^2+|a|^2}}x+i\frac{1}{|a|}y.$$

Then $W(f(C))=f(W(C))=\{z\in\mathbb{C}:|z|\leqslant\frac{1}{2}\}$. It follows from Lemma 3.6 that

$$W^n(f(C))=\left\{B\in\mathbb{M}_n\ ;\ \omega(B)\leqslant\frac{1}{2}\right\}.$$

Moreover, Since f is an affine transform, it is not hard to see that $W^n(f(C))=f(W^n(C))$. Therefore

$$W^n(C)=f^{-1}(f(W^n(C))=\left\{f^{-1}(B)\ ;\ B\in\mathbb{M}_n,\ \omega(B)\leqslant\frac{1}{2}\right\},$$

and so

$$W^n(A)=\left\{e^{i\theta}f^{-1}(B)+\frac{\lambda_1+\lambda_2}{2}I_n\ ;\ B\in\mathbb{M}_n,\ \omega(B)\leqslant\frac{1}{2}\right\}.$$

This completes the proof. $\square$

The remainder of this section will be devoted to obtain the matricial rangs of some specific higher order matrices, by using Theorem 3.7.

Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are C^*-convex subsets of $\mathbb{M}_k$. Recall that the C^*-convex hull of $\mathcal{K}_1 \cup \mathcal{K}_2$, denotes by $C^* - \text{conv}\{\mathcal{K}_1, \mathcal{K}_2\}$, is the smallest norm-closed C^*-convex set containing $\mathcal{K}_1 \cup \mathcal{K}_2$ [13]. More precisely,

$$C^* - \text{conv}\{\mathcal{K}_1, \mathcal{K}_2\} = \left\{ \sum_{i=1}^{\infty} X_i^* S_{ji} X_i : \sum_{i=1}^{\infty} X_i^* X_i = I_k, S_{ji} \in \mathcal{K}_j, j = 1, 2 \right\}.$$

We refer the reader to [13] for further results concerning this notion.

Lemma 3.8. *Let $A_1, A_2 \in \mathbb{M}_k$. Then*

$$W^k(A_1 \oplus A_2) = C^* - \text{conv}\{W^k(A_1), W^k(A_2)\}.$$

Proof. The k-th matricial range of $A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ has been described in the proof of [13, Lemma 18] by using Stinespring's theorem, as follows:

$$W^k(A_1 \oplus A_2) = \left\{ \sum_{i=1}^{\infty} X_i^* A_{ji} X_i : \sum_{i=1}^{\infty} X_i^* X_i = I_k, A_{ji} \in \{A_1, A_2\} \right\}.$$

Clearly,

$$W^k(A_1) \subseteq W^k(A_1 \oplus A_2) \text{ and } W^k(A_2) \subseteq W^k(A_1 \oplus A_2),$$

and since $W^k(A_1 \oplus A_2)$ is a C^*-convex set, we conclude that

$$C^* - \text{conv}\{W^k(A_1), W^k(A_2)\} \subseteq W^k(A_1 \oplus A_2).$$

On the other hand, it is known that $A_1 \in W^k(A_1)$ and $A_2 \in W^k(A_2)$. It follows that $W^k(A_1 \oplus A_2) \subseteq C^* - \text{conv}\{W^k(A_1), W^k(A_2)\}$. Therefore

$$W^k(A_1 \oplus A_2) = C^* - \text{conv}\{W^k(A_1), W^k(A_2)\}.$$

Lemma 3.9. *Let $X, Y \in \mathbb{M}_k$. Then*

$$\begin{aligned} W^k(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}) &= W^k(\begin{bmatrix} X+Y & 0 \\ 0 & X-Y \end{bmatrix}) \\ &= C^* - \operatorname{conv}\{W^k(X+Y), W^k(X-Y)\}. \end{aligned}$$

Proof. Let $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & -I_k \\ I_k & I_k \end{bmatrix}$ and let $S = \begin{bmatrix} X & Y \\ Y & X \end{bmatrix}$. It follows from Lemma 2.1 (ii) that

$$W^k(S) = W^k(U^*SU) = W^k(\begin{bmatrix} X+Y & 0 \\ 0 & X-Y \end{bmatrix}). \quad \square$$

Corollary 3.10. *If $X, Y \in \mathbb{M}_k$, then*

$$W^k(\begin{bmatrix} X & X \\ X & X \end{bmatrix}) = W^k(\begin{bmatrix} X & -X \\ -X & X \end{bmatrix}) = 2W^k(X),$$

and

$$W^k(\begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix}) = W^k(Y).$$

Similarly,

$$W^k(\begin{bmatrix} \lambda I_k & Y \\ -Y & \lambda I_k \end{bmatrix}) = W^k(\lambda I_k + Y) = \lambda I_k + W^k(Y),$$

and

$$\begin{aligned} W^k(\begin{bmatrix} \lambda I_k & Y \\ Y & \lambda I_k \end{bmatrix}) &= W^k(\begin{bmatrix} \lambda I_k + Y & 0 \\ 0 & \lambda I_k - Y \end{bmatrix}) \\ &= C^* - \operatorname{conv}\{\lambda I_k - W^k(Y), \lambda I_k + W^k(Y)\}. \end{aligned}$$

Example 3.11. Suppose that λ, μ, a, b are nonzero complex numbers. Let

$$X = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}, \quad Y = \begin{bmatrix} \mu & b \\ 0 & \mu \end{bmatrix} \textit{and} \quad Z = \begin{bmatrix} \mu & -a \\ 0 & \mu \end{bmatrix}.$$

If $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then Theorem 3.7 follows that

$$W^2(\begin{bmatrix} 0 & X \\ -X & 0 \end{bmatrix}) = \lambda I_2 + aW^2(T)$$

and

$$W^2(\begin{bmatrix} \lambda I_2 & Y \\ -Y & \lambda I_2 \end{bmatrix}) = (\lambda + \mu)I_2 + bW^2(T).$$

Moreover,

$$W^2(X + Y) = (\lambda + \mu)I_2 + (a + b)W^2(T).$$

and

$$W^2(X - Y) = (\lambda - \mu)I_2 + (a - b)W^2(T).$$

Then

$$W^2(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}) = C^* - \mathrm{conv}\{W^2(X + Y), W^2(X - Y)\},$$

and

$$\begin{aligned} W^2(\begin{bmatrix} X & Z \\ Z & X \end{bmatrix}) &= C^* - \mathrm{conv}\{W^2(X + Z), W^2(X - Z)\} \\ &= C^* - \mathrm{conv}\{(\lambda + \mu)I_2, \{(\lambda - \mu)I_2 + B : \omega(B) \leqslant |a|\}\}. \end{aligned}$$

Theorem 3.12. *Suppose that β is a nonzero complex number and $\{\lambda_k\}_{k=0}^{\infty}$ is a sequence of nonzero complex numbers. If $Y_2 := \begin{bmatrix} \lambda_0 & \beta \\ 0 & \lambda_0 \end{bmatrix}$ and*

$$Y_{2^{k+1}} = \begin{bmatrix} \lambda_k I_{2^k} & Y_{2^k} \\ -Y_{2^k} & \lambda_k I_{2^k} \end{bmatrix} \in \mathbb{M}_{2^{k+1}},$$

for all $k = 1, 2, \ldots$, then

$$\begin{aligned} W^{2^k}(Y_{2^{k+1}}) &= (\lambda_k + \ldots + \lambda_1 + \lambda_0)I_{2^k} + \beta W^{2^k}(T) \\ &= \left\{(\lambda_k + \ldots + \lambda_1 + \lambda_0)I_{2^k} + B : B \in \mathbb{M}_{2^k}, \omega(B) \leqslant \frac{|\beta|}{2}\right\}, \end{aligned}$$

for which $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Proof. It follows from Lemma 2.1 and Corollary 3.10 that

$$\begin{aligned} W^{2^k}(Y_{2^{k+1}}) &= W^{2^k}(\lambda_k I_{2^k} + \begin{bmatrix} 0 & Y_{2^k} \\ -Y_{2^k} & 0 \end{bmatrix}) \\ &= \lambda_k I_{2^k} + W^{2^k}(Y_{2^k}) \\ &= \lambda_k I_{2^k} + W^{2^k}(\lambda_{k-1} I_{2^{k-1}} + \begin{bmatrix} 0 & Y_{2^{k-1}} \\ -Y_{2^{k-1}} & 0 \end{bmatrix}) \\ &= (\lambda_k + \lambda_{k-1}) I_{2^k} + W^{2^k}(Y_{2^{k-1}}). \end{aligned}$$

Similarly,

$$\begin{aligned} W^{2^k}(Y_{2^{k-1}}) &= W^{2^k}(\lambda_{k-2} I_{2^{k-2}} + \begin{bmatrix} 0 & Y_{2^{k-2}} \\ -Y_{2^{k-2}} & 0 \end{bmatrix}) \\ &= \lambda_{k-2} I_{2^k} + W^{2^k}(Y_{2^{k-2}}). \end{aligned}$$

Hence

$$W^{2^k}(Y_{2^{k+1}}) = (\lambda_k + \lambda_{k-1} + \lambda_{k-2}) I_{2^k} + W^{2^k}(Y_{2^{k-2}}).$$

Continue this procedure to get

$$\begin{aligned} W^{2^k}(Y_{2^{k+1}}) &= (\lambda_k + \ldots + \lambda_1) I_{2^k} + W^{2^k}(Y_2) \\ &= (\lambda_k + \ldots + \lambda_1) I_{2^k} + W^{2^k}(\lambda_0 I_2 + \beta T) \\ &= (\lambda_k + \ldots + \lambda_1 + \lambda_0) I_{2^k} + \beta W^{2^k}(T) \\ &= \left\{ (\lambda_k + \ldots + \lambda_1 + \lambda_0) I_{2^k} + B : B \in \mathbb{M}_{2^k}, \omega(B) \leqslant \frac{|\beta|}{2} \right\}. \quad \square \end{aligned}$$

4. Matricial Ranges of Some 3×3 Matrices

In this section, utilizing the results of Section 2, we describe the matricial ranges of some special 3×3 matrices. Some inclusion results related to the numerical range of matrices have been studied in [4]. The following nice result will be applied in the sequel to achieve our goal.

Lemma 4.1. *[4, Theorem 4.3] Suppose that $A \in \mathbb{M}_3$ has a non-trivial reducing subspace and $B \in \mathbb{B}(\mathscr{H})$. Then $W(B) \subseteq W(A)$ if and only if B has a dilation, that is unitarily similar to $A \otimes I$.*

Theorem 4.2. *Let $A \in \mathbb{M}_3$ has a non-trivial reducing subspace and let $W(A) = \{z \in \mathbb{C} \ ; \ |z| \leqslant r\}$ for some $r > 0$. Then*

$$W^n(A) = \{B \in \mathbb{M}_n \ ; \ \omega(B) \leqslant r\} \quad (n \geqslant 1).$$

Proof. If $B \in W^n(A)$, then $B = \Phi(A)$ for some unital completely positive mapping $\Phi : C^*(A) \to \mathbb{M}_n$. The Stinespring theorem implies that there is an isometry V and a unital $*$-homomorphism π such that $\Phi(A) = V^*\pi(A)V$. Hence

$$\omega(B) = \omega(\Phi(A)) = \omega(V^*\pi(A)V) \leqslant \omega(\pi(A)) \leqslant \omega(A) \leqslant r.$$

Therefore

$$W^n(A) \subseteq \{B \in \mathbb{M}_n \ ; \omega(B) \leqslant r\}.$$

Now assume that $B \in \mathbb{M}_n$ and $\omega(B) \leqslant r$. Then $W(B) \subseteq W(A)$. Lemma 4.1 concludes that B has a dilation that is unitarily similar to $A \otimes I$, i.e. there is an isometry V such that $B = V^*(A \otimes I)V$. Define $\Phi : C^*(A) \to \mathbb{M}_n$ by $\Phi(X) = V^*(X \otimes I)V$. Then Φ is a unital completely positive linear mapping with $\Phi(A) = B$. Therefore $B \in W^n(A)$. This completes the proof. $\square$

Example 4.3. Consider the matrix $C_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{bmatrix}$ for some $r \in \mathbb{C}$. Then C_r has a reducing subspace, namely, $\operatorname{span}\{(0,0,1)\}$. It is known that the numerical range of C_r is the convex hull of $\{z \in \mathbb{C} \ ; \ |z| \leqslant \frac{1}{2}\}$ and the single point set $\{r\}$ [10, Theorem 2.2]. Consequently, if $|r| \leqslant \frac{1}{2}$, then

$$W(C_r) = \left\{ z \in \mathbb{C} \ ; \ |z| \leqslant \frac{1}{2} \right\}.$$

Theorem 4.2 now can be applied to get

$$W^n(C_r) = \left\{ B \in \mathbb{M}_n \ ; \ \omega(B) \leqslant \frac{1}{2} \right\}.$$

Example 4.4. Let $\lambda, \mu, b \in \mathbb{C}$ such that $b \neq 0$ and $|\mu - \lambda| \leqslant \frac{|b|}{2}$. Consider

$$A = \begin{bmatrix} \lambda & b & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

Assume that $\frac{\bar{b}(\mu-\lambda)}{|b|^2} = re^{i\theta}$ so that $r = \frac{|\mu-\lambda|}{|b|} \leqslant \frac{1}{2}$. The matrix $e^{-i\theta}\frac{\bar{b}(A-\lambda I_3)}{|b|^2}$ is unitarily equivalent to

$$C_r = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{bmatrix}.$$

Hence

$$W^n(\bar{b}(A - \lambda I_3)) = e^{i\theta}|b|^2 W^n(C_r),$$

whence

$$W^n(A) = \left\{ B + \lambda I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{|b|}{2} \right\}.$$

Similarly, if $\lambda, \mu, c \in \mathbb{C}$ such that $c \neq 0$, $|\mu - \lambda| \leqslant \frac{|c|}{2}$ and

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ c & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

then

$$W^n(A) = \left\{ B + \lambda I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{|c|}{2} \right\}.$$

Lemma 4.5. *Let $T \in \mathbb{B}(\mathscr{H})$ be a quadratic operator and let n be a positive integer. Then there is a matrix $C \in \mathbb{M}_2$ such that $W^n(T) = W^n(C)$.*

Proof. By Lemma 3.1, $W^1(T)$ is an elliptical disc. Then the elliptical range theorem follows that there is a matrix $C \in \mathbb{M}_2$ such that $W^1(T) =$

$W(C)$. Since every $C \in \mathbb{M}_2$ is quadratic, using Lemma 3.1 yields that

$$\begin{aligned} W^n(T) &= \left\{B \in \mathbb{M}_n \ ; \ W(B) \subseteq W^1(T)\right\} \\ &= \{B \in \mathbb{M}_n \ ; \ W(B) \subseteq W(C)\} \\ &= W^n(C). \quad \square \end{aligned}$$

The matricial ranges of some quadratic operators is given in the next theorem.

Theorem 4.6. *Let $T \in \mathbb{B}(\mathscr{H})$ be a quadratic operator and let n be a positive integer. If $W^1(T) = \{z \in \mathbb{C} \ ; \ |z - z_0| \leqslant r\}$ for some $r > 0$ and $z_0 \in \mathbb{C}$, then*

$$W^n(T) = \{B + z_0 I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant r\}.$$

Proof. Lemma 4.5 implies that there exists a matrix $C \in \mathbb{M}_2$ such that $W^1(T) = W(C)$ and $W^n(T) = W^n(C)$. Moreover, since

$$W(C) = W^1(T) = \{z \in \mathbb{C} \ ; \ |z - z_0| \leqslant r\},$$

it follows from Lemma 3.6 that

$$W^n(C) = \{B + z_0 I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant r\}. \quad \square$$

Example 4.7. Let $\lambda, \mu, \alpha, \beta \in \mathbb{C}$ for which $\alpha \neq 0, \beta \neq 0$ and let n be a positive integer. Consider the quaratic matrix

$$A = \begin{bmatrix} \lambda & \alpha & \beta \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}.$$

It has been shown in [10, Theorem 2.3] that $W(A)$ is an ellipse with the foci λ, μ and minor axis of length $d := \sqrt{|\alpha|^2 + |\beta|^2}$. Hence the elliptical range theorem implies that there is a 2×2 matrix $C = \begin{bmatrix} \lambda & d \\ 0 & \mu \end{bmatrix}$ such that $W(A) = W(C)$. Lemma 4.5 concludes that $W^n(A) = W^n(C)$. Moreover, applying Theorem 3.7 we get

$$W^n(A) = W^n(C) = \left\{ e^{i\theta} f^{-1}(B) + \frac{\lambda + \mu}{2} I_n \ ; \ B \in \mathbb{M}_n, \ \omega(B) \leqslant \frac{1}{2} \right\},$$

in which $\frac{\lambda-\mu}{2} = re^{i\theta}$ and $f : \mathbb{C} \to \mathbb{C}$ is defined by

$$f(z = x + iy) = \frac{1}{\sqrt{4r^2 + |d|^2}} x + i\frac{1}{|d|} y.$$

As an special case we have the following example.

Example 4.8. Let $\lambda, \alpha, \beta \in \mathbb{C}$ such that $\alpha \neq 0, \beta \neq 0$. Consider the quadratic matrix

$$A = \begin{bmatrix} \lambda & \alpha & \beta \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

By [10, Corollary 2.5] (see also [17, Theorem 2.1]) we have

$$W(A) = \{z \in \mathbb{C} : |z - \lambda| \leqslant \frac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}\}.$$

Hence Corollary 4.6 follows that

$$W^n(A) = \{B + \lambda I_n : B \in \mathcal{M}_n, \omega(B) \leqslant \frac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}\}.$$

A similar argument can be applied to obtain the matricial ranges of the matrices

$$\begin{bmatrix} \lambda & 0 & \beta \\ 0 & \lambda & \gamma \\ 0 & 0 & \mu \end{bmatrix} \text{and} \begin{bmatrix} \lambda & 0 & \beta \\ 0 & \lambda & \gamma \\ 0 & 0 & \lambda \end{bmatrix},$$

for all complex numbers $\lambda, \mu, \beta, \gamma$ such that $\beta \neq 0, \gamma \neq 0$.

The same arguments as in the proof of Theorem 3.12 actually lead to the following result.

Theorem 4.9. *Suppose that α, β are nonzero complex numbers and $\{\lambda_k\}_{k=0}^{\infty}$ is a sequence of nonzero complex numbers. If* $S = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *$Y_0 = \lambda_0 I_3 + S$ and*

$$Y_k = \begin{bmatrix} \lambda_k I_{3k} & Y_{k-1} \\ -Y_{k-1} & \lambda_k I_{3k} \end{bmatrix},$$

for all $k = 1, 2, \ldots,$ *then*

$$W^{3\times 2^k}(Y_k) = (\lambda_k + \ldots + \lambda_1 + \lambda_0) I_{3\times 2^{k-1}} + W^{3\times 2^k}(S).$$

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Mahdi Dehghani
Assistant Professor of Mathematics
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Kashan
P.O. Box 87317-53153
Kashan, Iran
E-mail: m.dehghani@kashanu.ac.ir and e.g.mahdi@gmail.com

Mohsen Kian
Department of Mathematics
Assiatant Professor of Mathematics
Faculty of Basic Sciences
University of Bojnord
P. O. Box 1339, Bojnord 94531, Iran
E-mail: kian@ub.ac.ir and kian_tak@yahoo.com