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# A Generalization of Dual Bi-Periodic Fibonacci Quaternions

**E. Tan**
Ankara University

**I. Gök***
Ankara University

**Abstract.** The purpose of the paper is to present a new generalization of the dual Fibonacci quaternions called the generalized dual bi-periodic Fibonacci quaternions. This new generalization allows us to state dual quaternion sequences as a unique sequence. Furthermore, we give the generating function, the Binet formula, the norm value of these dual quaternions, and some basic properties of them.

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## 1. Introduction

It is well known that the algebra of quaternions

$$\mathbf{H} = \{\sum_{i=0}^{3} a_i e_i : a_i \in \mathbb{R}\} \cong \mathbb{C}^2,$$

is defined as a four-dimensional vector space over $\mathbb{R}$ having a basis $e_0 \cong 1, e_1 \cong i, e_2 \cong j$ and $e_3 \cong k$, which satisfies the following multiplication

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*Corresponding author

rules:

$$\begin{aligned} e_l^2 &= -1,\ l \in \{1,2,3\}; \\ e_1e_2 &= -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2. \end{aligned}$$

The dual numbers are invented by Clifford as an extension of the real numbers, and defined by

$$d = a + \varepsilon a^*,$$

where $\varepsilon$ is the dual unit with $\varepsilon \neq 0, \varepsilon^2 = 0$, and $a, a^* \in \mathbb{R}$. Similar to real quaternions, the dual quaternions are defined by taking dual numbers instead of real numbers as a coefficient. The algebra of dual quaternions

$$\mathbf{D} = \left\{ q + \varepsilon q^* : q, q^* \in \mathbf{H},\ \varepsilon \neq 0, \varepsilon^2 = 0 \right\},$$

is a Clifford algebra and has some interesting applications on mechanics, mathematics, and physics. Especially, the dual quaternions are important to represent rigid body motions in 3D-space. For the detailed information related to dual quaternions and their applications, we refer to [3, 11, 13].

There are several studies on different types of sequences over quaternion algebra and dual quaternion algebra. In [9], Horadam introduced the $n-$th Fibonacci quaternion, $F_n$, as

$$F_n = f_ne_0 + f_{n+1}e_1 + f_{n+2}e_2 + f_{n+3}e_3,$$

where $f_n$ is the $n-$th Fibonacci number and defined by

$$f_0 = 0, f_1 = 1,\ f_n = f_{n-1} + f_{n-2},\ n \geqslant 2.$$

Similarly, Nurkan and Güven [10] introduced the $n-$th dual Fibonacci number and $n-$th dual Fibonacci quaternion as

$$\widetilde{f_n} = f_n + \varepsilon f_{n+1} \quad \text{and} \quad \widetilde{F_n} = F_n + \varepsilon F_{n+1},$$

respectively. Here $f_n$ is the $n-$th Fibonacci number, $F_n$ is the $n-$th Fibonacci quaternion. For a survey on researches related to different types of sequences over quaternion, dual quaternion, octonion and dual octonion algebras, we refer to [2, 5, 6, 7, 9, 10, 12, 14, 15, 16, 18, 19, 20], and for a recent study on the general linear second order recurrences we refer to [8].

In this paper, we find the motivation from the paper [1] and present a further generalization of the dual Fibonacci and the dual Lucas quaternions, called as, the generalized dual bi-periodic Fibonacci quaternions. Further, we obtain the generating function, the Binet formula, the norm value and some basic properties of these dual quaternions. Note that, this new generalization allows us to state several dual quaternion sequences as a unique sequence. By taking appropriate values for the generalized dual bi-periodic Fibonacci quaternions, we can easily see several dual quaternion sequences as a special case.

First consider the sequence of the generalized bi-periodic Fibonacci quaternions $\{W_n\}$, which is defined in [16] as:

$$W_n = \sum_{l=0}^{3} w_{n+l} e_l, \tag{1}$$

where $\{w_n\}$ is the sequence which is introduced in [4] as:

$$w_n = \begin{cases} aw_{n-1} + w_{n-2}, & \text{if } n \text{ is even} \\ bw_{n-1} + w_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geqslant 2, \tag{2}$$

with arbitrary initial conditions $w_0, w_1$ where $w_0, w_1, a, b$ are nonzero numbers. For the basic properties of the sequence $\{w_n\}$, see [17]. Since the sequence $\{w_n\}$ is a generalization of the bi-periodic Fibonacci and Lucas sequences, hence we can obtain a further generalization of the bi-periodic Fibonacci and Lucas quaternions as well. For example, if we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the bi-periodic Fibonacci quaternions in [14] , and if we take the initial conditions $w_0 = 2$ and $w_1 = b$, by switching $a$ and $b$, we get the bi-periodic Lucas quaternions in [15]. Finally, we want to emphasize that we can obtain several well-known quaternion sequences in the literature for particular values of $a$ and $b$.

We need the following results which can be found in [16].

For $n>0$, the Binet formula for the sequence $\{w_n\}$ and $\{W_n\}$ can be given by

$$w_n=\frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2}\rfloor}}\left(A\alpha^{n-1}-B\beta^{n-1}\right)\quad , \tag{3}$$

and

$$W_n=\begin{cases}\frac{1}{(ab)^{\lfloor \frac{n}{2}\rfloor}}\left(A\alpha^{*}\alpha^{n-1}-B\beta^{*}\beta^{n-1}\right), & \text{if } n \text{ is even}\\ \frac{1}{(ab)^{\lfloor \frac{n}{2}\rfloor}}\left(A\alpha^{**}\alpha^{n-1}-B\beta^{**}\beta^{n-1}\right), & \text{if } n \text{ is odd}\end{cases}, \tag{4}$$

respectively. Here $\alpha:=\frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta:=\frac{ab-\sqrt{a^2b^2+4ab}}{2}$ are the roots of the polynomial $x^2-abx-ab=0$, $\zeta(n)=n-2\left\lfloor\frac{n}{2}\right\rfloor$ is the parity function, i.e., $\zeta(n)=0$ when $n$ is even and $\zeta(n)=1$ when $n$ is odd. We assume that $a^2b^2+4ab\neq 0$, and we have $\alpha+\beta=ab$, $\alpha-\beta=\sqrt{a^2b^2+4ab}$, $\alpha\beta=-ab$. Additionally, $A, B, \alpha^*, \beta^*, \alpha^{**}$, and $\beta^{**}$ are defined as follows:

$$\begin{aligned}A &:= \frac{\alpha w_1+bw_0}{\alpha-\beta},\ B:=\frac{\beta w_1+bw_0}{\alpha-\beta},\\ \alpha^* &:= \sum_{l=0}^{3}\frac{a^{\zeta(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\alpha^l e_l,\ \beta^*:=\sum_{l=0}^{3}\frac{a^{\zeta(l+1)}}{(ab)^{\lfloor\frac{l}{2}\rfloor}}\beta^l e_l,\\ \alpha^{**} &:= \sum_{l=0}^{3}\frac{a^{\zeta(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\alpha^l e_l,\ \beta^{**}:=\sum_{l=0}^{3}\frac{a^{\zeta(l)}}{(ab)^{\lfloor\frac{l+1}{2}\rfloor}}\beta^l e_l.\end{aligned}$$

The generating function of the generalized bi-periodic Fibonacci quaternions is

$$G(t)=\frac{W_0+(W_1-bW_0)\,t+(a-b)\sum_{s=0}^{3}R(t,s)\,e_s}{1-bt-t^2}$$

where

$$\begin{aligned}R(t,s) &:= \left(f(t)-\sum_{k=1}^{\lfloor\frac{s+1}{2}\rfloor}w_{2k-1}t^{2k-1}\right)t^{1-s},\\ f(t) &:= \sum_{n=1}^{\infty}w_{2n-1}t^{2n-1}=\frac{w_1t+(bw_0-w_1)\,t^3}{1-(ab+2)\,t^2+t^4}.\end{aligned}$$

The norm of the generalized bi-periodic Fibonacci quaternions is

$$Nr\left(W_n\right) := W_n\overline{W_n}, \tag{5}$$

where $\overline{W_n} := w_n e_0 - w_{n+1}e_1 - w_{n+2}e_2 - w_{n+3}e_3$ is the conjugate of the generalized bi-periodic Fibonacci quaternions.

## 2. Generalized Dual Bi-Periodic Fibonacci Quaternions

In the present section, we introduce the generalized dual bi-periodic Fibonacci numbers and the generalized dual bi-periodic Fibonacci quaternions. Also, we state some basic properties of these dual quaternions.

**Definition 2.1.** *The generalized dual bi-periodic Fibonacci numbers* $\{\widetilde{w}_n\}$ *are defined by*

$$\widetilde{w}_n = w_n + \varepsilon w_{n+1}, \tag{6}$$

*where* $w_n$ *is the* $n-th$ *generalized bi-periodic Fibonacci number in (3).*

**Definition 2.2.** *The generalized dual bi-periodic Fibonacci quaternions* $\{\widetilde{W}_n\}$ *are defined by*

$$\widetilde{W}_n = W_n + \varepsilon W_{n+1} = \sum_{i=0}^{3} \widetilde{w}_{n+i}e_i, \tag{7}$$

*where* $\widetilde{w}_n$ *is the* $n-th$ *generalized dual bi-periodic Fibonacci number.*

The addition, the subtraction, and the multiplication of any two generalized dual bi-periodic Fibonacci quaternions $\widetilde{W}_n$ and $\widetilde{V}_n$ can be given as follows:

$$\widetilde{W}_n \pm \widetilde{V}_n = \left(W_n + V_n\right) \pm \varepsilon\left(W_{n+1} + V_{n+1}\right) \tag{8}$$

$$\widetilde{W}_n\widetilde{V}_n = \left(W_n V_n\right) + \varepsilon\left(W_n V_{n+1} + W_{n+1}V_n\right). \tag{9}$$

The norm of the generalized dual bi-periodic Fibonacci quaternions is

$$Nr\left(\widetilde{W}_n\right) := \widetilde{W}_n\overline{\widetilde{W}_n}, \tag{10}$$

where $\overline{\widetilde{W_n}} := \overline{W_n} + \varepsilon \overline{W_{n+1}}$ is the conjugate of the generalized dual bi-periodic Fibonacci quaternions.

In the following, we give several different sequences which are special cases of $\{\widetilde{W_n}\}$ :

1. If we take the initial conditions $w_0 = 0$ and $w_1 = 1$, we get the dual bi-periodic Fibonacci quaternions in [1]. It is obvious that, under these conditions if we take $a = b = 1$, we get the dual Fibonacci quaternions in [10].

2. If we take the initial conditions $w_0 = 2$ and $w_1 = b$, we get the dual bi-periodic Lucas quaternions. It is obvious that, under these conditions if we take $a = b = 1$, we get the dual Lucas quaternions in [10].

3. If we take the initial conditions $a = b$ in $\{w_n\}$, we get the dual Horadam quaternion numbers in [8] with the restriction of $q = 1$. Differ from the dual Horadam quaternions, we can obtain different dual quaternion sequences for the distinct values of $a$ and $b$ in the sequence $\{\widetilde{W_n}\}$.

Now, we give the generating function of the generalized dual bi-periodic Fibonacci quaternions.

**Theorem 2.3.** *The generating function for the generalized dual bi-periodic Fibonacci quaternions* $\widetilde{W_n}$ *is*

$$\begin{aligned}\widetilde{G}(t) &= \frac{W_0 + (W_1 - bW_0)\,t + (a-b)\sum_{s=0}^{3} R(t,s)\,e_s}{1 - bt - t^2} \\ &\quad + \varepsilon\left(\frac{W_1 + (W_2 - bW_1)\,t + (a-b)\sum_{s=0}^{3} R(t,s+1)\,e_s}{1 - bt - t^2}\right), \qquad (11)\end{aligned}$$

*where* $R(t,s)$ *and* $f(t)$ *are defined in (5) and (5), respectively.*

**Proof.** The formal power series representation of the generating function for $\{\widetilde{W_n}\}$ is

$$\widetilde{G}(t) = \sum_{n=0}^{\infty} \widetilde{W_n} t^n = \widetilde{W_0} + \widetilde{W_1} t + \widetilde{W_2} t^2 + \cdots + \widetilde{W_n} t^n + \cdots .$$

Note that,

$$bt\widetilde{G}\left(t\right)=b\widetilde{W}_0t+b\widetilde{W}_1t^2+\cdots+b\widetilde{W}_nt^{n+1}+\cdots,$$

and

$$t^2\widetilde{G}\left(t\right)=\widetilde{W}_0t^2+\widetilde{W}_1t^3+\cdots+\widetilde{W}_nt^{n+2}+\cdots.$$

Thus, we have

$$\begin{aligned}\left(1-bt-t^2\right)\widetilde{G}\left(t\right)&=\widetilde{W}_0+\left(\widetilde{W}_1-b\widetilde{W}_0\right)t+\sum_{n=2}^{\infty}\left(\widetilde{W}_n-b\widetilde{W}_{n-1}-\widetilde{W}_{n-2}\right)t^n\\&=\left(W_0+\left(W_1-bW_0\right)t\right)+\varepsilon\left(W_1+\left(W_2-bW_1\right)t\right)\\&+\sum_{n=2}^{\infty}\left(W_n-bW_{n-1}-W_{n-2}\right)t^n+\varepsilon\sum_{n=2}^{\infty}\left(W_{n+1}-bW_n-W_{n-1}\right)t^n\\&=W_0+\left(W_1-bW_0\right)t+\sum_{n=2}^{\infty}\left(W_n-bW_{n-1}-W_{n-2}\right)t^n\\&+\varepsilon\left(W_1+\left(W_2-bW_1\right)t+\sum_{n=2}^{\infty}\left(W_{n+1}-bW_n-W_{n-1}\right)t^n\right).\end{aligned}$$

By using the definition of the functions $R\left(t,s\right)$ and $f\left(t\right)$, we obtain the desired result. □

Now we will state the Binet formula for the generalized dual bi-periodic Fibonacci quaternions and so derive some well-known properties.

**Theorem 2.4.** *The Binet formula for the generalized dual bi-periodic Fibonacci quaternion is*

$$\widetilde{W_n}=\begin{cases}\frac{1}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\left[A\alpha^{n-1}\left(\alpha^*+\varepsilon\alpha^{**}\alpha\right)-B\beta^{n-1}\left(\beta^*+\varepsilon\beta^{**}\beta\right)\right], & \text{if } n \text{ is even}\\ \frac{1}{(ab)^{\left\lfloor\frac{n+1}{2}\right\rfloor}}\left[A\alpha^{n}\left(-\alpha^{**}\beta+\varepsilon\alpha^*\right)-B\beta^{n}\left(-\beta^{**}\alpha+\varepsilon\beta^*\right)\right], & \text{if } n \text{ is odd,}\end{cases}\tag{12}$$

*where $A,B,\alpha^*,\beta^*,\alpha^{**}$, and $\beta^{**}$ defined in (5)-(5).*

**Proof.** By using the definition of the generalized dual bi-periodic Fibonacci sequence $\{\widetilde{W}_n\}$ in (7) and the Binet formula of the sequence $\{w_n\}$ in (3), we can easily obtained the desired result. □

By using the Binet formula for the generalized dual bi-periodic Fibonacci quaternion sequences, we obtain the following identity.

**Theorem 2.5.** *(Catalan's like identity) For nonnegative integer number $n$ and even integer $r$, such that $r \leqslant n$, we have*

$$\begin{aligned}
&\widetilde{W}_{n-r}\widetilde{W}_{n+r}-\widetilde{W}_n^2\\
&=\begin{cases}
\frac{AB(\alpha^r-\beta^r)}{(\alpha\beta)^{r+1}}\left[(\alpha^*\beta^*\beta^r-\beta^*\alpha^*\alpha^r)\right. & \\
\left.+\varepsilon\left(\beta\left(\alpha^*\beta^{**}\beta^r-\beta^{**}\alpha^*\alpha^r\right)+\alpha\left(\alpha^{**}\beta^*\beta^r-\beta^*\alpha^{**}\alpha^r\right)\right)\right], & \text{if } n \text{ is even}\\
\frac{AB(\alpha^r-\beta^r)}{(\alpha\beta)^r}\left[(\alpha^{**}\beta^{**}\beta^r-\beta^{**}\alpha^{**}\alpha^r)\right. & \\
\left.+\varepsilon\left(\beta^{-1}\left(\beta^{**}\alpha^*\alpha^r-\alpha^*\beta^{**}\beta^r\right)+\alpha^{-1}\left(\beta^*\alpha^{**}\alpha^r-\alpha^{**}\beta^*\beta^r\right)\right)\right], & \text{if } n \text{ is odd}
\end{cases}.
\end{aligned}$$

**Proof.** For even $n$, by using the definition of the substraction and the multiplication of any two generalized dual bi-periodic Fibonacci quaternions in (8) and (9), we have

$$\begin{aligned}
&\widetilde{W}_{n-r}\widetilde{W}_{n+r}-\widetilde{W}_n^2\\
=\ &\left(W_{n-r}W_{n+r}-W_n^2\right)+\varepsilon\left(W_{n-r}W_{n+r+1}+W_{n-r+1}W_{n+r}-W_{n+1}W_n-W_nW_{n+1}\right).
\end{aligned}$$

Then by using the Catalan's like identity for the generalized bi-periodic Fibonacci quaternions in [16, Theorem 3], and the Binet formula of the sequence $\{W_n\}$ in (4), we get

$$\begin{aligned}
&\widetilde{W}_{n-r}\widetilde{W}_{n+r}-\widetilde{W}_n^2\\
=\ &\frac{AB\left(\alpha^r-\beta^r\right)}{(\alpha\beta)^{r+1}}\left[\alpha^*\beta^*\beta^r-\beta^*\alpha^*\alpha^r\right]\\
&+\varepsilon\frac{AB}{(ab)^n}\left[\left(\alpha^*\beta^{**}\alpha^{n-1}\beta^n\left(1-\left(\frac{\beta}{\alpha}\right)^r\right)+\beta^{**}\alpha^*\alpha^{n-1}\beta^n\left(1-\left(\frac{\alpha}{\beta}\right)^r\right)\right)\right.\\
&\left.+\left(\alpha^{**}\beta^*\alpha^n\beta^{n-1}\left(1-\left(\frac{\beta}{\alpha}\right)^r\right)+\beta^*\alpha^{**}\alpha^n\beta^{n-1}\left(1-\left(\frac{\alpha}{\beta}\right)^r\right)\right)\right]
\end{aligned}$$

$$\begin{aligned}
=\ &\frac{AB\left(\alpha^r-\beta^r\right)}{(\alpha\beta)^{r+1}}\left[(\alpha^*\beta^*\beta^r-\beta^*\alpha^*\alpha^r)\right.\\
&\left.+\varepsilon\left(\beta\left(\alpha^*\beta^{**}\beta^r-\beta^{**}\alpha^*\alpha^r\right)+\alpha\left(\alpha^{**}\beta^*\beta^r-\beta^*\alpha^{**}\alpha^r\right)\right)\right].
\end{aligned}$$

Similarly, it can be proven for odd $n$. □

If we take $r=2$ in the above theorem we obtain the following result which gives the Cassini's like identity for the generalized dual bi-periodic Fibonacci quaternions.

**Corollary 2.6.** *(Cassini's like identity) For $n \geqslant 1$, we have*

$$\begin{aligned} & \widetilde{W}_{2(n-1)}\widetilde{W}_{2(n+1)} - \widetilde{W}_{2n}^2 \\ = & \frac{AB\left(\alpha^2-\beta^2\right)}{(\alpha\beta)^3}\left[\left(\alpha^*\beta^*\beta^2-\beta^*\alpha^*\alpha^2\right)\right. \\ & \left.+\varepsilon\left(\beta\left(\alpha^*\beta^{**}\beta^2-\beta^{**}\alpha^*\alpha^2\right)+\alpha\left(\alpha^{**}\beta^*\beta^2-\beta^*\alpha^{**}\alpha^2\right)\right)\right] \end{aligned}$$

*and the identity in [16, Corollary 1] gives us*

$$\begin{aligned} & \widetilde{W}_{2(n-1)}\widetilde{W}_{2(n+1)} - \widetilde{W}_{2n}^2 \\ = & W_0W_4 - W_2^2 \\ & +\varepsilon\frac{AB\left(\alpha^2-\beta^2\right)}{(\alpha\beta)^3}\left(\beta\left(\alpha^*\beta^{**}\beta^2-\beta^{**}\alpha^*\alpha^2\right)+\alpha\left(\alpha^{**}\beta^*\beta^2-\beta^*\alpha^{**}\alpha^2\right)\right). \end{aligned}$$

Next, we give the norm of the generalized dual bi-periodic Fibonacci quaternions.

**Theorem 2.7.** *The norm of the generalized dual bi-periodic Fibonacci quaternions is*

$$Nr\left(\widetilde{W}_n\right) = T(n) + T(n+1) + \varepsilon 2a\left(S(n)+S(n+2)\right)$$

*where*

$$\begin{aligned} T(n) &:= \frac{a^{2\zeta(n+1)}}{(ab)^{2\left\lfloor\frac{n+2}{2}\right\rfloor}}\left[A^2\alpha^{2n}\left(\alpha^2+\beta^2\right)-4AB(\alpha\beta)^{n+1}+B^2\beta^{2n}\left(\alpha^2+\beta^2\right)\right], \\ S(n) &:= \frac{1}{(ab)^n}\left(A^2\alpha^{2n-1}(\alpha+2)+B^2\beta^{2n-1}(\beta+2)\right). \end{aligned}$$

**Proof.** By using the definition of the norm of the generalized dual bi-periodic Fibonacci quaternions in (10), we have

$$Nr\left(\widetilde{W}_n\right) = \widetilde{W}_n\overline{\widetilde{W}_n} = W_n\overline{W_n} + \varepsilon\left(W_n\overline{W_{n+1}} + W_{n+1}\overline{W_n}\right),$$

where $\overline{\widetilde{W_n}} := \overline{W_n} + \varepsilon \overline{W_{n+1}}$ is the conjugate of the generalized dual bi-periodic Fibonacci quaternion $\widetilde{W_n}$.

From [16, Theorem 6], we have

$$\begin{aligned} W_n \overline{W_n} &= w_n^2 + w_{n+1}^2 + w_{n+2}^2 + w_{n+3}^2 \\ &= T(n) + T(n+1), \end{aligned}$$

where

$$T(n) := \frac{a^{2\zeta(n+1)}}{(ab)^{2\left\lfloor \frac{n+2}{2} \right\rfloor}} \left[ A^2\alpha^{2n}\left(\alpha^2+\beta^2\right) - 4AB(\alpha\beta)^{n+1} + B^2\beta^{2n}\left(\alpha^2+\beta^2\right) \right].$$

And by using the Binet formula of the sequence $\{w_n\}$, we have

$$\begin{aligned} & W_n \overline{W_{n+1}} + W_{n+1}\overline{W_n} \\ = \; & 2w_{n+1}(w_n + w_{n+2}) + 2w_{n+3}(w_{n+2} + w_{n+4}) \\ = \; & \frac{2a}{(ab)^{n+3}} \Big[ (ab)^2 \left(A\alpha^n - B\beta^n\right)\left(A\alpha^{n-1}\left((ab)+\alpha^2\right) - B\beta^{n-1}\left((ab)+\beta^2\right)\right) \\ & + \left(A\alpha^{n+2} - B\beta^{n+2}\right)\left(A\alpha^{n+1}\left((ab)+\alpha^2\right) - B\beta^{n+1}\left((ab)+\beta^2\right)\right)\Big] \\ = \; & \frac{2a}{(ab)^{n+3}} \Big[ (ab)^3 \left(A\alpha^n - B\beta^n\right)\left(A\alpha^{n-1}(\alpha+2) - B\beta^{n-1}(\beta+2)\right) \\ & + (ab)\left(A\alpha^{n+2} - B\beta^{n+2}\right)\left(A\alpha^{n+1}(\alpha+2) - B\beta^{n+1}(\beta+2)\right)\Big] \\ = \; & \frac{2a}{(ab)^n}\left(A^2\alpha^{2n} - 2AB(\alpha\beta)^n + B^2\beta^{2n}\right) + \frac{4a}{(ab)^n}\left(A^2\alpha^{2n-1} + AB(\alpha\beta)^n + B^2\beta^{2n-1}\right) \\ & + \frac{2a}{(ab)^{n+2}}\left(A^2\alpha^{2n+4} - 2AB(\alpha\beta)^{n+2} + B^2\beta^{2n+4}\right) \\ & + \frac{4a}{(ab)^{n+2}}\left(A^2\alpha^{2n+3} + AB(\alpha\beta)^{n+2} + B^2\beta^{2n+3}\right) \\ = \; & 2a\Bigg[\frac{1}{(ab)^n}\left(A^2\alpha^{2n-1}(\alpha+2) + B^2\beta^{2n-1}(\beta+2)\right) \\ & + \frac{1}{(ab)^{n+2}}\left(A^2\alpha^{2n+3}(\alpha+2) + B^2\beta^{2n+3}(\beta+2)\right)\Bigg]. \end{aligned}$$

If we substitute the value

$$S(n) := \frac{1}{(ab)^n}\left(A^2\alpha^{2n-1}(\alpha+2) + B^2\beta^{2n-1}(\beta+2)\right),$$

into the above equation, we get

$$W_n\overline{W_{n+1}} + W_{n+1}\overline{W_n} = 2a\left(S\left(n\right) + S\left(n+2\right)\right).$$

Thus we obtain the desired result. □

Note that, if we take the initial conditions $w_0 = 0, w_1 = 1$ and $a = b = 1$, we get the norm of the dual Fibonacci quaternion

$$Nr\left(\widetilde{F}_n\right) = 3\left(\widetilde{f}_{2n+3} + \varepsilon f_{2n+4}\right),$$

which is given in [10, Theorem 2]. By using this formula, one can get an advantage to calculate the norm of any generalized dual Fibonacci quaternions.

Finally, we give some summation formulas for the generalized dual bi-periodic Fibonacci quaternions.

**Theorem 2.8.** *(Summation formulas) For $n \geqslant 1$, we have*

$$(i)\sum_{r=0}^{n-1}\widetilde{W}_{2r} = \frac{\widetilde{W}_{2n} - \widetilde{W}_{2n-2}}{ab} - \frac{A\alpha^*\beta^2 - B\beta^*\alpha^2}{(ab)^2} + \varepsilon\frac{A\alpha^{**}\beta - B\beta^{**}\alpha}{ab},$$

$$(ii)\sum_{r=0}^{n-1}\widetilde{W}_{2r+1} = \frac{\widetilde{W}_{2n+1} - \widetilde{W}_{2n-1}}{ab} + \frac{A\alpha^{**}\beta - B\beta^{**}\alpha}{ab} - \varepsilon\frac{A\alpha^* - B\beta^*}{ab}.$$

**Proof.** $(i)$ By using the equation (12), we have

$$\sum_{r=0}^{n-1}\widetilde{W}_{2r} = \sum_{r=0}^{n-1}\frac{1}{(ab)^{\left\lfloor\frac{2r+1}{2}\right\rfloor}}\left[A\alpha^{2r-1}\left(\alpha^* + \varepsilon\alpha^{**}\alpha\right) - B\beta^{2r-1}\left(\beta^* + \varepsilon\beta^{**}\beta\right)\right]$$

$$= A\left(\alpha^* + \varepsilon\alpha^{**}\alpha\right)\sum_{r=0}^{n-1}\frac{\alpha^{2r-1}}{(ab)^r} - B\left(\beta^* + \varepsilon\beta^{**}\beta\right)\sum_{r=0}^{n-1}\frac{\beta^{2r-1}}{(ab)^r}$$

$$= A\left(\alpha^* + \varepsilon\alpha^{**}\alpha\right)\alpha^{-1}\sum_{r=0}^{n-1}\left(\frac{\alpha^2}{ab}\right)^r - B\left(\beta^* + \varepsilon\beta^{**}\beta\right)\beta^{-1}\sum_{r=0}^{n-1}\left(\frac{\beta^2}{ab}\right)^r$$

$$= A\left(\alpha^{*}+\varepsilon\alpha^{**}\alpha\right)\alpha^{-1}\frac{\left(\frac{\alpha^{2}}{ab}\right)^{n}-1}{\frac{\alpha^{2}}{ab}-1}-B\left(\beta^{*}+\varepsilon\beta^{**}\beta\right)\beta^{-1}\frac{\left(\frac{\beta^{2}}{ab}\right)^{n}-1}{\frac{\beta^{2}}{ab}-1}$$

$$= A\left(\alpha^{*}+\varepsilon\alpha^{**}\alpha\right)\left(\frac{\alpha^{2n-2}}{(ab)^{n}}-\frac{1}{\alpha^{2}}\right)-B\left(\beta^{*}+\varepsilon\beta^{**}\beta\right)\left(\frac{\beta^{2n-2}}{(ab)^{n}}-\frac{1}{\beta^{2}}\right)$$

$$= \frac{A\left(\alpha^{*}+\varepsilon\alpha^{**}\alpha\right)\alpha^{2n-2}-B\left(\beta^{*}+\varepsilon\beta^{**}\beta\right)\beta^{2n-2}}{(ab)^{n}} - \frac{A\left(\alpha^{*}+\varepsilon\alpha^{**}\alpha\right)\beta^{2}-B\left(\beta^{*}+\varepsilon\beta^{**}\beta\right)\alpha^{2}}{(ab)^{2}}$$

$$= \frac{A\alpha^{*}\alpha^{2n-2}-B\beta^{*}\beta^{2n-2}}{(ab)^{n}}-\frac{A\alpha^{*}\beta^{2}-B\beta^{*}\alpha^{2}}{(ab)^{2}} +\varepsilon\left(\frac{A\alpha^{**}\alpha^{2n-1}-B\beta^{**}\beta^{2n-1}}{(ab)^{n}}-\frac{A\alpha^{**}\alpha\beta^{2}-B\beta^{**}\beta\alpha^{2}}{(ab)^{2}}\right).$$

By making some necessary calculations, we obtain

$$\begin{aligned}\sum_{r=0}^{n-1}\widetilde{W}_{2r} &= \frac{W_{2n}-W_{2n-2}}{ab}-\frac{A\alpha^{*}\beta^{2}-B\beta^{*}\alpha^{2}}{(ab)^{2}} +\varepsilon\left(\frac{W_{2n+1}-W_{2n-1}}{ab}+\frac{A\alpha^{**}\beta-B\beta^{**}\alpha}{ab}\right)\\ &= \frac{\left(W_{2n}+\varepsilon W_{2n+1}\right)-\left(W_{2n-2}+\varepsilon W_{2n-1}\right)}{ab} -\frac{A\alpha^{*}\beta^{2}-B\beta^{*}\alpha^{2}}{(ab)^{2}}+\varepsilon\frac{A\alpha^{**}\beta-B\beta^{**}\alpha}{ab}\\ &= \frac{\widetilde{W}_{2n}-\widetilde{W}_{2n-2}}{ab}-\frac{A\alpha^{*}\beta^{2}-B\beta^{*}\alpha^{2}}{(ab)^{2}}+\varepsilon\frac{A\alpha^{**}\beta-B\beta^{**}\alpha}{ab}.\end{aligned}$$

It can be proven for $(ii)$ by using the similar procedure. □

Note that, if we take the initial conditions $w_0 = 0$ and $w_1 = 1$ in Theorem 2.8, we get the summation formulas for the dual bi-periodic Fibonacci quaternions, and if we take the initial conditions $w_0 = 2$ and $w_1 = b$, by switching $a$ and $b$, in Theorem 2.8, we get the summation formulas for the dual bi-periodic Lucas quaternions.

## 3. Conclusion

In this paper, we defined the generalized dual bi-periodic Fibonacci quaternions which is a further generalization of the dual Fibonacci quaternions in [1]. By means of this new generalization, we have a unified approach to handle with many special dual quaternions in the literature. Also, the dual quaternions of unit length play an important role to represent rigid body motions in three dimensional space. The results of this paper would be interesting as an application in physics and robotic motion besides the theory of dual quaternions in mathematics.

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**Elif Tan**
Ph.D. of Mathematics
Department of Mathematics
Ankara University
Ankara, Turkey
E-mail: etan@ankara.edu.tr

**Ismail Gök**
Associate Professor of Mathematics
Department of Mathematics
Ankara University
Ankara, Turkey
E-mail: igok@science.ankara.edu.tr