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The Arithmetical Rank of k-Complete Ideals

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Abstract. We define the notions of algebraic and arithmetic derivation. As an application, we use the combinatorial decomposition of an ideal to provide a constructive method to find the algebraic invariants, as the arithmetical rank, for a family of squarefree monomial ideals, the k-complete ideals I_k^n, also known as squarefree Veronese ideals of degree k.

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1. Introduction

Let A be Noetherian commutative ring with identity. We say that some elements $r_1, \ldots, r_m$ in A generate an ideal I of A up to radical if

$$\sqrt{I} = \sqrt{(r_1, \ldots, r_m)}.$$

The smallest m with this property is called the *arithmetical rank* of I, denoted by $ara(I)$. Excellent reference for the arithmetical rank is [1]. Let $projdim_R(R/I)$ the *projective dimension* of R/I, i.e., the length of a minimal free resolution of R/I. Let $H_I^i(R)$ denote the $i-th$ local cohomology module of R with respect to I. The *cohomological dimension* of

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I is defined to be the natural number: $cd(I) = max\{i|H_I^i(R) \neq 0\}$. We shall throughout suppose that R is the polynomial ring $K[x_1, \ldots, x_n]$. From the expression of the local cohomology modules in terms of $\check{C}$ech complex, one can see that ([5, Theorem 5.4] Huneke-desigualdad) for all ideals I in a commutative Noetherian ring $cd(I) \leqslant ara(I)$. We recall that for I monomial ideal, $ara(I) = ara(\sqrt{I})$ with $\sqrt{I}$ a squarefree monomial ideal (See [1]). By Lyubeznik [9, Theorem 1]Lyubeznik-local-coho, for all squarefree monomial ideal I one has that $projdim(R/I) = cd(I)$. Therefore

$$ht(I) \leqslant projdim(R/I) = cd(I) \leqslant ara(I) \leqslant \mu(I). \tag{1}$$

Let us explain the organization of this paper. In Section 2, we introduce the notion of algebraic derivation and establish some results when this derivation is zero, one and two. In Section 3, we define the arithmetic derivation and we present some results when the ideal has arithmetic derivation zero and one. In Section 4 we recall the main results about Lyubeznik resolutions. Finally, in Section 5 we provide a constructive method to find the algebraic invariants of the family of k–complete ideals and we establish some results about the arithmetical rank, projective dimension and other invariants for this family.

2. Algebraic Derivation

We introduce the notion of algebraic derivation for an ideal I and we see that when this is equal to 0, 1 or 2 the equality $ara(I) = projdim(R/I)$ holds. We compute the algebraic derivation for the only example in the literature, the case when $projdim(R/I) \neq ara(I)$, and we obtain that its algebraic derivation is 7.

Definition 2.1. *For an ideal I in R, we define the* algebraic derivation *of a monomial ideal I by*

$$d_{alg}(I) = \mu(I) - ht(I).$$

Proposition 2.2. *Let I be a squarefree monomial ideal. If $d_{alg}(I) = 0$, then we have $ara(I) = projdim(R/I)$.*

Proof. $ht(I) = projdim(R/I) = cd(I) = ara(I) = \mu(I)$. $\square$

Theorem 2.3. ([7]) *Let I be a squarefree monomial ideal.*

(i) *If $\mu(I) - projdim(I) \leqslant 1$, then we have $ara(I) = projdim(R/I)$.*

(ii) *If $d_{alg}(I) = 2$, then we have $ara(I) = projdim(R/I)$.*

Corollary 2.4. Let I be a squarefree monomial ideal. If $d_{alg}(I) = 1$, then we have $ara(I) = projdim(R/I)$.

Remark 2.5. *In general, $projdim(R/I) \neq ara(I)$. The only example in the literature is the following monomial ideal. Let I_0 be a monomial ideal of $R = K[x_1x_2x_3x_4x_5x_6]$, defined by*

$$\langle x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_5,\ x_1x_4x_6,\ x_1x_5x_6,\\ x_2x_3x_6,\ x_2x_4x_5,\ x_2x_5x_6,\ x_3x_4x_5,\ x_3x_4x_6\rangle,$$

the Stanley-Reisner ideal of Reisner's triangulation of $P^2(\mathbb{R})$, with 6 vertices. Then $\mu(I_0) = 10$ and $ht(I_0) = 3$. If the characteristic of K is not 2, then $projdim(R/I_0) = ht(I_0)$ (i.e. I_0 is Cohen-Macaulay). But Z. Yan [14] showed ara $I_0 = 4$ using the étale cohomology. Therefore $projdim(R/I_0) < ara(I_0)$. Furthermore, $d_{alg}(I_0) = \mu(I_0) - ht(I_0) = 10 - 3 = 7$.

3. Arithmetic Derivation

We introduce the notion of arithmetic derivation for a monomial ideal I. We see that $ara(I) = projdim(R/I)$ holds when its arithmetic derivation is equal to 0 or 1.

Let $I = \langle \mu_1, \mu_2, \ldots \mu_f, \rangle$ be a monomial ideal in $R = K[x_1, \ldots, x_n]$ and

$$0 \to \bigoplus_j R(-j)^{\beta_{pj}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1j}} \to \bigoplus_j R(-j)^{\beta_{0j}} \to I \to 0,$$

a graded minimal free resolution of I over R. Here, p is called the *projective dimension* of I over R and denote it by $projdim(I)$. Put $\beta_i = \sum_j \beta_{i\,j}$. We have $projdim(R/I) = projdim(I) + 1$, $\beta_{i\,j}(I) = \beta_{(i+1)\,j}(R/I)$, $\beta_i(I) = \beta_{(i+1)}(R/I)$, $\mu(I) = \beta_0(I)$ and $\beta_{0\,j}(I) = |\{\mu_i : deg(\mu_i) = j\}|$. Recall the following relevant definitions :
The *initial degree* of I, $indeg(I) = min\{j : \beta_{0\,j}(I) \neq 0\}$.
The *relation type* of I, $rt(I) = max\{j : \beta_{0\,j}(I) \neq 0\}$.
The *(Castelnuovo–Mumford) regularity* of I, $reg(I) = max\{j - i : \beta_{i\,j}(I) \neq 0\}$.

We say that I has *linear resolution* if $reg(I) = indeg(I)$. The *arithmetic degree* of a squarefree monomial ideal I is $arithdeg(I) = |Ass(R/I)|$.
For squarefree monomial ideals, we have the following relations:

Theorem 3.1. ([4, 2]) *Let I be a squarefree monomial ideal. Then we have*

$$indeg(I) \leqslant reg(I) \leqslant arithdeg(I). \tag{2}$$

Definition 3.2. *We define the* arithmetic derivation $d_{arith}(I)$ *of a monomial ideal I by*

$$d_{arith}(I) = arithdeg(I) - indeg(I).$$

Theorem 3.3. *Let I be a squarefree monomial ideal.*

(i) *([10, 11]) If $d_{arith}(I) = 0$, then $ara(I) = projdim(R/I)$.*

(ii) *([7]) If $d_{arith}(I) = 1$, then $ara(I) = projdim(R/I)$.*

Note that for I_0, the ideal in Remark 2.5, we have

$$\begin{aligned} I_0 \;=\; & \langle x_1,x_2,x_3\rangle \cap \langle x_1,x_2,x_4\rangle \cap \langle x_1,x_2,x_5,x_6\rangle \cap \langle x_1,x_3,x_5\rangle \\ & \cap\langle x_1,x_4,x_6\rangle \cap \langle x_1,x_5,x_6\rangle \cap \langle x_2,x_3,x_4,x_5\rangle \cap \langle x_2,x_3,x_6\rangle \\ & \cap\langle x_2,x_4,x_5\rangle \cap \langle x_2,x_5,x_6\rangle \cap \langle x_3,x_4,x_5\rangle \cap \langle x_3,x_4,x_6\rangle \\ & \cap\langle x_1,x_3,x_4,x_6\rangle, \end{aligned}$$

where $arithdeg(I_0) = 13$, $indeg(I_0) = 3$ and $d_{arith}(I_0) = 10$.

4. Lyubeznik Resolution

In 1988, Lyubeznik [8] constructed a graded free resolution of R/I as a subcomplex of the Taylor resolution of R/I.

Definition 4.1. *Let $m_1, m_2, \ldots, m_\mu$ be an ordered sequence of μ monomials of R, let I be the ideal generated by these monomials. For all sequences $(i_1; i_2; \ldots; i_t)$, where $1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant \mu$, the symbol $u(i_1; i_2; \ldots; i_t)$ will be called L-admissible of dimension t if:*

$$m_q \text{ does not divide } lcm(m_{i_h}, m_{i_{h+1}}, \ldots, m_{i_t})$$

for all $h < t$ and $q < i_h$.

Set $L^0 = R$ and for all $t = 1, 2, \ldots, \mu$ let L^t be the free R-module generated by all L-admissible symbols of dimension t. Define the map $\partial_t : L^t \to L^{t-1}$ by setting

$$\partial_t(u(i_1; i_2; \ldots; i_t)) = \sum_{j=1}^{t} (-1)^{j+1} \frac{lcm(m_{i_1}, m_{i_2}, \ldots, m_{i_t})}{lcm(m_{i_1}, m_{i_2}, \ldots \widehat{m_{i_j}}, \ldots, m_{i_t})} u(i_1; i_2; \ldots; \widehat{i_j}; \ldots; i_t).$$

The *Lyubeznik resolution* of I is a subcomplex of the Taylor resolution of R/I generated by all L-admissible symbols.
For two L-admissible symbols $u(i_1; i_2; \ldots; i_s)$ and $u(j_1; j_2; \ldots; j_t)$, we say that

$$u(i_1; i_2; \ldots; i_s) \preceq u(j_1; j_2; \ldots; j_t)$$

if $i_1, i_2, \ldots, i_s$ is a subsequence of $j_1, j_2, \ldots, i_t$. Evidently, if $u(j_1; j_2; \ldots; j_t)$ is L-admissible, so are all smaller symbols. Hence every Lyubeznik resolution is uniquely determined by its maximal L-admissible symbols.

Definition 4.2. *A symbol $u(i_1; i_2; \ldots; i_t)$ is* stable *of I, if for all $1 \leqslant q \leqslant t$*

$$lcm(m_{i_1}, m_{i_2}, \ldots, m_{i_t}) \neq lcm(m_{i_1}, m_{i_2}, \ldots \widehat{m_{i_q}}, \ldots, m_{i_t}).$$

Note that if $u(i_1; i_2; \ldots; i_t)$ is stable, then also are all smaller L-admissible symbols.

Let $\mathtt{m}$ be the homogeneous maximal ideal of R, i.e., $\mathtt{m} = \langle x_1, x_2, \ldots, x_n \rangle$. $L_\bullet$ is minimal if and only if $\partial_t(L^t) \subseteq \mathtt{m}L^{t-1}$ for all t. By the construction of ∂_t, $L_\bullet$ is minimal if and only if for all maximal L-admissible symbols $u(i_1; i_2; \ldots; i_t)$, u is stable. We have the following proposition:

Proposition 4.3. *The Lyubeznik resolution of I with respect to some order of monomial generators is the minimal free resolution if and only if all maximal L-admissible symbols $u(i_1; i_2; \ldots; i_t)$ are stable. In particular,*

$$\beta_{t\,j}(R/I) = \beta_{t-1\,j}(I) \text{ and } \beta_{t-1\,j}(I) =$$

$$|\{u(i_1; i_2; \ldots; i_t) \,:\, u \text{ is } L\text{-admissible}; j = deg(lcm(m_{i_1}, m_{i_2}, \ldots, m_{i_t}))\}|.$$

Definition 4.4. *For a monomial ideal I, let $G(I)$ be its minimal set of monomial generators. If there is a total order on $G(I)$ such that the corresponding Lyubeznik resolution of I is a minimal free resolution of I, then I is called a* Lyubeznik ideal.

We define $L_\prec(I)$ as the length of Lyubeznik resolutions of I with respect to the order $\prec$. The most important theorem that relates to the Lyubeznik resolution and the arithmetical rank is as follows:

Theorem 4.5. [6] *Let I be a squarefree monomial ideal of R and $\prec$ be a monomial order. Then $ara(I) \leqslant L_\prec(I)$. In particular, if I is a Lyubeznik ideal, then $ara(I) = projdim(R/I)$.*

5. K-Complete Ideals

Finally, in this section we give a method to find the algebraic invariants of a family of monomial ideals, the k-complete ideals, which are indeed Lyubeznik ideals. As application we show that for each positive integer m, there exists an ideal I in this family with algebraic derivation m, and there is an ideal with arithmetic derivation m.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let $v_1, \ldots, v_q$ be the column vectors of a matrix $A = (a_{ij})$ whose entries are non-negative integers. For technical reasons, we shall always assume that the rows and columns of the matrix A are different from zero. As usual we use the notation $x^a := x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$. Consider the *monomial ideal*:

$$I = (x^{v_1}, \ldots, x^{v_q}) \subset R,$$

generated by $F = \{x^{v_1}, \ldots, x^{v_q}\}$.

A *clutter* $\mathcal{C}$, with finite vertex set $V = \{x_1, ..., x_n\}$ is a family of subsets of V, called edges, none of which is included in another. The set of vertices and edges of $\mathcal{C}$ are denoted by $V(\mathcal{C})$ and $E(\mathcal{C})$ respectively. For example, a simple graph (no multiple edges or loops) is a clutter. The *edge ideal* of $\mathcal{C}$, denoted by $I(\mathcal{C})$, is the ideal of R generated by all monomials $x_e = \prod_{x_i \in e} x_i$ such that $e \in E(\mathcal{C})$. The map

$$\mathcal{C} \longmapsto I(\mathcal{C}),$$

gives an one-to-one correspondence between the family of clutters and the family of squarefree monomial ideals. Edge ideals of graphs were introduced and studied in [13].

Let A be the *incidence matrix* of $\mathcal{C}$ whose column vectors are $v_1, \ldots, v_q$. The *set covering polyhedron* of $\mathcal{C}$ is given by:

$$Q(A) = \{x \in \mathbb{R}^n |\, x \geqslant 0;\, xA \geqslant \mathbf{1}\}.$$

A subset $C \subset V(\mathcal{C})$ is called a *minimal vertex cover* of $\mathcal{C}$ if: (i) every edge of $\mathcal{C}$ contains at least one vertex of C, and (ii) there is no proper subset of C with the first property. The map $C \mapsto \sum_{x_i \in C} e_i$ gives a bijection between the minimal vertex covers of $\mathcal{C}$ and the integral vectors of $Q(A)$, (see [12]). A polyhedron is called an *integral polyhedron* if it has only integral vertices. A clutter is called *d–uniform* or *uniform* if all its edges have exactly d vertices.

We recall the following result in algebraic combinatorics that relates the minimal vertex covers of a clutter with the primary decomposition of a monomial ideal.

Proposition 5.1. *Let $C_1, C_2, \ldots, C_s$ be the minimal vertex covers of a clutter $\mathcal{C}$. Then the primary decomposition of $I(\mathcal{C})$ is $\langle C_1\rangle \cap \langle C_2\rangle \cap \cdots \cap \langle C_s\rangle$, where $\langle C_i\rangle = \langle x_j | x_j \in C_i\rangle$.*

Now, we recall a family of squarefree monomial ideals as defined in [3] (also known as Veronese ideals), the *k-complete ideal* I_k^n with $k \leqslant n$,

$$I_k^n = \langle x_{i_1}x_{i_2}\cdots x_{i_k} | 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\rangle,$$

in $R = K[x_1, x_2, \ldots, x_n]$.

The corresponding induced clutter is $\mathcal{C}(I_k^n) = \{\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} | 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$. We obtain that the set of minimal vertex cover consisting of all the subset of $\{x_1, x_2, \ldots, x_n\}$ with cardinality $n-1+k$. It follows readily that the primary decomposition of I_k^n is

$$I_k^n = \bigcap_{1\leqslant i_1<i_2<\cdots<i_{n+1-k}\leqslant n} \langle x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1-k}}\rangle.$$

Hence $\mu(I_k^n) = \binom{n}{k}$, $rt(I_k^n) = k$, $indeg(I_k^n) = k$, $ht(I_k^n) = n+1-k$, $arithdeg(I_k^n) = \binom{n}{n+1-k}$.

We have immediately the following results

Proposition 5.2. *If m is a natural number, then there are two natural numbers $k \leqslant n$ such that*

$$d_{alg}(I_k^n) = m.$$

Proof. $d_{alg}(I_{m+1}^{m+2}) = \mu(I_{m+1}^{m+2}) - ht(I_{m+1}^{m+2}) = \binom{m+2}{m+1} - ((m+2)+1-(m+1)) = (m+2)-2 = m$. $\square$

Proposition 5.3. *If m is a natural number, then there are two natural numbers $k \leqslant n$ such that*

$$d_{arith}(I_k^n) = m.$$

Proof. $d_{arith}(I_2^{m+2}) = arithdeg(I_2^{m+2}) - indeg(I_2^{m+2}) = \binom{m+2}{m+2+1-2} - 2 = \binom{m+2}{m+1} - 2 = (m+2)-2 = m$. $\square$

Consider the lexicographical order on the variables $x_1 < x_2 < \cdots < x_n$ and the total order induced on the monomials $G(I_k^n) = \{x_{i_1}x_{i_2}\cdots x_{i_k} | 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$. The symbol $u(i_1; i_2; \ldots; i_t)$ will be L-admissible of dimension t if

($*$) For all j, the monomial i_j has a variable x which is not in the monomial i_{j-1}.

($\star$) There is at least one variable x in the monomial i_1 that is not found in all the monomials previous.

Therefore, all L-admissible symbol of dimension $n+1-k$ is necessarily maximal. In addition, any L-admissible symbol can be refined in a L-admissible symbol of dimension $n+1-k$. With this we can conclude the following

Lemma 5.4. *The symbol $u(i_1; i_2; \ldots; i_t)$ is a maximal L-admissible symbol if and only if*
$u(i_1; i_2; \ldots; i_t)$ satisfies the conditions ($*$), ($\star$) and $t = n+1-k$.

Proposition 5.5. *If I_k^n is a k-complete ideal, then*

$$ara(I_k^n) = projdim(R/I_k^n) = n+1-k.$$

Proof. Let be $u(i_1; i_2; \ldots; i_t)$ a maximal L-admissible symbol. By ($*$), each monomial m_{i_j} has a variable that is not in the other monomials, for all j. Hence

$$lcm(m_{i_1}, m_{i_2}, \ldots, m_{i_t}) \neq lcm(m_{i_1}, m_{i_2}, \ldots \widehat{m_{i_q}}, \ldots, m_{i_t}); \text{ for all } 1 \leqslant q \leqslant t.$$

Therefore, $u(i_1; i_2; \ldots; i_t)$ is stable. The Proposition 4.3 ensures that the corresponding Lyubeznik resolution of I_k^n is a minimal free resolution of I, i.e., I_k^n is a Lyubeznik ideal. By Theorem 4.5, $ara(I_k^n) = projdim(R/I_k^n)$. Furthermore,

$$projdim(R/I_k^n) = projdim(I_k^n) + 1 = \\ max\{i \mid \beta_{ij}(I_k^n) \neq 0 \text{ for some } j\} = n+1-k.$$

We conclude that

$$ara(I_k^n) = projdim(R/I_k^n) = n+1-k. \quad \square$$

We can summarize the above results in the following theorem.

Theorem 5.6. *For each natural number m, there are Lyubeznik ideals I and J such that $d_{alg}(I) = m$ and $d_{arith}(J) = m$.*

In this family of monomial ideals, the Equation (1) becomes the following equation

$$ht(I_k^n) = projdim(R/I_k^n) = cd(I_k^n) = \\ ara(I_k^n) = n + 1 - k \leqslant \mu(I_k^n) = \binom{n}{k}. \quad (3)$$

Now we will calculate the regularity of the ideal I_k^n.

Lemma 5.7. *If I_k^n is a k-complete ideal with $n = kq$, then $reg(I_k^n) = n + 1 - q$.*

Proof. The regularity of I_k^n is given by

$$reg(I_k^n) = max\{j - t : \beta_{t\,j}(I_k^n) \neq 0\},$$

equivalently

$$reg(I_k^n) = max\{j - t : u(i_1; i_2; \ldots ; i_{t+1}),$$

with

$$u\ L\text{-admissible}\ ; j = lcm(m_{i_1}, m_{i_2}, \ldots , m_{i_{t+1}}))\}.$$

We conclude that

$$reg(I_k^n) = n - (q - 1),$$

because the maximum is reached in the element

$$u(x_1 \cdots x_k; x_{k+1} \cdots x_{2k}; \ldots ; x_{(q-1)k+1} \cdots x_{qk}),$$

with $deg(lcm(u)) = n$. $\square$

Lemma 5.8. *If I_k^n is a k-complete ideal with $n = kq + r$ and $0 < r < q$, then $reg(I_k^n) = n - q$.*

Proof. Here the maximum is reached in the element

$$u(x_1 \cdots x_k; x_{k+1} \cdots x_{2k}; \ldots ; x_{(q-1)k+1} \cdots x_{qk}; x_{qk+1}x_{qk+2} \cdots x_n x_1 \cdots x_{k-r}),$$

with $deg(lcm(u)) = n$. Therefore, the regularity of I_k^n is given by $reg(I_k^n) = n - q$. $\square$

Proposition 5.9. *If I_k^n is a k-complete ideal, then $reg(I_k^n) = n+1-\lceil \frac{n}{k} \rceil$, where $\lceil \frac{n}{k} \rceil$ is the smallest integer not less than $\frac{n}{k}$.*

Proof. It follows directly from the Lemmas 5.7 and 5.8. $\square$

The following result can also be deduced from de fact that I_k^n is polymatroidal [3], which implies that this ideal has linear quotients.

Proposition 5.10. *The ideal I_k^n has linear resolution if and only if $k = n$ or $k = n - 1$.*

Proof. The ideal I_k^n has linear resolution if and only if $reg(I_k^n) = n + 1 - \lceil \frac{n}{k} \rceil = k = indeg(I_k^n)$. In the case $n = kq$, it has to be $reg(I_k^n) = n + 1 - q = k = indeg(I_k^n)$ if and only if $k = n$. In the other case, $n = kq + r$ with $0 < r < q$, it has to be $reg(I_k^n) = n - q = k = indeg(I_k^n)$ if and only if $k = n - 1$. $\square$

From the Equation (2) we obtain the following

$$indeg(I_k^n) = k \leqslant reg(I_k^n) = \\ n + 1 - \lceil \ \tfrac{n}{k} \ \rceil \leqslant arithdeg(I_k^n) = \binom{n}{n+1-k}. \quad (4)$$

To finish, we describe explicitly which are the elements $g_1, g_2, \ldots, g_{n+1-k}$ that generate the ideal $\sqrt{\langle g_1, g_2, \ldots, g_{n+1-k}\rangle} = \sqrt{I_k^n}$ in where the arithmetical rank is reached. Following an argument of Kimura in [6. p. 3629], we obtain that if we define elements

$g_1 = x_1 x_2 \cdots x_{k-1} x_k$,

$g_2 = x_1 x_2 \cdots x_{k-1} x_{k+1} + \sum m_{i_1} m_{i_2} \cdots m_{i_{n-k}}$

where the sum is over all the $u(i_1; i_2; \ldots ; i_{n-k})$ L-admissible with $i_1 \geqslant 3$,

$\vdots$

$g_l = x_1 x_2 \cdots x_{k-1} x_{k+(l-1)} + \sum m_{i_1} m_{i_2} \cdots m_{i_{n+1-k-(l-1)}}$

where the sum is over all the $u(i_1; i_2; \ldots ; i_{n+1-k-(l-1)})$ L-admissible with $i_1 \geqslant (l + 1)$,

$\vdots$

$g_{(n+1-k)-1} = x_1 x_2 \cdots x_{k-1} x_{n-1} + \sum m_{i_1} m_{i_2}$

where the sum is over all the $u(i_1;i_2)$ L-admissible with $i_1 \geqslant (n+1-k)$,
$g_{n+1-k} = x_1x_2\cdots x_{k-1}x_n + \sum m_{i_1}$
where the sum is over all the $i_1 \geqslant (n+1-k)+1$,
then

$$\sqrt{\langle g_1, g_2, \ldots, g_{n+1-k}\rangle} = \sqrt{I_k^n}.$$

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