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New Cross Entropy on Dempster-Shafer Theory Under Uncertainty

F. Khalaj
Robat Karim Branch, Islamic Azad University

M. Khalaj*
Robat Karim Branch, Islamic Azad University

R. Tavakkoli-Moghaddam
University of Tehran

E. Pasha
Kharazmi University

Abstract. This paper introduces a new generalized definition of cross entropy for finding a solution of a multi-criteria decision-making (MCDM) problem. New interpretation of a belief function in Dempster Shafer theory, known as evidence or belief function theory, under uncertainty is presented, which is affected in the decision-making process. Belief functions are determined as three elements (i.e., belief degree, uncertainty degree and disbelief degree). Then, a belief measure between an alternative and ideal alternative is calculated by cross entropy to rank alternatives and select the most desirable one. Finally, the efficiency of the proposed MCDM method is demonstrated by solving a numerical example.

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*Corresponding author

1. Introduction

Multi-criteria decision-making (MCDM) is the process to rank alternatives or find the best alternative among a set of feasible alternatives. Usually in real situations, decision-making process deals with the uncertainty and incomplete, imprecise or conflict information.

In different MCDM methods, one of the main steps is to evaluate the relative likelihood among alternatives, then a comparison of these alternatives is used to rank them. However, selection of the optimal alternative with respect to several criteria under uncertainty is usually based on insufficient information and judgment. Therefore, lot of studies have been devoted to MCDM problems with respect to uncertainty, which may be arisen with incorrect, incomplete or imprecise data and information. Uncertain information involved in decision making can be identified as epistemic and aleatory uncertainties [24]. In this revised manuscript, the proposed methodology is based on the epistemic uncertainty.

The classification of epistemic uncertainty in four general categories; 1) randomness, 2) incompleteness, 3) imprecision and 4) conflict evidence [24], different frameworks are introduced to rank alternatives and find a best solution in the decision-making process. The fuzzy set theory and Dempster-Shafer (D-S) or evidence theory are two main classes of uncertainty theories [3, 24]. The various properties and the associated relations of these basic forms are presented in some references, such as [10, 18, 25, 30, 31].

Thus, the MCDM methods are challenging tasks, among different theory (or methods) with respect to uncertainty handling. The incompleteness, imprecision and conflict evidence and data, are referred to a lack of knowledge or epistemic uncertainty [24]. The proper framework to handle imprecise information is the fuzzy logic. Additionally, a D-S framework is used, for incomplete information [22, 24]. Many existing studies are proposed with imprecise data in a fuzzy environment, (e.g., [2, 6, 7, 11, 15, 17]).

Because of the limitations of the traditional fuzzy set theory dealing with vagueness and uncertainty, several extensions and generalization (e.g., intuitionistic fuzzy sets, vague sets, interval-valued intuitionistic fuzzy sets, hesitant fuzzy set, neutrosophic sets, and single-valued neutrosophic sets [12, 13, 23, 28, 29] are proposed to fully described information in MCDM problems. The mathematical frames of these methods are effective tools to handle uncertainty; however, it not been demonstrated based on evidential reasoning. This reasoning is considered available evidence, which is accurate and exact; however it is incomplete and insufficient.

Thus, in order to improve or develop uncertainty presentation, some of the studies proposed as a combination of the fuzzy set theory and D-S theory (e.g.,[16, 26, 32]). In essential, most of the above-mentioned studies on belief function theory are the generalization of another technique and theories to solve an MCDM problem, especially fuzzy set theory.

However, these methods are effective tools to handle indeterminate and inconsistent information, in which most of them are based on approximate reasoning, imprecise data and information. Additionally, the mathematical frameworks of these methods are based on the opinion about a certain statement, which are provided based on knowledge, experience experts and expressed preferences for objects and possibility of the statement.

A cross-entropy method can be used as a useful tool to determine ideal alternative in comparing with each other in MCDM problems. It is defined to calculate the divergence and discrimination measure between alternatives and the absolute ideal solutions under an uncertain environment. It is an applicable index in MCDM problems and provides the construction to determine the rank of alternatives with respect to each criteria under uncertainty.

Accordingly, different extension and application of the cross entropy method are proposed in MCDM problems and fuzzy fields, such as [14, 27].

A large number of studies have been proposed to develop and improve the decision-making process in order to achieve more accurate results, based on mathematical reasoning and uncertain condition (e.g., entropy and cross entropy method). Entropy is a tool to measure the amount of uncertainty in random events and used in many fields and applications of statistical science and engineering [5]. Additionally, based on the concept of entropy, relative entropy was first defined by Kullback and Leibler [8], which is known as a different name, such as the Kullback-Leibler distance, cross entropy, information diverges and information for discrimination [21]. Cross entropy is an important and popular method that has been extended to deal with MCDM problems. It is originally proposed by Rubinstein [1] as an important factor, in order to determine importance sampling for estimating rare event probabilities. Then, it was extended as a Kullback-Leibler distance to present a divergence measure between a pair of the probability distriion [1] which has been applied to many areas.

Although, a lot of work has been proposed about cross entropy and its application in MCDM problems under uncertainty; however, most of them are proposed in fuzzy fields. They are effective methods based on imprecise information; however, little attention has been paid regarding the type of uncertainty with incomplete information. Therefore, in the existing methods and the extension concepts of cross entropy in MCDM methods, there are the following shortcomings:

1) Generally, the existing methods are not able to clearly account the uncertainty for incomplete decision results, when there is incomplete information.

2) Lots of methods based on the D-S theory, which are proposed to handle uncertainty with incomplete information, are integrated with other theory (e.g., fuzzy set theory). These are not defined and proposed in the framework of belief function theory straightly. In the other hand, so far, there is no investigation on the use of the cross entropy in MCDM problems based on fundamental belief function in the D-S theory directly.

Most of the above-mentioned references are studied about applications of the cross entropy using the D-S theory as a generalization of another technique or theory, especially fuzzy set theory. They propose the cross entropy method and its application under uncertainty. However, they are not straightly defined in the framework of the D-S theory under uncertainty and incomplete data. Furthermore, so far, there is no investigation on the mathematical framework to calculate the cross entropy measure between different belief degrees and between bodies of evidence directly, which is evaluated based on belief functions.

However, the relation between the D-S theory using different theory or mathematical framework can be rarely affected to improve the application of a cross entropy in various uncertainty fields; these do not involve a cross entropy measure or divergence measure among expert's belief degrees about an objective or hypothesis as a crucial matter in belief functions for an incomplete decision. Therefore, the cross entropy measure based on belief functions may play an important role to analyze an incomplete information between the objectives. It can be considered as a divergence between belief degrees, which are included; belief degrees of truth, falsity or unknown about a proposition, by using three belief functions (i.e., belief, uncertainty and disbelief functions), which is defined in the theory of Dempster-Shafer.

Thus, in this work, a new cross entropy for belief functions is defined and called the belief cross entropy. It is directly proposed in the framework of the D-S theory based on the belief function and a new definition of the belief measure for an objective. Then it is used to establish an MCDM method based on the cross entropy of belief functions. This paper is organized as follows. Section 2 presents some concepts of the D-S theory and belief functions. Section 3 proposes a new cross entropy between two belief functions as a belief cross entropy measure. Section 4 is presented a new MCDM based on the proposed belief cross entropy. Section 5 illustrates a numerical example. The comparison results of an illustrative example in Section 5 and two cross entropy measures for PMVNNs are summarized in Section 6. Finally, conclusion and some final remarks are given in the last section.

2. Dempster-Shafer Theory of Belief Functions

The Dempster-Shafer (D-S) theory (or belief function theory) is a generalization of the Bayesian theory introduced by Dempster in 1968 [4], and then it was improved by Shafer in 1976 [18]. This deals with incomplete data. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of N mutually exclusive hypotheses. All possible sub-sets of this set are constructed the discernment frame (or power set) defined by Ω or $2^X(2^N)$. Thus, the power set is included with the 2^N proposition A of Ω, where $\Omega = \{\varnothing, \{x_1\}, \{x_2\}, \{x_N\}, \cdots, \{x_1 \cup x_2\}, \cdots, X\}$, and $\varnothing$ is an empty set. There are three basic functions in this theory on the discernment frame. Important concepts in the D-S theory are discussed in the following definitions, included as the basic belief mass function, belief function, plausibility function and combination rule.

Definition 2.1. *Let*
$H_1 = \{x_1\}, H_2 = \{x_2\}, \cdots, H_N = \{x_N\}, \cdots, H_{2^N} = \{x_1, x_2, \cdots, x_n\}$, *are subset of the power set (or frame of discernment), which is called focal elements. The basic probability assignments (or belief mass functions) are denoted by* $m(H_i), (i = 1, 2, \cdots, 2^N)$ *to the subset* H_i *on the frame of discernment. It represents the level of evidence in support of hypotheses* H_i *as defined by:*

$$m(H_i) : \Omega \to [0, 1]. \tag{1}$$

The conditions of this function are as follows:

$$\sum_{H_i \in \Omega} m(H_i) = 1, \tag{2}$$

and

$$m(\varnothing) = 0. \tag{3}$$

Example 2.2. Let us assume that the decision maker reviews investment documents to choose the best option for investment based on evidence. The decision maker determines the highest level of relative evidence to support the hypothesis H and say 0.6 in the interval $[0, 1]$, which the best option to invest really occurred. In the meantime, he/she reports that different evidence and document have been manually prepared, which may indicate that the good or best investment does not actually occur. He/she assigns a level of support 0.2 to this hypothesis (i.e., $\sim$ H).

Using the BPA function, the decision maker can define the all over evidence by:

$$m(H) = 0.6, \quad m(\sim H) = 0.2, \quad m(H, \sim H) = 0.2,$$

The statement $m(H) = 0.6$ represents that the belief degree is 0.6 on scale $0 - 1$, where 'H' is true, while $m(\sim H) = 0.2$ is expected belief, where H is false or $\sim H$ is true based on the evidence and $m(H, \sim H) = 0.2$ expresses that the belief does not assign to any specific, but determined to the overall form $(H, \sim H)$, which represents uncertainty and ignorance.

Definition 2.3. *Let $H_i, H_j \in \Omega$,where $m(H_i) \geqslant 0$ and $m(H_i) \geqslant 0$. The belief function is an essential concept with regard to the D-S theory. It represents the exact belief that a person has obtained based on the evidence. It is given by:*

$$bel(H_i) = \sum_{H_j \subseteq \ H_i} m(H_j), \quad for\, any \quad H_i \in \Omega. \tag{4}$$

In the D-S theory, $bel(H_i) = 0$ denotes the deficiency of evidence about A and $bel(H_i) = 1$ represents that the occurrence of A is certain, while in probability theory $P(A) = 0$ presents A is impossible event and $P(A) = 1$ represents that event A is certainty true.

Example 2.4. Continuing the previous example $bel(H) = m(H) = 0.6$ reflects the exact support to the hypothesis 0.6 or on the other hand, the belief degree for the occurrence of hypotheses A is 0.6. Also, $bel(\sim H) = m(\sim H) = 0.2$ represents that hypothesis H has not occur, which is 0.2. $bel(H, \sim H) = m(H) + m(\sim H) + m(H, \sim H) = 0.6 + 0.2 + 0.2 = 1$. $bel(H, \sim H)$ is a belief that supports the hypotheses, do not report that either H or $\sim H$ is true or false.

Definition 2.5. *Let $H_i, H_j \in \Omega$, where $m(H_i) \geqslant 0$ and $m(H_i) \geqslant 0$. Plausibility of A expresses the total support that event A may be occurred, i.e. the whole amount of the belief that is potentially presented in A is defined by:*

$$pl(H_i) = \sum_{H_i \cap H_j \neq \varnothing} m(H_j), \quad for\, each \quad H_i \in \Omega. \tag{5}$$

Plausibility is a complementary belief in the proposition "not A" that is:

$$pl(H_i) = 1 - bel(\sim H_i), \tag{6}$$

$pl(H_i)$ expresses the maximum level of a belief that may be the support of the set H_i. $pl(H_i) = 1$ mention that A is possible and we do not have any evidence that 'not H_i' is true. On the other hand, $bel(\sim H_i) = 0$. $pl(H_i) = 0$ is similar to $p(H_i) = 1$, in which implies H_i that is improbable. It expresses that if H_i is not impossible, 'not H_i' is true for sure or $bel(\sim H_i) = 1$.

Example 2.6. In Example 2.2, the plausibility of the hypotheses that investment is not true or hypotheses A have not occurred, which can be represented by:

$$\begin{aligned} pl(H) &= m(H) + m(H, \sim H) = 0.6 + 0.2 = 0.8 \\ &= 1 - bel(\sim H) = 1 - 0.2 = 0.8, \\ pl(\sim H) &= m(\sim H) + m(H, \sim H) = 0.6 + 0.2 = 0.8 \\ &= 1 - bel(\sim H) = 1 - 0.2 = 0.8. \end{aligned}$$

Definition 2.7. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set of N mutually exclusive hypotheses, a belief set A of reference set X is characterized through the belief function or belief degree $bel_A(H_i)$, uncertainty function or uncertainty degree $u_A(H_i)$ and disbelief function or disbelief degree $bel_A(\sim H_i)$. Then a belief set A can be denoted by:*

$$A = \{bel_A(H_i), u_A(Hi), bel_A(\sim H_i) | H_i \in \Omega\}, \tag{7}$$

where $bel_A(H_i), u_A(Hi), bel_A(\sim H_i) \in [0, 1]$ for each $H_i \in \Omega$ and
$0 \leqslant bel_A(H_i) + u_A(Hi) + bel_A(\sim H_i) \leqslant 1$.

According to Definition 2.7 and Eq. 7, the belief set of H_i can also be described as shown in Fig. 1.

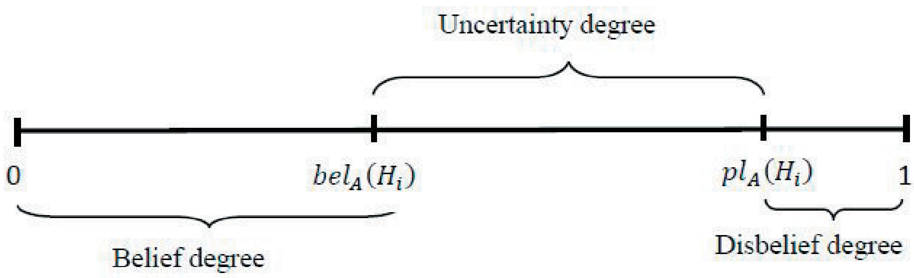

Figure 1. Description of a belief set.

Example 2.8. Let $X = \{x_1, x_2, \cdots, x_n\}$ be the belief set of hypotheses $H_4 = \{x_1, x_2\}$ with respect to the given mass assignment in Table 1. which is determined.

Table 1: BPA to the focal element or hypotheses.

H_i	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$
$m(H_i)$	0.1	0.2	0.1	0.1	0.1	0.3	0.1

$$\begin{aligned}
bel(H_4) &= bel(\{x_1, x_2\}) = m(\{x_1\}) + m(\{x_2\}) + m(\{x_1, x_2\}) \\
&= 0.1 + 0.2 + 0.1 = 0.4, \\
pl(H_4) &= pl(\{x_1, x_2\}) \\
&= m(\{x_1\}) + m(\{x_2\}) + m(\{x_1, x_2\}) \\
&+ m(\{x_1, x_3\}) + m(\{x_2, x_3\}) + m(\{x_1, x_2, x_3\}) \\
&= 0.1 + 0.2 + 0.1 + 0.1 + 0.3 + 0.1 = 0.9, \\
bel(\sim H_4) &= bel(\sim \{x_1, x_2\}) = 1 - pl(\{x_1, x_2\}) = 1 - 0.9, \\
u(H_4) &= u(\{x_1, x_2\}) = 1 - bel(\{x_1, x_2\}) - bel(\sim \{x_1, x_2\}) \\
&= 1 - bel(H_4) - bel(\sim H_4) = 1 - 0.4 - 0.1 = 0.5.
\end{aligned}$$

Therefore, the belief set of hypotheses $H_4 = \{x_1, x_2\}$ is defined as $A = (0.4, 0.5, 0.1)$.

Definition 2.9. *Two or more different bodies of evidence are combined by the Dempster's rule [4]. Accordingly, two or more belief set A and B can be aggregated with the corresponding belief functions by:*

$$bel_A(H_i) \oplus bel_B(H_i) = \frac{bel_A(H_i)bel_B(H_i) + bel_A(H_i)u_B(H_i) + bel_B(H_i)u_A(H_i)}{1 - bel_A(H_i)bel_B(\sim H_i) - bel_B(H_i)bel_A(\sim H_i)}. \tag{8}$$

$$bel_A(\sim H_i) \oplus bel_B(\sim H_i) = \frac{bel_A(\sim H_i)bel_B(\sim H_i) + bel_A(\sim H_i)u_B(H_i) + bel_B(\sim H_i)u_A(H_i)}{1 - bel_A(H_i)bel_B(\sim H_i) - bel_B(H_i)bel_A(\sim H_i)}. \tag{9}$$

$$u_A(H_i) \oplus u_B(H_i) = \frac{u_A(H_i)u_B(H_i)}{1 - bel_A(H_i)bel_B(\sim H_i) - bel_B(H_i)bel_A(\sim H_i)}. \tag{10}$$

3. Cross Entropy for a Divergence Measure Between Belief Sets

A considerable mentioned is that the cross-entropy measures for belief sets have not been studied, this section defines cross-entropy measures for belief sets based on the equivalenttrans formation function.

3.1 Basic concepts

A value of the cross entropy is measured the divergence among two probability distributions P and Q. Let $X = \{x_1, x_2, \cdots, x_n\}$, $P = \{p_1, p_2, \cdots, p_n\}$ and $Q = \{q_1, q_2, \cdots, q_n\}$, the cross entropy or Kullbak-Libler distance is defined as follows, when $n = 2$ and $P = \{p, 1-p\}$, $Q = \{q, 1-q\}$ [20]:

$$E(P, Q) = p(x_i) \ln \frac{p(x_i)}{q(x_i)} + \big(1 - p(x_i)\big) \ln \frac{p(x_i)}{q(x_i)}. \tag{11}$$

Because of undefined point $q(x_i) = 0$ and $p(x_i) = 0$ for each $x_i \in X$ Lin [9] proposed a new method to modify the cross entropy measure.

It indicates the discrimination degree of probability distributions P. From probability distributions Q, as follows:

$$E(P,Q)=\sum_{i=1}^{n}\left[p(x_i)\log_2\frac{p(x_i)}{\frac{1}{2}(p(x_i)+q(x_i))}\right.$$
$$\left.+(1-p(x_i)\log_2\frac{1-p(x_i)}{1-\frac{1}{2}(p(x_i)+q(x_i))}\right]. \tag{12}$$

Equation 12 with respect to its argument is not symmetric, thus a symmetric measure has also defined in Shang and Jiang [19] to obtain discriminations information by:

$$I(P,Q)=H(P,Q)+H(Q,P), \tag{13}$$

where $I(P,Q)\geqslant 0$ and $I(P,Q)=0$ only if $P=Q$.

3.2 Generalized cross entropy for belief sets

According to the basic concepts in Subsection 3.1, with the aim of the extension of a measure of the cross entropy into belief sets, we consider pair exploit information carried by both belief and disbelief functions. Let us consider $X=\{x_1,x_2,\cdots,x_n\}$ to be a finite set and $\Omega=\{H_1,H_2,\cdots,H_{2^N}\}$ are focal elements as alternatives or hypotheses and Ω to be a frame of discernment in the D-S theory. $A=\{bel_A(H_i),u_A(H_i),bel_A(\sim H_i))|H_i\in\Omega\}$, $B=\{bel_B(H_i),u_B(H_i),bel_B(\sim H_i))|H_i\in\Omega\}$ are two belief sets. Thus, by using Eq 12, the discrimination measure of A against Bis given by Eqs. 14 to 16. According to solely on the belief function, these equations are as the expected information, respectively.

$$E_{bel}(A,B)=\sum_{i=1}^{n}\left[bel_A(H_i)\log_2\frac{bel_A(H_i)}{bel_B(H_i)}\right.$$
$$\left.+\big(1-bel_A(H_i)\big)\log_2\frac{1-bel_A(H_i)}{1-\frac{1}{2}(bel_A(H_i)+bel_B(H_i))}\right], \tag{14}$$

$$E_u(A,B) = \sum_{i=1}^{n}\Big[u_A(H_i)\log_2\frac{u_A(H_i)}{u_B(H_i)} + \big(1-u_A(H_i)\big)\log_2\frac{1-u_A(H_i)}{1-\frac{1}{2}(u_A(H_i)+u_B(H_i))}\Big], \qquad (15)$$

$$E_{\sim bel}(A,B) = \sum_{i=1}^{n}\Big[bel_A(\sim H_i)\log_2\frac{bel_A(\sim H_i)}{bel_B(\sim H_i)} + \big(1-bel_A(\sim H_i)\big)\log_2\frac{1-bel_A(\sim H_i)}{1-\frac{1}{2}(bel_A(\sim H_i)+bel_B(\sim H_i))}\Big]. \qquad (16)$$

Therefore, a novel belief valued cross entropy measure between A and B is achieved as the summation of the individual quantities $E_{bel}(A,B)$, $E_u(A,B), E_{\sim bel}(A,B)$ as follows:

$$\begin{aligned} E(A,B) = {} & \sum_{i=1}^{n}\Big[bel_A(H_i)\log_2\frac{bel_A(H_i)}{bel_B(H_i)} \\ & + \big(1-bel_A(H_i)\big)\log_2\frac{1-bel_A(H_i)}{1-\frac{1}{2}(bel_A(H_i)+bel_B(H_i))}\Big] \\ & + \sum_{i=1}^{n}\Big[u_A(H_i)\log_2\frac{u_A(H_i)}{u_B(H_i)} \\ & + \big(1-u_A(H_i)\big)\log_2\frac{1-u_A(H_i)}{1-\frac{1}{2}(u_A(H_i)+u_B(H_i))}\Big] \\ & + \sum_{i=1}^{n}\Big[bel_A(\sim H_i)\log_2\frac{bel_A(\sim H_i)}{bel_B(\sim H_i)} \\ & + \big(1-bel_A(\sim H_i)\big)\log_2\frac{1-bel_A(\sim H_i)}{1-\frac{1}{2}(bel_A(\sim H_i)+bel_B(\sim H_i))}\Big]. \end{aligned} \qquad (17)$$

According to the Shannon entropy [18], $E(A,B) \geqslant 0$ and $E(A,B) = 0$ if only if $bel_A(H_i) = bel_B(H_i)$, $u_A(H_i) = u_B(H_i)$ and $bel_A(\sim H_i) = bel_B(\sim H_i)$. For any $H_i \in \Omega$. Moreover $E(A^c, B^c) = E(A,B)$, where A^c

and B^c are denoted as complementary sets of A and B, where based on Eq. 7, A^c and B^c are defined by:

$$A^c = \big\{bel_A(\sim H_i), u_A(H_i), bel_A(H_i)|H_i \in \Omega\big\}, \tag{18}$$

$$B^c = \big\{bel_B(\sim H_i), u_B(H_i), bel_B(H_i)|H_i \in \Omega\big\}. \tag{19}$$

Then the proposed belief cross entropy can be defined as a symmetric discrimination by:

$$I(A, B) = E(A^c, B^c) + E(A, B). \tag{20}$$

4. Proposed MCDM Model Based on the Cross Entropy of the Beliefs Value

In this section, the fundamental conception of cross entropy are considered to use in an MCDM model. Then, this concept is extended and proposed through a belief function. The belief function and measure can be handle the uncertainty environment in MCDM problems when the provided information is incomplete and inconsistence. Let A be a set of alternatives, which is discrete and infinite, where $A = \{A_1, A_2, \cdots, A_n\}$. Also, C is a set of criteria with finite elements, where $C = \{C_1, C_2, \cdots, C_m\}$. The criteria weight of each decision maker can be determined by $w_j \in [0, 1]$, where $\sum_{j=1}^{m} w_j = 1$. Thus, the belief set S_i of the alternative$A_i (i = 1, 2, \cdots, n)$ with respect to criteria $C_j (j = 1, 2, \cdots, m)$ is characterized by:

$$S_i = \big\{bel_{A_i}(C_j), u_{A_i}(C_j), bel_{A_i}(\sim C_j)|C_j \in C\big\}, \tag{21}$$

where $bel_{A_i}(C_j), u_{A_i}(C_j), bel_{A_i}(\sim C_j) \in [0, 1], (i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$; For the sake of simplicity, a belief value$\big(bel_{A_i}(C_j), u_{A_i}(C_j), bel_{A_i}(\sim C_j)\big)$ is denoted by the symbol $a_{ij} = (bel_{ij}, u_{ij}, \sim bel_{ij})$.

For example, let us consider the decision-making problem with two alternatives $(companyA_1, companyA_2)$ and three criteria $\big(C_1(cost), C_2$

$(risk), C_3(growth)\big)$. When the expert's opinion about company A_1 or alternative A_1 with respect to cost (criteria C_1) is asked, he/she may present his/her belief value based on evidence, which supports his/her opinion or past experience as $a_{11} = (0.5, 0.2, 0.3)$. In this case, the frame of discernment is $\{T, F\}$, where T means a true solution and F or $\sim T$ means a false solution for investment in company A_1 with respect to cost (criteria C_1), the BPAs value of the belief function supports the following hypothesis:

- Company A_1 is a proper and positive ideal solution for investment with respect to the cost criteria with a belief degree of 0.5.
- Company A_1 is a negative ideal solution for investment with respect to the cost criteria with a belief degree of 0.3.
- We do not know company A_1 is a positive or negative ideal solution with a belief degree of 0.2, thus the belief degree of our uncertainty is 0.2.

Accordingly, in continuation of the discussion, a belief value decision matrix $A = (a_{ij})_{n\times m}$ can be concluded by:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}$$

$$= \begin{pmatrix} (bel_{11}, u_{11}, \sim bel_{11}) & (bel_{12}, u_{12}, \sim bel_{12}) & \cdots & (bel_{1m}, u_{1m}, \sim bel_{1m}) \\ (bel_{21}, u_{21}, \sim bel_{21}) & (bel_{22}, u_{22}, \sim bel_{22}) & \cdots & (bel_{2m}, u_{2m}, \sim bel_{2m}) \\ \vdots & \vdots & \cdots & \vdots \\ (bel_{n1}, u_{n1}, \sim bel_{n1}) & (bel_{n2}, u_{n2}, \sim bel_{n2}) & \cdots & (bel_{nm}, u_{nm}, \sim bel_{nm}) \end{pmatrix}. \tag{22}$$

The MCDM procedure helps to detect the best alternative between a set of feasible alternative. Generally, the best alternative is chosen based on the shortest discrimination or distance from a favor or positive ideal solution. However, in the real world, the ideal solution does not exist, a useful construct can provide a framework to make a preference decision using prioritization the alternatives. Hence, the ideal value for selecting the alternative A_i with respect to criteria is $a_j^* = (bel^*, u^*, \sim bel^*) = (1, 0, 0)$. Thus, by using Eqs. 17 and 20, the weighted cross entropy can be obtained amongst alternatives A_i and A^*, which is the positive ideal alternative, as follows:

$$D_i(A^*, A_i) = \sum_{i=1}^{m} w_j \left[\log_2 \frac{1}{\frac{1}{2}(1 + bel_{ij})} + \log_2 \frac{1}{1 - \frac{1}{2}(u_{ij})} + \log_2 \frac{1}{1 - \frac{1}{2}(\sim bel_{ij})} \right]$$

$$+ \sum_{i=1}^{m} w_j \left[bel_{ij} \log_2 \frac{bel_{ij}}{\frac{1}{2}(1 + bel_{ij})} + (1 - bel_{ij}) \log_2 \frac{1 - bel_{ij}}{1 - \frac{1}{2}(1 + b_{ij})} \right]$$
$$+ \sum_{i=1}^{m} w_j \left[u_{ij} + (1 - u_{ij}) \log_2 \frac{1 - u_{ij}}{1 - \frac{1}{2} u_{ij}} \right] \tag{23}$$
$$+ \sum_{i=1}^{m} w_j \left[\sim bel_{ij} + (1 - \sim bel_{ij}) \log_2 \frac{1 - \sim bel_{ij}}{1 - \frac{1}{2}(\sim bel_{ij})} \right].$$

The smallest value of $D_i(A^*, A_i)$ is shown the alternative A_i is closer to the positive ideal alternative against other alternatives. Therefore, the alternative can be ranked to determine and choose the best one.

5. Illustrative Example

Now we concern a decision-making problem about a computer center to select a new information system according to a production companies. Suppose that there is a set of four production companies $A = \{A_1, A_2, A_3, A_4\}$ that remain in the candidate list. They are evaluated by means of the four criteria (C_1, C_2, C_3, C_4) as follows:

- The cost of hardware and software investment (C_1)
- The level of service (C_2)
- The quality and performance (C_3)

- The reliability and lifetime of hardware and software (C_4)

The proposed weight for four criteria are $w = (0.3, 0.3, 0.2, 0.2)$.
One expert is estimated each alternative $A_i (i = 1, 2, 3, 4)$ according to criteria$C_j (j = 1, 2, 3, 4)$ based on insufficient existence evidence and document. For example, he/she considers about alternative A_1 with respect to criterion C_1, say $(0.6, 0.2, 0.2)$ using opinion and knowledge, which may be insufficient. It means that evidence supports that company A_1 with respect to C_1 are acceptable with a belief degree of 0.6, in which it is not proper with a belief degree of 0.2 and is uncertain with 0.2. Accordingly, the extracted decision matrix based on a belief function is presented by:

$$A = \begin{pmatrix} (0.6, 0.2, 0.2) & (0.5, 0.2, 0.3) & (0.7, 0.1, 0.2) & (0.3, 0.2, 0.4) \\ (0.5, 0.2, 0.3) & (0.3, 0.2, 0.5) & (0.8, 0.1, 0.1) & (0.5, 0.3, 0.2) \\ (0.4, 0.2, 0.4) & (0.4, 0.3, 0.3) & (0.5, 0.1, 0.4) & (0.7, 0.2, 0.1) \\ (0.6, 0.2, 0.2) & (0.3, 0.3, 0.4) & (0.6, 0.3, 0.1) & (0.6, 0.3, 0.1) \end{pmatrix}.$$

By applying Eq. 20, we obtain the following cross entropy values between an alternative A_i $(i = 1, 2, 3, 4)$ and the positive ideal valueA^* :
$D(A_1, A^*) = 0.73941, D(A_2, A^*) = 0.81731, D(A_3, A^*) = 0.84588,$
$D(A_4, A^*) = 0.79330$. Hence, the ranking order of four alternatives with respect to four criteria, according to the cross-entropy is $A_3 < A_2 < A_4 < A_1$. Thus, the best alternative is A_1. The result is applicable because it can handle incomplete and inconsistence information using a belief function, which exists commonly in the real situation.

6. A Comparison Result

In order to show the validity and effectiveness of our proposed method of MCDM problems, our method using the cross entropy measure based on a belief set is compared with two cross-entropy measures for probability multi-valued neutrosophic sets (PMVNSs) [12] in a neutrosophic set. The defined cross entropy measure for PMVNNs can be found with Peng et.al. [12] and Wu et.al. [29].
In the table 2, The final ranking of alternatives obtained by the proposed

method and two similar important methods of investment is the same.

Table 2: Numerical results of three cross entropy measures.

Alternatives	$D(A, A^*)$	$CE_1(A, A^*)$	$CE_1(A, A^*)$
A_1	0.73941	0.276246	0.650854
A_2	0.81731	0.345415	0.787231
A_3	0.84588	0.351466	0.791818
A_4	0.79330	0.31527	0.72311
Ranking	$A_3 < A_2 < A_4 < A_1$	$A_3 < A_2 < A_4 < A_1$	$A_3 < A_2 < A_4 < A_1$

It is clear that the ranking orders of three similarity measures (i.e., cross entropy based on a belief degree in the D-S theory and two types of cross-entropy measures of PMVNSs) are the same ranking of all alternatives. Also, comparison of them is shown that the most desirable alternative for investing with respect to three criteria is Company A_1. Additionally, the decision results of different similarity measures based on new definition of the belief set are demonstrated that the proposed method under uncertainty and insufficient information are applicable and effective.

7. Conclusion

This paper defined a new aspect of three belief measures in the D-S theory based on their investigated properties. The new presented method about belief cross entropy was focused on a belief degree based on incomplete information and epistemic uncertainty using a belief function. Then a new multi-criteria decision-making (MCDM) problem was proposed based on a belief cross entropy. Three belief values were demonstrated as belief degree, disbelief degree and uncertainty degree in the structure of the cross entropy, which were considered the divergence a decision alternative from the positive ideal alternative in MCDM problems. Therefore, a discrimination amount of two belief sets could measure to rank alternatives based on the belief cross entropy, which was used to choose the best option and the most desirable one. The proposed method was logically reasonable as an approximate reasoning which can be applied in many fields (e.g decision-making), when we needed to use the D-S framework to handle uncertainty. Thus, it could provide a flexible environment to deal with incomplete information by using belief functions. Finally, a practical example illustrated to present the efficiency of the new decision-making method.

Then, the results of the proposed cross entropy measure in the D-S framework are compared with the existing similarity measure as PMVNN -s in neutrosophic logic. PMVNNs measure have similar elements with the concept of belief sets in the D-S theory and belief set can be considered as a special case of PMVNNs as a probability concept. Considering the ability of the neutrosophic logic and similarity of this logic with belief sets and cross entropy measure can be an interesting topic for future research. Furthermore, development of both methods for MCDM problems and reduction of the uncertainty can be considered as future studies.

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Fereshteh Khalaj
Instructor of Engineering
Department of Industrial Engineering
Robat Karim Branch, Islamic Azad University
Tehran, Iran
E-mail: Khalaj82@gmail.com

Mehran Khalaj
Assistance Professor of Engineering
Department of Industrial Engineering
Robat Karim Branch, Islamic Azad University
Tehran, Iran
E-mail: Mkhalaj@rkiau.ac.ir

Reza Tavakkoli-Moghaddam
Professor of Engineering
School of Industrial Engineering
College of Engineering
University of Tehran
Tehran, Iran
E-mail: Tavakoli@ut.ac.ir

Einollah Pasha
Professor of Mathematics
Department of Mathematics
Faculty of Mathematical Sciences and Computer
Kharazmi University
Tehran, Iran
E-mail: Pashaeinollah@yahoo.com