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Extended Generalized Lindley Distribution: Properties and Applications

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Abstract. In this study, we introduce a new model called the Extended Exponentiated Power Lindley distribution which extends the Lindley distribution and has increasing, bathtub and upside down shapes for the hazard rate function. It also includes the power Lindley distribution as a special case. Several statistical properties of the distribution are explored, such as the density, hazard rate, survival, quantile functions, and moments. Estimation using the maximum likelihood method and inference on a random sample from this distribution are investigated. A simulation study is performed to compare the performance of the different parameter estimates in terms of bias and mean square error. We apply a real data set to illustrate the applicability of the new model. Empirical findings show that proposed model provides better fits than other well-known extensions of Lindley distributions.

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1. Introduction

The statistical analysis and modeling of lifetime data are essential in almost all applied sciences such as, biomedical science, engineering, nance, and insurance, amongst others. A number of one-parameter continuous

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distributions for modelling lifetime data has been introduced in statistical literature including exponential, Lindley, gamma, lognormal, and Weibull. The exponential, Lindley and Weibull distributions are more popular than the gamma and lognormal distributions because the survival functions of the gamma and the lognormal distributions cannot be expressed in closed forms and both require numerical integration. The Lindley distribution is a very well-known distribution that has been extensively used over the past decades for modeling data in reliability, biology, insurance, and lifetime analysis. It was introduced by Lindley [18] to analyze failure time data, especially in applications of modeling stress-strength reliability. The motivation for introducing the Lindley distribution arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. It may also be mentioned that the Lindley distribution belongs to an exponential family and it can be written as a mixture of an exponential and a gamma distributions. This distribution represents a good alternative to the exponential failure time distributions that suffer from not exhibiting unimodal and bathtub shaped failure rates (Bakouch et al. [4]). The properties and inferential procedure for the Lindley distribution were studied by Ghitany et al. [10, 11]. They show via a numerical example that the Lindley distribution gives better modeling than the one based on the exponential distribution when hazard rate is unimodal or bathtub shaped. Furthermore, Mazucheli and Achcar [19] showed that many of the mathematical properties are more exible than those of the exponential distribution and proposed the Lindley distribution as a possible alternative to exponential or Weibull distributions. The need for extended forms of the Lindley distribution arises in many applied areas. The emergence of such distributions in the statistics literature is quite recent. For some extended forms of the Lindley distribution and their applications, the interested reader is referred to Kumaraswamy Lindley (Cakmakyapan and Ozel, [6]), beta odd log-logistic Lindley (Cordeiro et al., [8]), generalized Lindley (Nadarajah et al., [21]), quasi Lindley distributions (Shanker and Mishra, [21]).

The probability density function (pdf) and cumulative density function (cdf) of the power Lindley distribution are given respectively by

$$f(x) = \frac{\beta\lambda^2}{1+\lambda}(1+x^{\beta})\, x^{\beta-1}\, e^{-\lambda x^{\beta}},$$
$$F(x) = 1-(1+\frac{\lambda\, x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}^{\beta}}. \tag{1}$$

It can be seen that this distribution is a mixture of exponential and gamma distributions. Having only one parameter, the Lindley distribution does not provide enough exibility for analyzing different types of lifetime data. To increase the exibility for modeling purposes it will be useful to consider further alternatives to this distribution. Our purpose here is to provide a generalization that may be useful for more complex situations. Once the proposed distribution is quite exible in terms of pdf and hazard rate function (hrf), it may provide an interesting alternative for describing income distributions and can also be applied in actuarial science, nance, bioscience, telecommunications and modeling lifetime data. Therefore, goal is to introduce a new distribution using the Lindley distribution. Alizadeh et. al [1], introduced a new class of exponentiated distributions which called Extended Exponentiated distribution (EE-G). The cdf and pdf of this family are given by

$$F(x;\alpha,\gamma,\boldsymbol{\xi}) = \int_0^{\frac{G(x;\boldsymbol{\xi})^{\alpha}}{1-G(x;\boldsymbol{\xi})^{\gamma}}} \frac{dt}{(1+t)^2}\, dt = \frac{G(x;\boldsymbol{\xi})^{\alpha}}{G(x;\boldsymbol{\xi})^{\alpha}+1-G(x;\boldsymbol{\xi})^{\gamma}}, \tag{2}$$

$$f(x;\alpha,\gamma,\boldsymbol{\xi}) = \frac{g(x;\boldsymbol{\xi})\, G(x;\boldsymbol{\xi})^{\alpha-1}\, [\alpha+(\gamma-\alpha)\, G(x;\boldsymbol{\xi})^{\gamma}]}{[G(x;\boldsymbol{\xi})^{\alpha}+1-G(x;\boldsymbol{\xi})^{\gamma}]^2}, \tag{3}$$

where $\alpha,\gamma > 0$ are two shape parameters and $\boldsymbol{\xi}$ is the vector of parameters for baseline cdf G. For $\alpha = \gamma$, it contains exp-G family of distributions. Taking $G(x;\boldsymbol{\xi})$ as Lindley distribution with parameters λ, we introduce a new extension of Exponentiated Lindley distribution.

The article is outlined as follows: In Section 2, we introduce the EGL distribution and provide plots of the density and hazard rate functions. Shapes, quantile function, moments, and moment generating function are also obtained. Moreover, mean deviation, Lorenz and Bonferroni curves, order statistics. In Section 3, the asymptotic properties and extreme values are obtained. Estimation by the method of maximum likelihood and an explicit expression for the observed information matrix are presented in Section 4. The simulation study presented in Section 5. The applications to real data sets are considered in Section 6. Finally, Section 7 offers some concluding remarks.

2. Main Properties

2.1 Probability density and cumulative density functions

Inserting (1) in (2), the cdf of the EGL with three parameters ($\alpha, \gamma, \lambda > 0$) is defined as

$$F(x;\alpha,\gamma,\lambda)=\frac{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha}}{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\gamma}}. \quad (4)$$

The corresponding pdf of EGL is given by

$$f(x;\alpha,\gamma,\lambda)=\frac{\lambda^{2}(1+x)\mathrm{e}^{-\lambda\,\mathrm{x}}\left[1-(1+\frac{\lambda \mathrm{x}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha-1}\left\{\alpha+(\gamma-\alpha)\left[1-(1+\frac{\lambda \mathrm{x}}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\gamma}\right\}}{(1+\lambda)\left\{\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x}{1+\lambda})\mathrm{e}^{-\lambda\,\mathrm{x}}\right]^{\gamma}\right\}^{2}}, \quad (5)$$

where λ is a scale parameter α and γ are the shape parameters. Here, a and b govern the skewness of (5). A random variable X with the pdf (5) is denoted by $X \sim EGL(\alpha, \gamma, \lambda)$ where $\lambda, \alpha, \gamma > 0$ are four shape parameters and $x > 0$. It is easy to see that:

- For $\alpha = \gamma$, we obtain Exponentiated Lindley.
- For $\alpha = \gamma = 1$, we obtain Lindley.

Some of the possible shapes of the density function (5) for the selected parameter values are illustrated in Figure 1. As seen in Figure 1, the density function can take various forms depending on the parameter values. It is evident that the EGL distribution is much more flexible than the Lindley distribution, i.e. the additional shape parameter allows for a high degree of flexibility of the EGL distribution. Both unimodal and monotonically decreasing and increasing shapes appear to be possible.

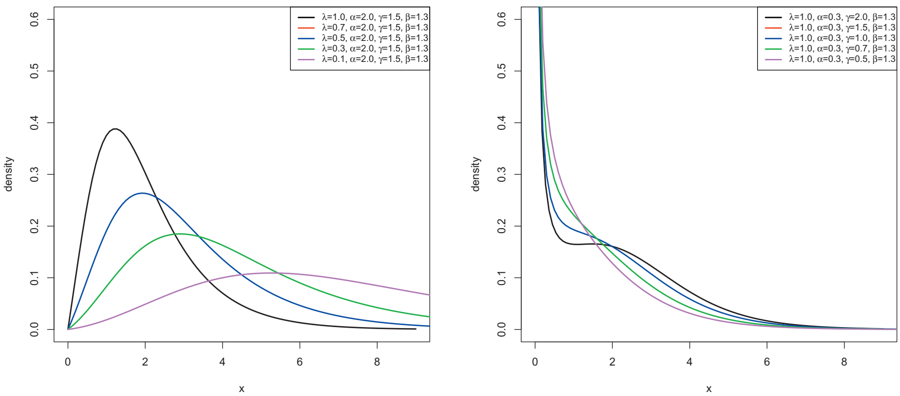

Figure 1. Plots of Pdfs of the EGL model for selected λ, α and γ.

2.2 Survival and hazard rate functions

Central role is played in the reliability theory by the quotient of the pdf and survival function. We obtain the survival function corresponding to (4) as

$$S(x;\lambda,\alpha,\gamma) = \frac{1-\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\gamma}}{\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\gamma}}. \tag{6}$$

In reliability studies, the hrf is an important characteristic and fundamental to the design of safe systems in a wide variety of applications. Therefore, we discuss these properties of the EGL distribution. The hrf of X takes the form

$$h(x;\lambda,\alpha,\gamma) = \frac{\lambda^2(1+x)e^{-\lambda x}\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha-1}\left\{\alpha+(\gamma-\alpha)\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\gamma}\right\}}{\left\{\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\alpha}+1-\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\gamma}\right\}\left\{1-\left[1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right]^{\gamma}\right\}}. \tag{7}$$

Plots of the hrf of the EGL distribution for several parameter values are displayed in Figure 2.

Figure 2 shows that the hrf of the EGL distribution can have very flexible shapes, such as increasing, decreasing, bathtub followed by upside down bathtub, and bathtub shapes for the selected values of the model parameters. This attractive flexibility makes the hrf of the EGL distribution useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

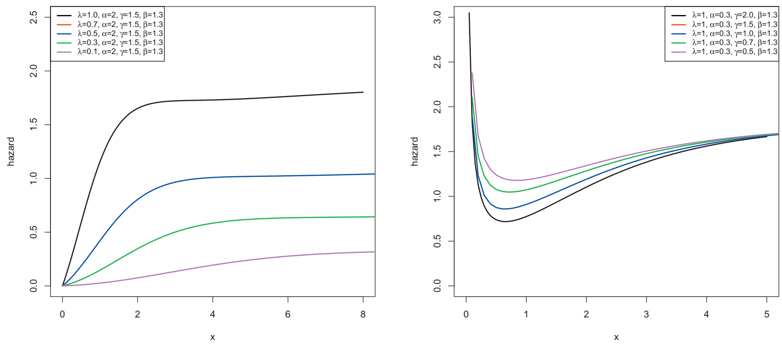

Figure 2. Plots of Pdfs of the EGL model for selected λ, α and γ.

2.3 Mixture representations for the pdf and cdf

In this subsection, we provide alternative mixture representations for the pdf and cdf of X. Some useful expansions for (4) can be derived by using the concept of power series. We have

$$\begin{aligned}
[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\alpha} &= \sum_{i=1}^{\infty}(-1)^{i}\binom{\alpha}{i}[1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{i} \\
&= \sum_{i=1}^{\infty}\sum_{k=0}^{i}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k} \\
&= \sum_{k=0}^{\infty}\sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k} \\
&= \sum_{k=0}^{\infty}a_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k},
\end{aligned}$$

where $a_k = a_k(\alpha) = \sum_{i=k}^{\infty}(-1)^{i+k}\binom{\alpha}{i}\binom{i}{k}$. Also

$$[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\alpha}+1-[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{\gamma} = \sum_{k=0}^{\infty}b_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k},$$

where $b_0 = a_0(\alpha)+1-a_0(\gamma)$ and $b_k = a_k(\alpha)-a_k(\gamma)$ for $k \geqslant 1$. Then using the ratio of two power series, we can write

$$\begin{aligned}
F(x) &= \frac{\sum_{k=0}^{\infty}a_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k}}{\sum_{k=0}^{\infty}b_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k}} \\
&= \sum_{k=0}^{\infty}c_k[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^{k}, \qquad (8)
\end{aligned}$$

where $c_0 = \frac{a_0}{b_0}$ and for $k \geqslant 1$,

$$c_k = \frac{1}{b_0}[a_k - \frac{1}{b_{_0}} \sum_{r=1}^{k} b_r c_{k-r}] \tag{9}$$

Equation (8) shows that me can write the cdf of EGL an a Linear combination of generalized lindly distribution. Then we can write

$$f(x) \quad = \quad \sum_{k=0}^{\infty} c_{k+1} \frac{(k+1)\lambda^2(1+x)}{1+\lambda} e^{-\lambda x}[1-(1+\frac{\lambda}{1+\lambda}x)e^{-\lambda x}]^k.$$

2.4 Moments and moment generating function

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis).

Now we obtain ordinary moments and the moment generating function of the EGL distribution. We define and compute

$$A(a_1,a_2,a_3,a_4;\lambda) = \int_0^\infty x^{a_1}\,(1+x)^{a_2}\,\mathrm{e}^{-a_3\,x}\left[1-(1+\frac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_4} \mathrm{d}x. \tag{10}$$

Using generalized binomial expansion, one can obtain

$$A(a_1,a_2,a_3,a_4;\lambda) = \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l \binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}(\frac{\lambda}{1+\lambda})^l \times \frac{\Gamma(a_1+1+k+r)}{(\lambda\,l+a_3)^{a_1+1+k+r}}. \tag{11}$$

Next, the nth moment of the EGL distribution is given by

$$E\,[X^n] = \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty} k\,c_k\;A(n,1,\lambda,k;\lambda). \tag{12}$$

For integer values of k, let $\mu'_k = E(X^k)$ and $\mu = \mu'_1 = E(X)$, then one can also find the kth central moment of the EGL distribution through the following well-known equation

$$\mu_k = E(X-\mu)^k = \sum_{r=0}^{k}\binom{k}{r}\mu'_r(-\mu)^{k-r}. \tag{13}$$

The moment generating function of a random variable provides the basis of an alternative route to analytical results compared with working directly with its pdf and cdf. Using (12) and (13), we obtain

$$M_X(t) = E\left[\mathrm{e}^{\mathrm{t\,X}}\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\,A(k+1,\lambda,0,\lambda-t).$$

Using (13), the variance, skewness and kurtosis measures can be obtained. Skewness measures the degree of the long tail and kurtosis is a measure of the degree of tail heaviness. For the EGL distribution, The skewness can be computed as

$$S = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3}{(\mu_2' - \mu_1'^2)^{3/2}},$$

and the kurtosis is based on octiles as

$$K = \frac{\mu_4}{\mu_2^2} = \frac{\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4}{\mu_2' - \mu_1'^2}.$$

When the distribution is symmetric $S = 0$, and when the distribution is right (or left) skewed $S > 0 (or\ S < 0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

We present first four ordinary moments, skewness and kurtosis of the EGL distribution for various values of the parameters in Table 1. Plots for skewness and kurtosis are presented in Figure 3.

Next, we define and compute

$$B(a_1,a_2,a_3,a_4;y,\lambda) = \int_0^y x^{a_1}\,(1+x)^{a_2}\,\mathrm{e}^{-a_3\,x}\left[1-(1+\frac{\lambda}{1+\lambda}\,x)\mathrm{e}^{-\lambda x}\right]^{a_4}\mathrm{d}x. \tag{14}$$

From the generalized binomial expansion, we have

$$\begin{aligned} &B(a_1,a_2,a_3,a_4;a,\lambda)\\ &= \sum_{l,r=0}^{\infty}\sum_{k=0}^{l}(-1)^l\binom{a_4}{l}\binom{l}{k}\binom{a_2}{r}(\frac{\lambda}{1+\lambda})^l \times \frac{\gamma(a_1+1+k+r,\frac{y}{\lambda\,l+a_3})}{(\lambda\,l+a_3)^{a_1+1+k+r}}, \end{aligned} \tag{15}$$

where $\gamma(\lambda,z) = \int_0^z t^{\lambda-1}\,\mathrm{e}^{-t}\,dt$ denotes the incomplete gamma function. Now, the nth incomplete moment of the EGL distribution is found to be

$$m_n(y) = E\left[X^n\,|X<y\right] = \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)\,c_{k+1}\;B(n,1,\lambda,k,y;\lambda). \tag{16}$$

Table 1: Moments, skewness, and kurtosis of the EGL dist. for the some parameter values.

λ	α	γ	μ_1'	μ_2'	μ_3'	μ_4'	*Skewness*	*Kurtosis*
0.5	0.5	0.5	2.1268	10.4546	76.2091	724.9688	1.9899	1.9899
0.5	0.5	1.0	2.8458	16.6597	134.1046	1349.641	1.5157	1.5157
0.5	0.5	3.0	4.1036	31.3523	299.9124	3374.622	0.9432	0.9432
0.5	2.0	0.5	3.3017	15.4462	99.46779	851.2992	1.9048	1.9048
0.5	2.0	1.0	4.0045	22.2918	163.9227	1530.386	1.5694	1.5694
0.5	2.0	3.0	5.2516	37.7282	339.4111	3653.751	1.0728	1.0728
2.0	0.5	0.5	0.4135	0.45722	0.783860	1.788817	2.3382	2.3382
2.0	0.5	1.0	0.5716	0.75289	1.413329	3.384414	1.7824	1.7824
2.0	0.5	3.0	0.8617	1.49044	3.300667	8.747468	1.1248	1.1248
2.0	2.0	0.5	0.6464	0.65531	0.981019	2.028222	2.1655	2.1655
2.0	2.0	1.0	0.8063	0.98555	1.678982	3.744753	1.7687	1.7687
2.0	2.0	3.0	1.0979	1.76532	3.675197	9.345144	1.2118	1.2118

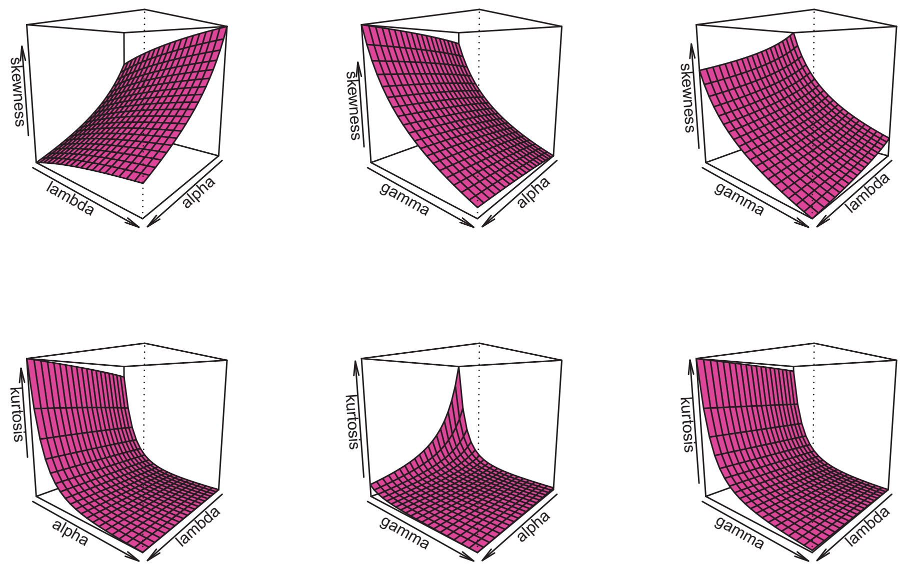

Figure 3. Hazard Rate Function of the EGL model for selected λ, α and γ.

2.5 Mean deviations

The amount of scatter in a population may be measured to some extent by deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, defined by

$$\delta_1(X) = \int_0^\infty |x-\mu|\, f(x)\,\mathrm{d}x,$$

and

$$\delta_2(X) = \int_0^\infty |x-M|\, f(x)\,\mathrm{d}x,$$

respectively, where $\mu = E(X)$ and $M = \text{Median}(X) = Q(0.5)$ denotes the median and $Q(p)$ is the quantile function. The measures $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the relationships

$$\delta_1(X) = 2\mu\, F(\mu) - 2\int_0^\mu x\, f(x)\,\mathrm{d}x,$$

and

$$\delta_2(X) = \mu - 2\int_0^M x\, f(x)\,\mathrm{d}x.$$

Finally we have

$$\delta_1(X) = 2\mu\, F(\mu) - \frac{\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)c_{k+1}\, A(\beta,1,\lambda,k;\lambda,\beta),$$

and

$$\delta_2(X) = \mu - \frac{2\beta\,\lambda^2}{1+\lambda}\sum_{k=0}^{\infty}(k+1)c_{k+1}\, B(\beta,1,\lambda,k;M,\lambda,\beta).$$

2.6 Bonferroni and Lorenz curves

The Bonferroni and Lorenz curves have applications in economics as well as other fields like reliability, medicine and insurance. Let $X \sim EGL(\lambda,\alpha,\gamma)$ and $F(x)$ be the cdf of X, then the Bonferroni curve of the EGL distribution is given by

$$B(F(x)) = \frac{1}{\mu\, F(x)}\int_0^x t f(t)\mathrm{d}t,$$

where $\mu = E(Y)$. Therefore, from (15), we have

$$B(F(x)) = \frac{1}{\mu\, F(x)} \times \frac{\beta\,\lambda^2}{1+\lambda} \sum_{k=0}^{\infty}(k+1)c_{k+1}\, B(\beta, 1, \lambda, k; x, \lambda, \beta).$$

The Lorenz curve of the EGL distribution can be obtained using the relation

$$L(F(x)) = F(x)B(F(x)) = \frac{1}{\mu} \times \frac{\beta\,\lambda^2}{1+\lambda} \sum_{k=0}^{\infty}(k+1)c_{k+1}\, B(\beta, 1, \lambda, k; x, \lambda, \beta).$$

2.7 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Suppose $X_1, \ldots, X_n$ is a random sample from any EGL distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = K\, f(x)\, F^{i-1}(x)\, \{1 - F(x)\}^{n-i} = K \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} f(x)\, F(x)^{j+i-1},$$

where $K = 1/B(i, n-i+1)$. We use the result of Gradshteyn and Ryzhik [14] for a power series raised to a positive integer n (for $n \geqslant 1$)

$$\left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} d_{n,i}\, u^i, \tag{17}$$

where the coefficients $d_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation (with $d_{n,0} = a_0^n$)

$$d_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, d_{n,i-m}. \tag{18}$$

We can show that the density function of the ith order statistic of any EGL distribution can be expressed as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}\, f_{GL}(x; \lambda, r+k+1), \tag{19}$$

where $f_{GL}(x; \lambda, \beta, r+k+1)$ denotes the density function of generalized Lindley distribution with parameters λ and $r+k+1$,

$$m_{r,k} = \frac{n!\,(r+1)\,(i-1)!\, c_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\, f_{j+i-1,k}}{(n-i-j)!\, j!}.$$

Here, c_r is given by (9) and the quantities $f_{j+i-1,k}$ can be determined given that $f_{j+i-1,0} = c_0^{j+i-1}$ and recursively we have:

$$f_{j+i-1,k} = (k\, c_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\, c_m\, f_{j+i-1,k-m}, k \geqslant 1.$$

Equation (19) is the main result of this section. It reveals that the pdf of the ith order statistic is a triple linear combination of exponentiated Lindley distributions. Therefore, several mathematical quantities of these order statistics like ordinary and incomplete moments, factorial moments, mgf, mean deviations and others can be derived using this result.

3. Asymptotic Properties and Extreme Value

One of the main usage of the idea of an asymptotic distribution is in providing approximations to the cumulative distribution functions of the statistical estimators. Moreover, the extreme value theory is a branch of statistics dealing with the extreme deviations from the median of probability distributions. It seeks to assess, from a given ordered sample of a given random variable, the probability of events that are more extreme than any previously observed. Extreme value analysis is widely used in many disciplines,

3.1 Asymptotic properties

The asymptotic of cdf, pdf and hrf of the EGL distribution as $x \to 0$ are, respectively, given by

$$\begin{aligned}
&F(x) \sim (\lambda x)^{\alpha} \quad as \quad x \to 0,\\
&f(x) \sim \alpha\lambda^{\alpha}x^{\alpha-1} \quad as \quad x \to 0,\\
&h(x) \sim \alpha\lambda^{\alpha}x^{\alpha-1} \quad as \quad x \to 0.
\end{aligned}$$

The asymptotic of cdf, pdf and hrf of the EGL distribution as $x \to \infty$ are, respectively, given by

$$\begin{aligned}
&1 - F(x) \sim \frac{\gamma\lambda}{1+\lambda} x e^{-\lambda x} \quad as \quad x \to \infty,\\
&f(x) \sim \frac{\gamma\lambda^2}{1+\lambda} x e^{-\lambda x} \quad as \quad x \to \infty,\\
&h(x) \sim \lambda x \quad as \quad x \to \infty.
\end{aligned}$$

These equations show the effect of parameters on the tails of the EGL distribution.

3.2 Extreme value

Let $X_1, ..., X_n$ be a random sample from (5) and $\bar{X} = (X_1 + ... + X_n)/n$ denote the sample mean, then by the usual central limit theorem, the distribution of $\sqrt{n}(\bar{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$. Sometimes one would be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. For (4), it can be seen that

$$\lim_{t \to 0} \frac{F(t\,x)}{F(t)} = x^{\alpha},$$

and

$$\lim_{t \to \infty} \frac{1 - F(t\,x)}{1 - F(t)} = \mathrm{e}^{-\alpha\lambda\,x}.$$

Thus, it follows from Theorem 1.6.2 in Leadbetter et al. [16] that there must be norming constants $a_n > 0, b_n, c_n > 0$ and d_n such that

$$Pr\left[a_n(M_n - b_n) \leqslant x\right] \to \mathrm{e}^{-\mathrm{e}^{-\lambda\alpha\,x}},$$

and

$$Pr\left[a_n(m_n - b_n) \leqslant x\right] \to 1 - \mathrm{e}^{-x^{\alpha}},$$

as $n \to \infty$. Using Corollary 1.6.3 of Leadbetter et al. [16], we can obtain the form of normalizing constants a_n, b_n, c_n and d_n.

4. Estimation

Several approaches for parameter estimation have been proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider estimation of the unknown parameters of the EGL distribution by the method of maximum likelihood. Let $x_1, x_2, ..., x_n$ be observed values from the EGL distribution with parameters α, γ and λ. The log-likelihood function for $(\alpha; \gamma; \lambda\ )$ is given by

$$\begin{aligned} \ell_n \quad &= \quad 2n\log(\lambda) + \sum_{i=1}^{n} \log(1 + x_i) - \lambda \sum_{i=1}^{n} x_i + (\alpha + 1) \sum_{i=1}^{n} \log k_i \\ &\quad + \sum_{i=1}^{n} \log(\alpha + (\gamma - \alpha)k_i^{\alpha}) - 2\sum_{i=1}^{n} \log(k_i^{\alpha} + 1 - k_i^{\gamma}), \end{aligned}$$

where

$$k_i = 1 - (1 + \frac{\lambda}{1+\lambda} x_i) e^{-\lambda x_i}.$$

The derivatives of the log-likelihood function with respect to the parameters α, γ and λ are given respectively, by

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \sum_{i=1}^{n} \log k_i + \sum_{i=1}^{n} \frac{1 - k_i^{\alpha-1}(\alpha + k_i)}{\alpha + (\gamma - \alpha) k_i^{\alpha}} - 2 \sum_{i=1}^{n} \frac{\alpha k_i^{\alpha-1}}{k_i^{\alpha} + 1 - k_i^{\alpha}} \\
\frac{\partial \ell_n}{\partial \gamma} &= \sum_{i=1}^{n} \frac{k_i^{\alpha}}{\alpha + (\gamma - \alpha) k_i^{\alpha}} + 2 \sum_{i=1}^{n} \frac{k_i^{\gamma} \log(k_i)}{k_i^{\alpha} + 1 - k_i^{\gamma}} \\
\frac{\partial \ell_n}{\partial \lambda} &= \frac{2n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha - 1) \sum_{i=1}^{n} \frac{k_i^{(\lambda)}}{k_i} + \sum_{i=1}^{n} \frac{\alpha(\gamma - \alpha) k_i^{\alpha-1} k_i^{(\lambda)}}{\alpha + (\gamma + \alpha) k_i^{\alpha}} \\
&\quad -2 \sum_{i=1}^{n} \frac{\alpha k_i^{(\lambda)} k_i^{\alpha-1} - \gamma k_i^{(\lambda)} k_i^{\gamma-1}}{k_i^{\alpha} + 1 - k_i^{\gamma}},
\end{aligned}$$

where

$$k_i^{(\lambda)} = \frac{\partial k_i}{\partial \lambda} = x_i e^{-\lambda x_i} \left[1 + \frac{\lambda}{\lambda + 1} x_i + \frac{1}{(1+\lambda)^2} \right].$$

The maximum likelihood estimates (MLEs) of $(\alpha; \gamma; \lambda)$, say $(\widehat{\alpha}; \widehat{\gamma}; \widehat{\lambda})$, are the simultaneous solution of the equations $\frac{\partial \ell_n}{\partial \alpha} = 0$; $\frac{\partial \ell_n}{\partial \gamma} = 0$; $\frac{\partial \ell_n}{\partial \lambda} = 0$.

For estimating the model parameters, numerical iterative techniques should be used to solve these equations. We can investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. Let $\theta = (\alpha, \gamma, \lambda)^T$, then the asymptotic distribution of $\sqrt{n}(\theta - \hat{\theta})$ is $N_4(0, K(\theta)^{-1})$, under standard regularity conditions (see Lehmann and Casella, 1998, pp. 461-463), where $K(\theta)$ is the expected information matrix. The asymptotic behavior remains valid if $K(\theta))$ is superseded by the observed information matrix multiplied by $1/n$, say $I(\theta)/n$, approximated by $\hat{\theta}$, i.e. $I(\hat{\theta})/n$. We have

$$I(\theta) = - \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\gamma} & I_{\alpha\lambda} \\ I_{\gamma\alpha} & I_{\gamma\gamma} & I_{\gamma\lambda} \\ I_{\lambda\alpha} & I_{\lambda\gamma} & I_{\lambda\lambda} \end{bmatrix},$$

where

$$I_{\alpha\alpha} = \frac{\partial^2 \ell_n}{\partial \alpha^2}; \qquad I_{\alpha\gamma} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial \alpha \partial \gamma},$$

$$I_{\gamma\lambda} = I_{\lambda\gamma} = \frac{\partial^2 \ell_n}{\partial\gamma\partial\lambda}; \qquad I_{\alpha\lambda} = I_{\alpha\gamma} = \frac{\partial^2 \ell_n}{\partial\alpha\partial\lambda}.$$

5. Simulation Study

The performance of the maximum likelihood method is evaluated for estimating the EGL parameters using a Monte Carlo simulation study. The coverage probabilities (CPs), mean square error (MSES) and the bias of the parameter estimates, estimated average lengths (ALs) are calculated. We generate $N = 10,000$ samples of sizes $n = 50, 55, ..., 500$ from the EGL distribution with $\alpha = 2$, $\gamma = 5$, $\lambda = 0.5$. Let $(\widehat{\alpha}, \widehat{\gamma}, \widehat{\lambda})$ be the MLEs of the new model parameters and $(s_{\widehat{\alpha}}, s_{\widehat{\gamma}}, s_{\widehat{\lambda}})$ be the standard errors of the MLEs. The estimated biases and MSEs are given by

$$\widehat{Bias}_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon_i} - \epsilon),$$

and

$$\widehat{MSE}_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon_i} - \epsilon)^2,$$

for $\epsilon = \alpha, \gamma, \lambda$. The CPs and ALs are given, respectively, by

$$CP_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\epsilon_i} - 1.95996 s_{\hat{\epsilon_i}}, \hat{\epsilon_i} + 1.95996 s_{\hat{\epsilon_i}}),$$

and

$$AL_\epsilon(n) = \frac{3.919928}{N}\sum_{i=1}^{N} s_{\hat{\epsilon_i}}.$$

Figure 4 displays the numerical results for the above measures. We conclude below results from these plots:

- ✓ The estimated biases decrease when the sample size n increases,
- ✓ The estimated MSEs decay toward zero as n increases,
- ✓ The CPs are near 0.95 and approach the nominal value when the sample size increases,
- ✓ The ALs decrease for all parameters when the sample size increases.

These results reveal the consistency property of the MLEs.

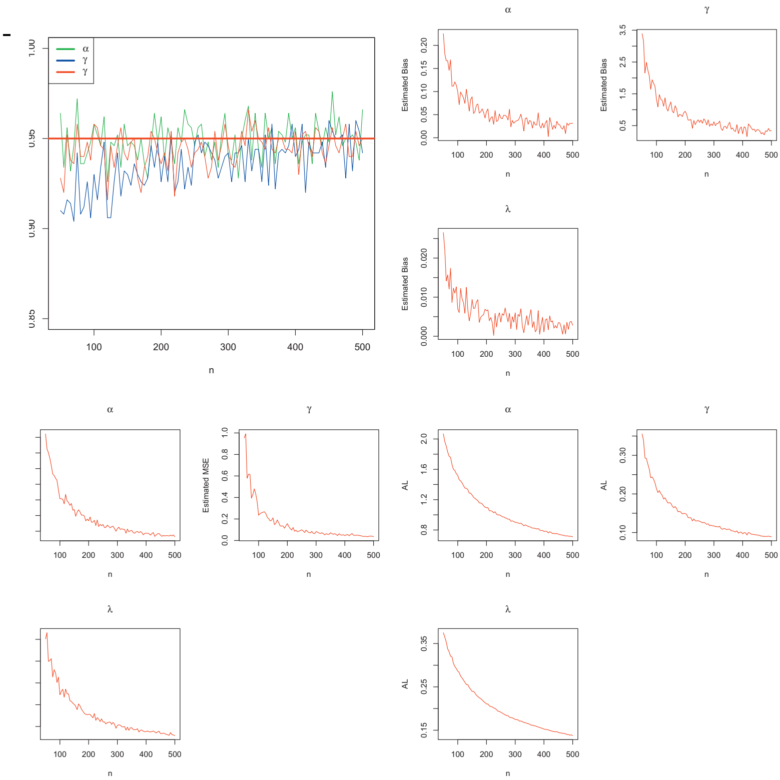

Figure 4. Estimated CPs, biases, MSEs and ALs for the selected parameter values.

6. Application

In this section, we illustrate the fitting performance of the EGL distribution using two real data sets. For the purpose of comparison, we fitted the following models to show the fitting performance of EGL distribution by means of real data set:

- Gamma Distribution, $G(\alpha, \lambda)$.
- Weibull Distribution, $W(\alpha, \lambda)$.
- Generalized Exponential Distribution, $GE(\alpha, \lambda)$, with distribution function given by
$$F(x) = \left(1 - e^{-\lambda x}\right)^{\alpha}.$$

- Lindley Distribution, $L(\lambda)$.
- Power Lindley distribution, $PL(\beta, \lambda)$.
- Generalized Lindley, $GL(\alpha, \lambda)$, (Nadarajah et al. [21], with distribution function given by

$$F(x) = \left(1 - (1 + \frac{\lambda\, x}{1+\lambda})\mathrm{e}^{-\lambda\, \mathrm{x}}\right)^{\alpha}.$$

- Beta Lindley, $BL(\alpha, \beta, \lambda)$, with distribution function given by

$$F(x) = \int_0^{L(x,\lambda)} t^{\alpha-1}(1-t)^{\beta-1}dt.$$

- Exponentiated power Lindley distribution, $EPL(\alpha, \beta, \lambda)$, with distribution function given by

$$F(x) = \left(1 - (1 + \frac{\lambda\, x^{\beta}}{1+\lambda})\mathrm{e}^{-\lambda\, \mathrm{x}^{\beta}}\right)^{\alpha}.$$

- Odd log-logistic power Lindley distribution $OLL-PL(\alpha, \beta, \lambda)$, (Alizadeh et al. [1]), with distribution function given by

$$F(x) = \frac{PL(x, \beta, \lambda)^{\alpha}}{PL(x, \beta, \lambda)^{\alpha} + (1 - PL(x, \beta, \lambda))^{\alpha}}.$$

- Kumaraswamy Power Lindley, $KPL(\alpha, \beta, \gamma, \lambda)$ (Broderick et al. [22])

$$F(x) = 1 - (1 - PL(x, \beta, \lambda)^{\alpha})^{\gamma}.$$

Estimates of the parameters of EGL distribution, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each dataset. We have also considered the Kolmogorov-Smirnov (K-S) statistic and its corresponding p-value and the minimum value of the minus log-likelihood function (-Log(L)) for the sake of comparison. Generally speaking, the smaller values of AIC, BIC, W^* and A^*, the better fit to a data set. All the computations were carried out using the software R.

Note that initial values of model parameters are quite important to obtain the correct MLEs of parameters. To avoid local minima problem, we first obtain the parameter estimate of the Lindley distribution. Then, the estimated parameter of the Lindley distribution is used as the initial value of the parameter of the PL and GL distributions. Then, the estimated parameters of PL distribution, λ and β, is used as the initial values of the EGL distribution. This approach is quite useful to obtain correct parameter estimates of extended models.

6.1 First application

Fonseca and Franca [9] studied the soil fertility in influence and the characterization of the biologic fixation of N2 for the Dimorphandra wilsonii rizz growth. For 128 plants, they made measures of the phosphorus concentration in the leaves. The data, which have also been analyzed by Bidram and Nekoukhou [5], are listed in Table 3.

Table 2: First data set.

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23	0.24	0.20
0.08	0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09	0.10	0.12	0.12	0.10
0.09	0.17	0.19	0.21	0.18	0.26	0.19	0.17	0.18	0.20	0.24	0.19	0.21	0.22	0.17
0.08	0.08	0.06	0.09	0.22	0.23	0.22	0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12
0.15	0.08	0.12	0.09	0.14	0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13
0.11	0.11	0.11	0.08	0.22	0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15
0.17	0.14	0.12	0.18	0.14	0.18	0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11	0.11
0.14	0.07	0.07	0.19	0.17	0.18	0.16	0.19	0.15	0.07	0.09	0.17	0.10	0.08	0.15
0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13							

The ML estimates of the parameters and the goodness-of-fit test statistics for the real data set is presented in Table 3 and 4 respectively. As we can see, the smallest values of AIC, BIC, A^{*}, W^{*} and $-l$ statistics and the largest p-values belong to the EGL distribution. Therefore the EGL distribution outperforms the other competitive considered distribution in the sense of this criteria.

Table 3: Parameter ML estimates and theirs standard errors (in parentheses) for first data set.

$Model$	α	β	γ	λ
$Gamma(\lambda)$	6.6141	–	–	46.9801
	(0.8067			(5.9537)
$Weibull(\lambda)$	2.8184	–	–	0.1584
	(0.1919)			(0.0052)
$GE(\lambda)$	21.1353	–	–	10.6090
	(1.648)			(2.0334)
$Lindley(\lambda)$	–	–	–	7.901
				(0.6343)
$GL(\alpha,\lambda)$	10.5303	–	–	21.9739
	(2.0203)			(1.6518)
$PL(\beta,\lambda)$	–	2.8183	–	180.7664
		(0.1919)		(60.2260)
$BL(\alpha,\beta,\lambda)$	6.3358	45.4292	–	1.5111
	(0.8070	(64.7580)		(1.5113)
$EPL(\alpha,\beta,\lambda)$	1.5150	3.5946	–	37.6386
	(0.5103)	(2.8255)		(22.0225)
$OLLL(\alpha,\lambda)$	2.9955	–	–	6.0085
	(0.2183)			(0.1971)
$OLLPL(\alpha,\beta,\lambda)$	1.1537	2.4982	–	99.0435
	(0.3756)	(0.7218)		(134.6303)
$KPL(\alpha,\beta,\gamma,\lambda)$	6.6698	0.9258	3.0713	10.7987
	(3.2657)	(0.0598)	(1.2352)	(14.3619)
$EGL(\alpha,\gamma,\lambda)$	19.3599	–	69.6518	30.4042
	(6.1287)		(53.5897)	(3.6772)

In addition, the profile log-likelihood functions of the EGL distribution are plotted in Figure 4. These plots reveal that the likelihood equations of the EGL distribution have solutions that are maximizers.

Here, we also applied likelihood ratio (LR) tests. The LR tests can be used for comparing the EGL distribution with its sub-models. For example, the test of $H_0 : \alpha = \gamma$ against $H_1 : \alpha \neq \gamma$ is equivalent to comparing the EGL and GL distributions with each other. For this test, the LR statistic can be calculated by the following relation

$$LR = 2\left[l(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\lambda}) - l(\widehat{\alpha^*}, 1, \widehat{\gamma^*}, \widehat{\lambda^*})\right],$$

Table 4. Goodness-of-fit test statistics for first data set.

$Model$	W^*	A^*	AIC	BIC	$CAIC$	$-l$
$Gamma(\lambda)$	0.1334	0.7575	-389.8834	-384.1794	-389.7874	196.9417
$Weibull(\lambda)$	0.2071	1.1569	-385.6297	-379.9256	-385.5337	194.8149
$GE(\lambda)$	0.1325	0.8167	-388.0439	-382.3398	-387.9479	196.0219
$Lindley(\lambda)$	0.1332	0.7556	-245.3218	-242.4697	-245.2900	123.6609
$GL(\alpha,\lambda)$	0.1323	0.8146	-388.0868	-382.3827	-387.9908	196.0434
$PL(\beta,\lambda)$	0.2071	1.1569	-385.6296	-379.9255	-385.5336	194.8148
$BL(\alpha,\beta,\lambda)$	0.1348	0.7607	-387.9407	-379.3846	-387.7471	196.9703
$EPL(\alpha,\beta,\lambda)$	0.1378	0.7824	-387.4424	-378.8863	-387.2489	196.7212
$OLLL(\alpha,\lambda)$	0.1883	1.0973	-383.0465	-377.3424	-382.9505	193.5232
$OLLPL(\alpha,\beta,\lambda)$	0.1986	1.1065	-383.8543	-375.2982	-383.6607	194.9271
$KPL(\alpha,\beta,\gamma,\lambda)$	0.1406	0.8002	-385.0695	-373.6613	-384.7443	196.5347
$EGL(\alpha,\gamma,\lambda)$	0.0818	0.4857	-391.0308	-382.4747	-390.8373	198.5154

where $\widehat{\alpha^*}, \widehat{\gamma^*}$ and $\widehat{\lambda^*}$ are the ML estimators of α, γ and λ, respectively, obtained under H_0. Under the regularity conditions and if H_0 is assumed to be true, the LR test statistic converges in distribution to a chi square with r degrees of freedom, where r equals the difference between the number of parameters estimated under H_0 and the number of parameters estimated in general, (for $H_0 : \beta = 1$, we have $r = 1$). Table 5 gives the LR statistics and the corresponding p-values.

Table 5. The LR test results for first data set.

	Hypotheses	LR	p-value
EGL versus Lindley	$H_0 : \alpha = \gamma = 1$	149.709	0
EGL versus GL	$H_0 : \alpha = \gamma$	4.944	0.02618

From Table 5, we observe that the computed p-values are too small so we reject all the null hypotheses and conclude that the EGL fits the first data better than the considered sub-models according to the LR criterion.

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 5. Figure 5 suggests that the EGL fits the skewed data very well.

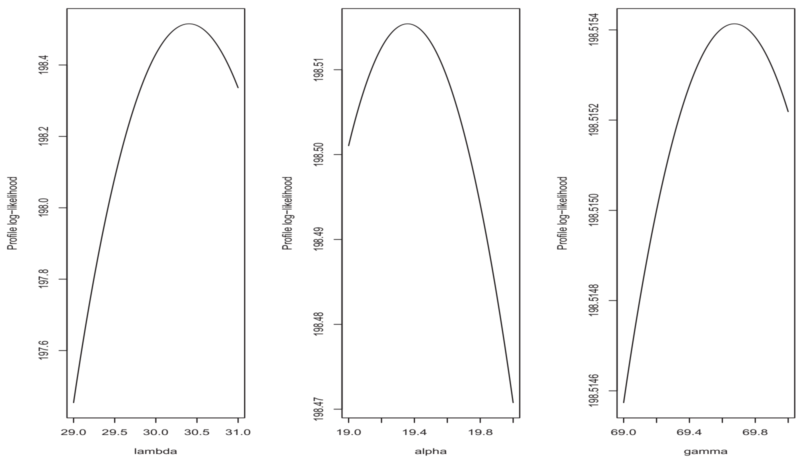

Figure 5. The profile log-likelihood functions of the EGL distribution for first data set.

6.2 Second application

And the second data set refers to the failure times of 20 mechanical components reported in Murthy et al. [20] (2004) and, more recently, in Cordeiro et al. [7].

Table 6: Second data set.

0.067	0.068	0.076	0.081	0.084	0.085	0.085	0.086	0.089	0.098
0.098	0.114	0.114	0.115	0.121	0.125	0.131	0.149	0.160	0.485

Table 6 and 7 present the parameter ML estimates and the goodness-of-fit test statistics for the second data, respectively. We see that the EGL distribution outperforms the considered models according to the considered goodness-of-fit criteria. In addition, the profile log-likelihood functions of the EGL distribution are plotted in Figure 6. These plots reveal that the likelihood equations of the EGL distribution have solutions that are maximizers.

Table 7: Parameter ML estimates and theirs standard errors (in parentheses) for second data set.

Model	α	β	γ	λ
$Gamma(\lambda)$	4.2425	–	–	34.9036
	(1.2923)			(11.287)
$Weibull(\lambda)$	1.6421	–	–	0.1375
	(0.2312)			(0.0200)
$GE(\lambda)$	27.755	–	–	13.8298
	(6.1088)			(8.3796)
$Lindley(\lambda)$	–	–	–	9.0461
				(1.8549)
$GL(\alpha,\lambda)$	13.7448	–	–	28.6133
	(8.3240)			(6.1151)
$PL(\beta,\lambda)$	–	1.6418	–	26.8802
		(0.2319)		(11.3786)
$BL(\alpha,\beta,\lambda)$	149.850	0.2595	–	76.0261
	(91.9383)	(0.0667)		(0.0028)
$EPL(\alpha,\beta,\lambda)$	0.38720	3240.02	–	21.0939
	(.0833)	(3557.231)		(3.5196)
$OLLL(\alpha,\lambda)$	3.1777	–	–	7.4678
	(0.6124)			(0.5514)
$OLLPL(\alpha,\beta,\lambda)$	52.3755	0.0639	–	1.2568
	(151.8376)	(0.1862)		(0.4314)
$KPL(\alpha,\beta,\gamma,\lambda)$	12.3697	1.3518	0.2669	98.2790
	(6.6157)	(0.5236)	(0.0720)	56.6321
$EGL(\alpha,\gamma,\lambda)$	7.6749	–	0.0661	10.8218
	(2.9370)		(0.1647)	(5.8422)

Here, we also applied likelihood ratio (LR) tests for the second data set. The LR test results for the second data set are given in Table 7. The null hypotheses are all rejected in favor of the EGL distribution since the p-values are less than 0.01.

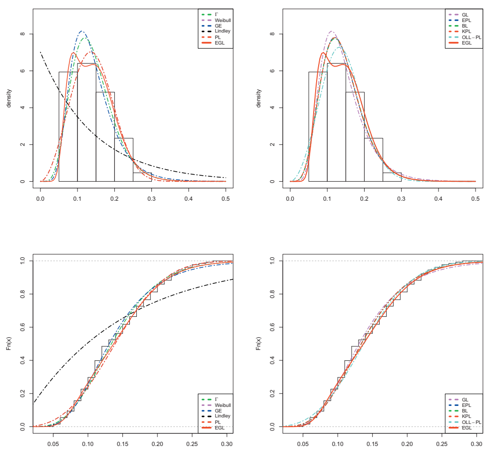

Figure 6. Fitted densities of distributions for second data set.

In addition, the profile log-likelihood functions of the EGL distribution are plotted in Figure 5. These plots reveal that the likelihood equations of the EGL distribution have solutions that are maximizers.

We also plotted the fitted pdfs and cdfs of the considered models for the sake of visual comparison, in Figure 6. Figure 6 suggests that the EGL fits the skewed data very well.

Table 8: Goodness-of-fit test statistics for second data set.

$Model$	W^*	A^*	AIC	BIC	$CAIC$	$-l$
$Gamma(\lambda)$	0.2919	1.9067	-55.78189	-53.79043	-55.07601	29.8909
$Wiebull(\lambda)$	0.3970	2.4519	-48.84565	-46.85418	-48.13977	26.4228
$GE(\lambda)$	0.2919	1.9067	-55.78189	-53.79043	-55.07601	29.8909
$Lindley(\lambda)$	0.2946	1.9209	-42.40305	-41.40732	-42.18083	22.2015
$GL(\alpha,\lambda)$	0.1767	1.2565	-61.87834	-59.88687	-61.17246	32.9392
$PL(\beta,\lambda)$	0.3981	2.4575	-48.81023	-46.81876	-48.10434	26.4051
$BL(\alpha,\beta,\lambda)$	0.1120	0.8509	-65.95215	-62.96496	-64.45215	35.9761
$EPL(\alpha,\beta,\lambda)$	0.0780	0.6086	-68.00055	-65.01335	-66.50055	37.0003
$OLLL(\alpha,\lambda)$	0.1649	1.1781	-62.48433	-60.49287	-61.77845	33.2422
$OLLPL(\alpha,\beta,\lambda)$	0.0904	0.6783	-66.10777	-63.12057	-64.60777	36.0539
$KPL(\alpha,\beta,\gamma,\lambda)$	0.2010	1.4022	-57.58809	-53.60516	-54.92142	32.7941
$EGL(\alpha,\gamma,\lambda)$	0.0663	0.4917	-68.71485	-65.72766	-67.21485	37.3574

Table 9: The LR test results for second data set.

	Hypotheses	LR	p-value
EGL versus Lindley	$H_0 : \alpha = \gamma = 1$	30.3118	< 0.0001
EGL versus GL	$H_0 : \alpha = \gamma$	8.8365	0.00295

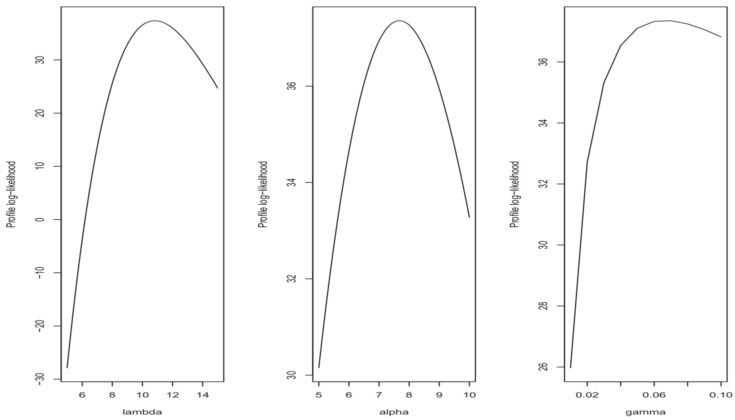

Figure 7. The profile log-likelihood functions of the EGL distribution for second data.

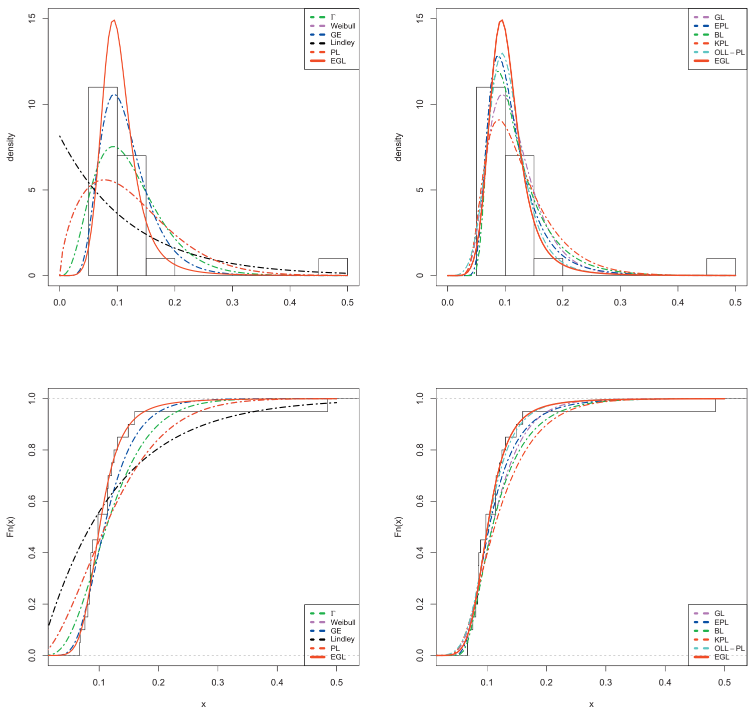

Figure 8. Fitted densities of distributions for second data set.

7. Conclusion

In this paper, a new distribution called Extended Generalized Lindley(EGL) distribution was introduced. The statistical properties of the EGL distribution including the hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation for the model parameters are given. Simulation studies was conducted to examine the performance of the new EGL distribution. We also presented applications of this new model to a real life data set in order to illustrate the usefulness of the distribution.

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