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Non-Commutative Hypervaluation on Division Hyperrings

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Abstract. In this paper, the notion of a non-commutative hypervaluation as a generalization of a non-commutative valuation is introduced and some results are proved in this respect. Also, we define the notion of a non-commutative hypervaluation over a Krasner hyperfield and some basic results and characterizations are obtained. Finally, we introduce the non-commutative discrete hypervaluation of a hyperfield and investigate the important properties.

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1. Motivation

The concept of a hypergroup was introduced in 1934 by a French mathematician Marty [23], at the 8th congress of Scandinavian mathematicians. Then many mathematicians have worked and studied on this new field and developed it. The canonical hypergroup is a special type of

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hypergroups. Mittas in 1970 [26] introduced the notion of a canonical hypergroup and then studied mainly by many authors, for example Mittas [27, 28], Corsini [9, 10, 11, 12] and De Salvo [19]. The basic results of hypergroup theory are found in [12, 13, 31]. The quasi canonical hypergroup is as a generalization of a canonical hypergroup which satisfy all conditions of a canonical hypergroup except commutativity, was introduced by Bonansinga [4] and it was studied by Bonansinga and Corsini [5, 6] and Massouros [24]. This class of hypergroups was studied by Comer independently and he named them polygroups [7, 8]. The theory of polygroups is reviewed in the book [17]. The notion of a hyperfield and a hyperring was first introduced by Krasner [21, 22] and then studied by many authors [18, 31]. There are several kinds of hyperrings that can be defined on a non-empty set R. If the addition $+$ is hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called Krasner hyperring under supplementary conditions [16]. De Salvo [18] studied hyperrings in which the addition and the multiplication were hyperoperations, called them general hyperrings. General hyperrings were also studied by Rahnamai Barghi [29] and Asokkumar and Velrajan [1, 2, 3].

A comprehensive review of hyperrings theory and its applications can be seen in the book, written by Davvaz and Leoreanu Fotea [15].

Davvaz and Salasi [14], introduced the notion of a hypervaluation on a hyperring R. For this, as in the classical case we need a mapping from R onto a totally ordered group G. The non-commutative valuation was introduced by Schilling [3], which is a natural generalization of a valuation on a field. Schilling extended the concept of a valuation on a field to that of a division ring. Recently, Mirdar and Anvariyeh introduced the notion of a hypervaluation of a hyperfield onto a totally ordered canonical hypergroup and obtain some related basic results [25]. The aim of this paper is to extend the theory of non-commutative valuation rings to the hypervaluation theory. For this purpose, a non-commutative hypervaluation on a division hyperring to a totally ordered polygroup is defined and some of their properties described. Then we introduce a non-commutative hypervaluation over a Krasner hyperfield to a totally ordered polygroup. Also a discrete hypervaluation on a Krasner hyperfield is introduced, and then some related properties are investigated.

2. Preliminaries and Basic Definitions

We shall present in this section the basic definitions and facts of the hyperstructure theory which will be used in the subsequent work. First, we recall here some of the fundamental concepts of hypergroup theory.

Let H be a non-empty set and $\circ : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. The couple $(H, \circ)$ is called a *hypergroupoid*. If A and B are non-empty subsets of H and $x \in H$, then $A \circ B = \bigcup\limits_{a \in A, b \in B} a \circ b,\ A \circ x = A \circ \{x\},\ x \circ B = \{x\} \circ B$.

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, we have $(a \circ b) \circ c = a \circ (b \circ c)$, which means that $\bigcup\limits_{u \in a \circ b} u \circ c = \bigcup\limits_{v \in b \circ c} a \circ v$.

A semihypergroup $(H, \circ)$ is called a *hypergroup* if for all $a \in H$, we have $a \circ H = H \circ a = H$. A non-empty subset K of a hypergroup $(H, \circ)$ is called a subhypergroup if it is a hypergroup. Hence, a non-empty subset K of a hypergroup $(H, \circ)$ is a subhypergroup if for all a of K we have $a \circ K = K \circ a$.

A semihypergroup $(H, \circ)$, is called a *canonical hypergroup* if: (1) it is commutative; (2) there exists $0 \in H$ such that $0 \circ a = a \circ 0 = a$ for every $a \in H$; (3) for every $a \in H$ there exists a unique element $-a \in H$ such that $0 \in a \circ (-a)$; (4) it is reversible, which means that if $c \in a \circ b$ implies $a \in c \circ (-b)$ and $b \in (-a) \circ c$. A *polygroup* is a system $< P, \cdot, e, ^{-1} >$, where $e \in P, ^{-1}$ is a unitary operation on $P, \cdot$ maps $P \times P$ onto the non-empty subsets of P, and the following axioms hold for all $a, b, c \in P$: (1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$; (2) $e \cdot a = a \cdot e = a$; (3) $a \in b \cdot c$ implies $b \in a \cdot c^{-1}$ and $c \in b^{-1} \cdot a$.

A *Krasner hyperring* is an algebraic structure $(R, \circ, \cdot)$ which satisfies satisfies the following axioms: (1) $(R, \circ)$ is a canonical hypergroup; (2) relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element, i.e., $a \cdot 0 = 0 \cdot a = 0$; (3) the multiplication is distributive with respect to the hyperoperation $\circ$, i.e., for all $a, b, c \in R$ we have $a \cdot (b \circ c) = a \cdot b \circ a \cdot c$ and $(a \circ b) \cdot c = a \cdot c \circ b \cdot c$.

A Krasner hyperring $(R, \circ, \cdot)$ is called a *Krasner hyperfield* if $(R \setminus \{0\}, \cdot)$ is a group (notice that this group is not necessary abelian).

Example 2.1. Let $(\mathbb{Z}, +, \cdot)$ be the ring of integer numbers. We define the following hyperaddition on $\mathbb{Z}$:

$$\begin{cases} a \circ b = \{a, b, a+b\} & if\ a \neq -b,\ a, b \in \mathbb{Z} \setminus \{0\}, \\ a \circ (-a) = \mathbb{Z} & for\ all\ a \in \mathbb{Z} \setminus \{0\}, \\ a \circ 0 = 0 \circ a = a & for\ all\ a \in \mathbb{Z}. \end{cases}$$

It is easy to see that $(\mathbb{Z}, \circ, \cdot)$ is a Krasner hyperring.

Now, we express the definition of hyperring (general).

A *hyperring* is an algebraic structure $(R, \circ, \cdot)$ which satisfies the following axioms: (1) $(R, \circ)$ is a canonical hypergroup; (2) $(R, \cdot)$ is a semihypergroup having 0 as a obsorbing element; (3) the multiplication is distributive with respect to the hyperoperation $\circ$.

A hyperring $(R, \circ, \cdot)$ is called *commutative* (with unit element) if $(R, \cdot)$ is a commutative semihypergroup (with unit element). A hyperring R is called a *division hyperring*, if $(R \setminus \{0\}, \cdot)$ is a hypergroup and a hyperring R is called a *hyperdomain* if $0 \in a \cdot b$ implies that $a = 0$ or $b = 0$ for all $a, b \in R$. Let $(R, \circ, \cdot)$ be a hyperring and A be a nonempty subset of R, then A is said to be *subhyperring* of R if $(A, \circ, \cdot)$ is itself a hyperring. A subhyperring I of a hyperring R is a left (right) *hyperideal* of R if $r \cdot a \subseteq I$ $(a \cdot r \subseteq I)$ for all $r \in R, a \in I$. A subhyperring I is called a hyperideal if I is both a left and a right hyperideal. A proper hyperideal M of R is a *maximal hyperideal* if there is no hyperideal I such that $M \subseteq I \subseteq R$. A hyperring R is called *local hyperring* if it has only one maximal hyperideal. A hyperideal P is called a *prime hyperideal*, if $a \cdot b \subseteq P$ implies $a \in P$ or $b \in P$.

A hyperring R is said to satisfy the *ascending (resp. descending) chain conditions* if for every ascending (resp. descending) sequence $I_1 \subseteq I_2 \subseteq$

$\cdots$ (resp. $I_1 \supseteq I_2 \supseteq \cdots$) of hyperideals of R, there exists a natural number n such that $I_n = I_k$ for all $n \geqslant k$. If R satisfies the ascending (resp. descending) chain condition, we say R is a *Noetherian (resp. Artinian)* hyperring.

Example 2.2. Consider the Krasner hyperring $(\mathbb{Z}, \circ, \cdot)$ in Example 2.1 and let I be a hyperideal of $\mathbb{Z}$. Define $\odot : \mathbb{Z}\times\mathbb{Z} \to \mathcal{P}^*(\mathbb{Z})$ by $a\odot b = ab\circ I$ for $a, b \neq 0$ and $a \odot b = \{0\}$ if $a = 0$ or $b = 0$. It is easy to check that $(\mathbb{Z}, \circ, \odot)$ is a division hyperring.

Example 2.3. Let $(\mathbb{Q}, +, \cdot)$ be the field of real numbers and consider subgroup $\mathbb{N}$ of its multiplicative semigroup. Then, an easy verification shows that $\mathbb{Q}/\mathbb{N}$ with the hyperaddition and the multiplication given by

$$\alpha\mathbb{N} \oplus \beta\mathbb{N} = \{\gamma\mathbb{N} | \gamma \in \alpha\mathbb{N} + \beta\mathbb{N}\},\ \alpha\mathbb{N} \odot \beta\mathbb{N} = \{\alpha\beta\mathbb{N}, -\alpha\beta\mathbb{N}\},$$

is a division hyperring.

Example 2.4. Consider the division hyperring $(\mathbb{Q}/\mathbb{N}, \oplus, \odot)$ in Example 2.3 and let x be an element which does not belong to $\mathbb{Q}/\mathbb{N}$. By a polynomial in x over $\mathbb{Q}/\mathbb{N}$ we mean any expression of the form $f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{k=0}^{\star} a_k x^k$, in which all a_k belong to $\mathbb{Q}/\mathbb{N}$ and symbol $\star$ means that only a finite number of a_k are non-zero, where $f(x)$ is a series with non-zero constant term. Denote the set of all polynomials in x over $\mathbb{Q}/\mathbb{N}$ by $\mathbb{Q}/\mathbb{N}[x]$. Instead of $xb = bx$ for $b \in \mathbb{Q}/\mathbb{N}$, we shall now stipulate that $xb = -bx$. Now, we define the following hyperoperations in $\mathbb{Q}/\mathbb{N}[x]$:

For all $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{m} b_k x^k$ elements of $\mathbb{Q}/\mathbb{N}[x]$, we consider

$$f(x) \circ g(x) = \{\sum_{k=0}^{r} c_k x^k | c_k \in a_k \oplus b_k\},$$

$$f(x) \otimes g(x) = \{\sum_{k=0}^{n+m} c_k x^k | c_k \in \sum_{i+j=k} (-1)^i a_i \odot b_j\}.$$

It is not difficult to see that the hyperstructure $(\mathbb{Q}/\mathbb{N}[x], \circ, \otimes)$ is a non-commutative division hyperring.

Example 2.5. We repeat the construction of Example 2.4, but we replace the hyperring $\mathbb{Q}/\mathbb{N}$, by non-commutative division hyperring $\mathbb{Q}/\mathbb{N}[x]$. Moreover, we consider an element y which does not belong to $\mathbb{Q}/\mathbb{N}[x]$. It can be easily checked that $\mathbb{Q}/\mathbb{N}[x][y] := \mathbb{Q}/\mathbb{N}[x, y]$ is a non-commutative division hyperring. We remark furthermore that each element of $\mathbb{Q}/\mathbb{N}[x, y]$ can be expressed as a $f(x, y) = \sum_{i,j=0}^{\star} c_{i,j} x^i y^j$, in which all $c_{i,j}$ belong to $\mathbb{Q}/\mathbb{N}$.

In the following, we express the definitions of totally ordered polygroups, which will be used frequently.

A polygroup $(P, +)$ is called *totally ordered* if there exists a binary relation $\leqslant$ in P such that for all $\alpha, \beta, \gamma, \delta \in P$.

1. $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$;
2. $\alpha \leqslant \alpha$;
3. if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$ then $\alpha = \beta$;
4. if $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ then $\alpha \leqslant \gamma$;
5. if $\alpha \leqslant \beta$ then $\delta + \alpha + \gamma \leqslant \delta + \beta + \gamma$, where for $A, B \subseteq P$, $A \leqslant B$ means that for all $\alpha \in A$ there exists $\beta \in B$ and for all $\beta \in B$ there exists $\alpha \in A$ such that $\alpha \leqslant \beta$.

Example 2.6. The field of real numbers $(\mathbb{R}, +, \cdot)$ with the hyperoperation like the Example 2.1, is a canonical hypergroup. $(\mathbb{R}, \circ)$ is a totally ordered canonical hypergroup with the relation $a \leqslant b$ if and only if $(b \circ (-a)) \cap \mathbb{R}^+ \neq \emptyset$. By considering the order relation $\leqslant$ in the above example, express the next example of totally ordered polygroup.

Example 2.7. Consider the canonical hypergroup $(\mathbb{R}, \circ)$ and let

$$P = \mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}.$$

Assume that e is an arbitrary fixed element of $\mathbb{R}$. We define the hyperoperation $*$ on P as follows:

$$(x_1, y_1) * (x_2, y_2) = \{(x, y) | x \in x_1 \circ x_2, y \in e^{x_2} y_1 \circ y_2\}.$$

It is easy to see that $(P, *)$ is a polygroup, where $(0, 0)$ is the identity of P and $(-x, -e^{-x}y)$ is the inverse of (x, y) in P. This polygroup is a totally ordered polygroup with the relation $(a_1, b_1) \leqslant_P (a_2, b_2)$ if $a_1 < a_2$ or $e^{-a_1}b_1 < b_2$ for $a_1 = a_2$.

3. Non-Commutative Hypervaluation on a Division Hyperring

In this section, we develop the concept of valuation on a division ring to a division hyperring. For this first define non-commutative hypervaluation and then we prove some basic results.

For simplicity, we express hypervaluation instead of non-commutative hypervaluation.

Definition 3.1. *Let $(P, +)$ be a totally ordered polygroup with order relation $\leqslant$, and ∞ a symbol satisfying the rules $\alpha + \infty = \infty + \alpha = \infty + \infty = \infty \geqslant \alpha$ for all $\alpha \in P$. Let $(D, +, \cdot)$ be a division hyperring. A hypervaluation on D is a map $\nu : D \to P \cup \{\infty\}$ satisfying the following axioms:*

1. $\nu(a) \leqslant \infty$;
2. $\nu(a) = \infty$ *if and only if* $a = 0$;
3. $\nu(-a) = \nu(a)$;
4. $\nu(a \cdot b) \subseteq \nu(a) + \nu(b)$;
5. $c \in a + b \Longrightarrow \nu(c) \geqslant \min\{\nu(a), \nu(b)\}$;

for any $a, b \in D$.

If in condition (4) equality is satisfied then we say that ν is a *good hypervaluation*.

The hypervaluation sending all $a \neq 0$ to 0 is called the *trivial hypervaluation*. Every valuation is a hypervaluation.

Remark 3.2. *Let p be a fixed prime number. If $x\mathbb{N} \in \mathbb{Q}/\mathbb{N}$ other than 0, we can write $x\mathbb{N}$ in the form $x\mathbb{N} = p^{\alpha}\frac{a}{b}\mathbb{N}$, where $a, b \in \mathbb{Z}, p \nmid a, b$ and $\alpha \in \mathbb{Z}$.*

Example 3.3. Consider the division hyperring $(\mathbb{Q}/\mathbb{N}, \oplus, \odot)$ in Example 2.3 and totally ordered canonical hypergroup $(\mathbb{R}, \circ)$ in Example 2.6. It is easy to see that the map $\nu_p : \mathbb{Q}/\mathbb{N} \longrightarrow \mathbb{R} \cup \{\infty\}$, given by $\nu_p(x\mathbb{N}) = \alpha$ for all $x\mathbb{N} \in \mathbb{Q}/\mathbb{N} \setminus \{0\}$ and $\nu_p(0) = \infty$ is a hypervaluation.

Example 3.4. Suppose $P = \mathbb{R} \times \mathbb{R}$ is the totally ordered polygroup which was mentioned in Example 2.2 and $D = \mathbb{R}/\mathbb{N}[[x, y]]$ is the non-commutative division hyperring with the hyperaddition and hyperproduct defined in Example 2.5. We define the map $\nu : \mathbb{R}/\mathbb{N}[[x, y]] \longrightarrow \mathbb{R}\times\mathbb{R}\cup\{\infty\}$ given by $\nu(0) = \infty$ and $\nu(f(x, y)) = (\nu_p(c_{0,0}), \min\{\nu_p(c_{i,j})\})$. One can easily check that ν is a hypervaluation.

Lemma 3.5. *Let ν be a good hypervaluation and be a surjective mapping. If P is non-commutative then D is certainly non-commutative division hyperring.*

Proof. This follows, from ν is surjective and equality in (4). $\square$

Lemma 3.6. *If $\nu(a) \neq \nu(b)$, then $\nu(c) = \min\{\nu(a), \nu(b)\}$, for any $c \in a + b$.*

Proof. Let $c \in a + b$, $\nu(c) > \{\nu(a), \nu(b)\}$ and $\nu(a) < \nu(b)$. Then $\nu(c) > \min\{\nu(a), \nu(b)\} = \nu(a)$. Since $c \in a + b$, we have $a \in c - b$, hence $\nu(a) \geqslant \min\{\nu(c), \nu(b)\} > \nu(a)$, which is a contradiction. $\square$

Proposition 3.7. *Let D be a division hyperring with hypervaluation ν and multiplicative hypergroup $D^\star$. Then*

$$U = \{a \in D^\star | \nu(a) = 0\},$$

is a subhypergroup of $D^\star$.

Proof. For every $u_1, u_2 \in U$, we have $\nu(u_1 \cdot u_2) \subseteq \nu(u_1) + \nu(u_2) = 0$ and $\nu(u_2 \cdot u_1) \subseteq \nu(u_2) + \nu(u_1) = 0$, i.e. $u_1 \cdot u_2, u_2 \cdot u_1 \subseteq U$. Moreover, there exist $a, b \in D$ such that $u_1 \in a \cdot u_2$ and $u_1 \in u_2 \cdot b$ and consequently $\nu(u_1) \in \nu(a) + \nu(u_2)$ and $\nu(u_1) \in \nu(u_2) + \nu(b)$, which implies $\nu(a) = 0$ and $\nu(b) = 0$, hence $a, b \in U$. Thus U is a subhypergroup of $D^\star$. $\square$

Proposition 3.8. *Let $(P, +, \leqslant)$ be a totally ordered polygroup, and let $\nu : D \to P \cup \{\infty\}$ be a hypervaluation of a division hyperring D. Then*

$$R = \{a \in D | \nu(a) \geqslant 0\},$$

is a subhyperring of D.

Proof. Let $a, b \in R$, then $\nu(a), \nu(b) \geqslant 0$, we must show that $a - b \subseteq R$ and $a \cdot b \subseteq R$. For every $c \in a - b, \nu(c) \geqslant \min\{\nu(a), \nu(b)\} \geqslant 0$, which means that $a - b \subseteq R$. Also, we have $\nu(a \cdot b) \subseteq \nu(a) + \nu(b) \geqslant 0$, thus $a \cdot b \subseteq R$. $\square$

Definition 3.9. *A subhyperring R of a division hyperring D is called a hypervaluation hyperring of D if there is a totally ordered polygroup P and a hypervaluation $\nu : D \to P \cup \{\infty\}$ of D such that $R = \{a \in D | \nu(a) \geqslant 0\}$.*

Example 3.10. Any division hyperring is a hypervaluation hyperring.

Lemma 3.11. *Let R be a hypervaluation hyperring associated to the hypervaluation ν we put*

$$M = \{a \in D | \nu(a) > 0\} = R \setminus U.$$

M is a prime left (and prime right) hyperideal of R.

Proof. Let $a, b \in M$ and $r \in R$. Then

- for all $c \in a + b$, $\nu(c) \geqslant \min\{\nu(a), \nu(b)\} > 0$, that is $a + b \subseteq M$.
- $\nu(ra) \subseteq \nu(r) + \nu(a) > 0$ and $\nu(ar) \subseteq \nu(a) + \nu(r) > 0$, that is, $ra, ar \subseteq M$.

Thus, M is a hyperideal of R. Show that M is the prime hyperideal of R. Let $a, b \in R$ be elements such that $a \cdot b \subseteq M$ and $a \notin M$. Since $a \in R$ and $a \notin M$, then $\nu(a) = 0$. From $a \cdot b \subseteq M$, we have $\nu(a \cdot b) > 0$ and so $0 < \nu(a \cdot b) \subseteq \nu(a) + \nu(b)$, which implies $b \in M$. $\square$

Lemma 3.12. *Let R be the hypervaluation hyperring of a division hyperring D with a hypervaluation ν, and $a, b \in R$. Then the following statements are equivalent:*

1. $a \in bc_1$ *with* $c_1 \in R$;
2. $a \in c_2 b$ *with* $c_2 \in R$;
3. $\nu(a) \geqslant \nu(b)$.

Proof. $(1), (2) \Rightarrow (3)$ Suppose $a \in bc_1, c_2b$ with $c_1, c_2 \in R$. Then $\nu(a) \in \nu(bc_1) \subseteq \nu(b) + \nu(c_1) = \nu(c_2) + \nu(b) \geqslant \nu(b)$. Hence $\nu(a) \geqslant \nu(b)$.
$(3) \Rightarrow (1), (2)$ Suppose $\nu(a) \geqslant \nu(b)$ and $a, b \neq 0$. Since $a, b \in D^\star$ and $D^\star$ is a hypergroup there exists $c_1, c_2 \in D^\star$ such that $a \in c_1 b$ and $a \in bc_2$. Let $c_1, c_2 \notin R$. Then $\nu(c_1) < 0$ and $\nu(c_2) < 0$. Hence $\nu(c_1) + \nu(b) < \nu(b)$ and $\nu(b) + \nu(c_2) < \nu(b)$, therefore $\nu(a) < \nu(b)$ which is contradiction. Now suppose that $b = 0$. Then $\nu(b) = \infty$ and so $\nu(a) = \infty$, hence $a = 0$. This means that a is again both a left and a right multiple of b. In the case that $a = 0$ clearly satisfied. □

The following fact is an immediate consequence of Lemma 3.11.

Corollary 3.13. *Let R be the hypervaluation hyperring of a division hyperring D with a hypervaluation ν. Then $aR \subseteq bR$ or $bR \subseteq aR$ for any $a, b \in R$.*

In the following theorem we show that every right hyperideal of a hypervaluation hyperring is a left hyperideal and conversely.

Theorem 3.14. *Let R be the hypervaluation hyperring of a division hyperring D with a hypervaluation ν. Each hyperideal of R is two sided.*

Proof. Suppose that I is a left hyperideal of R, that is, $rI \subseteq I$, for any $r \in R$. Let $x \in \sum_{i=1}^{n} y_i r_i$ be an arbitrary element of the set IR, where $y_i \in I, r_i \in R$. Then $\nu(y_i r_i) \subseteq \nu(y_i) + \nu(r_i) \geqslant \nu(y_i)$. Consequently, by Lemma 3.11, $y_i r_i \subseteq b_i y_i$ for some $b_i \in R$. Therefore $x \in \sum_{i=1}^{n} b_i y_i \subseteq RI \subseteq I$. Thus I is a right hyperideal. □

Theorem 3.15. *Let R be the hypervaluation hyperring of a division hyperring D with a hypervaluation ν. Then any finitely generated hyperideal of R is principal.*

Proof. Let $I = a_1R + a_2R + \cdots + a_nR$, where $a_i \in R$. Since R is a hypervaluation hyperring then we can choose among the elements $a_1, \cdots, a_n$ an element with the minimal value. Without loss of generality, we can consider that $\nu(a_i) \geqslant \nu(a_1)$ for all i. Then by Lemma 3.11, this means that $a_iR \subseteq a_1R$. So $I = a_1R$. $\square$

Lemma 3.16. *Let R be a hyperring with a division hyperring D. If the set of right (left) principal hyperideals of R is linearly ordered by inclusion, then the set of all hyperideals of R is linearly ordered by inclusion.*

Proof. Let I and J be right hyperideals of R. Suppose that I is not contained in J. Choose a nonzero element $a \in I \setminus J$. Let b be any element of J. Since $a \notin J, a \notin bR$, and so $aR \nsubseteq bR$. Therefore, by assumption, $bR \subseteq aR \subseteq I$. It follows that $J \subseteq I$. $\square$

Theorem 3.17. *Let R be a hypervaluation hyperring of some hypervaluation ν on division hyperring D. Then the set of all hyperideals of R is linearly ordered by inclusion.*

Proof. Let $a, b \in R$ and $\nu(a) \geqslant \nu(b)$. Then from Lemma 3.11 it follows that $a \in bR$ and $a \in Rb$. Therefore $aR \subseteq bR$ and $Ra \subseteq Rb$. So the set of right (left) principal hyperideals of R is linearly ordered by inclusion. Hence from Lemma 3.12, the set of all hyperideals of R is linearly ordered by inclusion. $\square$

4. Non-Commutative Hypervaluation Over Krasner Hyperfield

In this section we consider the hyperfield (Krasner) as a special case of division hyperring and non-commutative hypervaluation over hyperfield considered and expressed some properties of them. Note that all of the above results are satisfied.

Definition 4.1. *Let $(P, +)$ be a totally ordered polygroup with order relation $\leqslant$ and $(F, +, \cdot)$ be a hyperfield. A hypervaluation on F is surjective map $\nu : F \to P \cup \{\infty\}$ satisfying the following axioms:*

1. *$\nu(a) \leqslant \infty$;*

2. *$\nu(a) = \infty$ if and only if $a = 0$;*
3. *$\nu(-a) = \nu(a)$;*
4. *$\nu(a \cdot b) \in \nu(a) + \nu(b)$;*
5. *$c \in a + b \Longrightarrow \nu(c) \geqslant \min\{\nu(a), \nu(b)\}$;*

for any $a, b \in F$.

Lemma 4.2. *Let $a, b \in F^{\star}$, then at least one of the following cases holds $\{ab^{-1}, b^{-1}a\} \subseteq R$ or $\{a^{-1}b, ba^{-1}\} \subseteq R$.*

Proof. Suppose that $ab^{-1} \in R$. Then $b^{-1}a \in R$ by Lemma 3.11. Now assume that $ab^{-1} \notin R$. Then by the definition of R, $\nu(ab^{-1}) < 0$. Consequently $\nu(b) - \nu(a) > 0$ and thus $a^{-1}b \in R$ and $ba^{-1} \in R$. □

Lemma 4.3. *Let R be a hypervaluation hyperring of a hyperfield F with respect to hypervaluation ν on F. If $M = R \setminus U$, where $U = U(R)$ is the group of units R, then $dRd^{-1} = R$ and $dMd^{-1} = M$ for any $d \in F^{\star}$.*

Proof. Suppose that $dRd^{-1} \nsubseteq R$ for some $d \in F^{\star}$. Then there is an element $x = dyd^{-1} \in dRd^{-1}$ with $y \in R$ and $x \notin R$. Therefore $\nu(x) < 0$ and $\nu(y) \geqslant 0$. On the other hand $y^{-1} = d^{-1}x^{-1}d$, and so $\nu(y^{-1}) \in \nu(d^{-1}) + \nu(x^{-1}) + \nu(d) > \nu(d^{-1}) + \nu(d)$, hence $\nu(y^{-1}) > 0$. This contradiction shows that $dRd^{-1} = R$, for any $d \in F^{\star}$. Now suppose that $dMd^{-1} \nsubseteq M$, for some $d \in F^{\star}$. Then there is an element $x = dyd^{-1} \in dMd^{-1}$ with $y \in M$ and $x \notin M$. Since $dMd^{-1} \subset dRd^{-1} = R$ and $R = M \cup U, x \in U$. So $\nu(y^{-1}) \in \nu(d^{-1}) + \nu(d) \geqslant 0$, hence $\nu(y^{-1}) = 0$. This contradiction shows that $dMd^{-1} = M$. □

In the next theorem we express the equivalent definition of a hypervaluation hyperring.

Theorem 4.4. *Let R be a subhyperring of a hyperfield F. Then the following are equivalent:*

1. *R is a hypervaluation hyperring with respect to some hypervaluation ν on F.*
2. *$dRd^{-1} = R$ for any $d \in F^{\star}$ and for any element $x \in F^{\star}$ either $x \in R$ or $x^{-1} \in R$.*

Proof. $1 \Rightarrow 2$ $dRd^{-1} = R$ for any $d \in F^\star$, by Lemma 4.3. Suppose $x \in F^\star$ and $x \notin R$ which means that $\nu(x) < 0$. Then $0 = \nu(1) = \nu(x \cdot x^{-1}) \in \nu(x) + \nu(x^{-1})$, hence $\nu(x^{-1}) = -\nu(x) \geqslant 0$. Thus $x^{-1} \in R$.
$2 \Rightarrow 1$ Suppose that $U = U(R)$ be the group of units R. Let $u \in U$ and $d \in F^\star$. Then $x = dud^{-1} \in R$ and $x^{-1} = d^{-1}u^{-1}d \in R$. Therefore $x, x^{-1} \in U$, hence $dUd^{-1} = U$ for any $d \in F^\star$. Let $M = R \setminus U$ and $d \in F^\star$. We can show that $dMd^{-1} = M$. Assume that $dMd^{-1} \neq M$. Hence there exists an element $x = dyd^{-1} \in dMd^{-1}$ with $y \in M$ and $x \notin M$. Note that $x \in R$, therefore $x \in U$ and $y = d^{-1}xd \in U$. So $y \in M \cap U = \emptyset$. This contradiction shows that $d^{-1}Md = M$. Since U is a subgroup of $F^\star$, we can consider $P = F^\star//U$ as an polygroup and define the map $\nu : F \to P \cup \{\infty\}$ such that $\nu(d) = UdU$ for each $d \in F^\star$, $\nu(u) = U$ for any $u \in U$ and $\nu(0) = \infty$. Then ν is a surjective map with $ker\nu = U$. Let $a, b \in F^\star$ and $\nu(x) \leqslant \infty$ for all $x \in F$. The polygroup P to be a totally ordered, if $\nu(a) > \nu(b)$ in case ab^{-1} and $b^{-1}a$ lie in M. With these assumptions ν is a hypervaluation of F with hypervaluation hyperring R. $\square$

5. Non-Commutative Discrete Hypervaluation

The aim of this section is to define non-commutative discrete hypervaluation and prove that every discrete hypervaluation hyperring is Noetherian hyperring.

Definition 5.1. *A discrete hypervaluation on hyperfield F is a surjective function $\nu : F \to \mathbb{Z} \cup \{\infty\}$ satisfying*

1. $\nu(a) = \infty$ *if and only if* $a = 0$;
2. $\nu(a \cdot b) = \nu(a) + \nu(b)$;
3. $c \in a + b \Longrightarrow \nu(c) \geqslant \min\{\nu(a), \nu(b)\}$;

A subhyperring R of a hyperring F is called the *discrete hypervaluation hyperring* of ν if $R = \{a \in F | \nu(a) \geqslant 0\}$.

Lemma 5.2. *Let R be a discrete hypervaluation hyperring of a hyperfield F with respect to discrete hypervaluation ν. Then $U = U(R)$, where*

$U(R)$ is the group of discrete hypervaluation units of F, and $U = \{a \in D | \nu(a) = 0\}$.

Proof. Suppose that $u \in U(R)$, then there is element $w \in U(R)$ such that $1 = uw$. Therefore $0 = \nu(uw) = \nu(u) + \nu(w)$. So $\nu(u) = \nu(w) = 0$ since $\nu(u) \geqslant 0$ and $\nu(w) \geqslant 0$. Conversely, suppose $u \in U$. Then $u^{-1} \in F^{\star}$ and $\nu(u^{-1}) = -\nu(u) = 0$. Hence $u, u^{-1} \in R$, which means that $u \in U(R)$. $\square$

Theorem 5.3. *Let R be a discrete hypervaluation hyperring of hyperfield D with respect to a discrete hypervaluation ν. Let t be a fixed element of R with $\nu(t) = 1$. Then*

1. *R is a local hyperdomain with the nonzero maximal hyperideal $M = \{a \in R | \nu(a) > 0\}$.*
2. *Any nonzero element $a \in R$ has a unique representation in the form $a = t^n u = wt^n$, for some $u, w \in U(R)$, and $n \in \mathbb{Z}, n \geqslant 0$.*
3. *Any one-sided hyperideal I of R is a two-sided hyperideal and has the form $I = t^n R = Rt^n$ for some $n \in \mathbb{Z}, n \geqslant 0$. R is a principal hyperideal hyperring (Recall that a hyperring R is called a principal hyperideal hyperring if each one-sided hyperideal of R is principal). In particular, $M = tR = Rt$, and $I = M^n = t^n R = Rt^n$.*
4. *R is a Noetherian hyperring.*

Proof. (1) Since a discrete hypervaluation hyperring is a particular case of a division hyperring, follows from Lemma 3.11, M is hyperideal of R. Show that M is the maximal hyperideal of R. Suppose that I is an hyperideal of R such that $M \subset I \subseteq R$. Since $M = R \setminus U$ where $U = \{a \in K | \nu(a) = 0\}$, there is a $u \in I$ such that $\nu(u) = \nu(u^{-1}) = 0$ and $u^{-1} \in R$. Consequently, $1 = uu^{-1} \in I$. Thus, $I = R$, i.e. M is a maximal hyperideal of R. Since $M = R \setminus U$, therefore R is a local hyperring, and M is the unique maximal hyperideal of R.

(2) Let t be a fixed element of R with $\nu(t) = 1$, and $a \in R$ with $\nu(a) = n \geqslant 0$. Then $t \in M$, and $\nu(at^{-n}) = \nu(a) - n = 0 = \nu(t^{-n}a)$. Therefore

from Lemma 5.2 it follows that $at^{-n} = u \in U(R)$ and similarly $t^{-n}a = w \in U(R)$. Hence $a = ut^n = t^n w$.

(3) Since R is a hypervaluation hyperring, any one-sided hyperideal of R is two sided. Suppose that I is a hyperideal of R. Choose in I an element a with a minimal value $\nu(a) = n$. Then $a = t^n u = wt^n$ with $u, w \in U(R)$. Therefore $t^n R \subseteq I$ and $Rt^n \subseteq I$. Let $b \in I$, then $b = t^m w$ with $m \geqslant n$. So $\nu(t^{-n}b) \geqslant 0$, hence $t^{-n}b \in R$ and $b \in t^n R$. Therefore $I = t^n R$. Analogously, $I = Rt^n$. In particular, since $t \in M$, $M = tR = Rt$, and $M^n = t^n R = Rt^n = I$.

(4) This follows from (3), since every hyperideal of R is finitely generated. □

Definition 5.4. *The hyperdomain R is said to be Euclidean hyperdomain if there is a function $N : R \to \mathbb{N} \cup \{0\}$ such that for any two elements a and b of R with $b \neq 0$, there exist elements q and r in R with $a \in bq + r$ where $r = 0$ or $N(r) < N(b)$.*

Theorem 5.5. *Discrete hypervaluation hyperring are Euclidean.*

Proof. We define $N : R \to \mathbb{N} \cup \{0\}$ by $N(0) = 0$ and $N(r) = \nu(r)$ if $0 \neq r \in R$. To show that Euclidean property holds, suppose $a, b \in R$ and $b \neq 0$. We have to find $q, r \in R$ with $a \in bq + r$ and $r = 0$ or $N(r) < N(b)$. If $\nu(a) \geqslant \nu(b)$ then $\nu(ab^{-1}) \geqslant 0$. So $q = b^{-1}a \in R$ by Lemma 4.2, and we can let $r = 0$. Suppose $\nu(a) < \nu(b)$. This case is easy $q = 0$ and $r = a$. □

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