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New Fixed Point Results under Generalized C-Distance in tvs-Cone B-Metric Spaces with an Application to Systems of Fredholm Integral Equations

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Abstract. In this paper, we define a generalized c-distance in tvs-cone b-metric spaces and introduce some results about its properties. Then we prove some new fixed point and common fixed point results (with the underlying cone which is not normal). Respective results concerning mappings without periodic points are also deduced. Some examples are presented to validate our obtained results. An application to system of Fredholm integral equations is presented.

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1. Introduction

Ordered normed spaces and cones have many applications in applied mathematics. Hence, fixed point theory in K-metric and K-normed spaces

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was developed in the mid-20th century (see [7, 21]). In 2007, Huang and Zhang [11] reintroduced such spaces under the name of cone metric spaces by substituting the set of real numbers by an ordered normed space and obtained some fixed point results. Topological vector space-valued version of these spaces was treated in [9, 13] (see also the references contained therein). On the other hand, the concept of b-metric space (or metric-type space) was studied by Bakhtin [2] and Czerwik [6]. Then analogously with definition of a b-metric space, Cvetković et al. [5] defined cone b-metric spaces or (cone metric-type spaces) and proved several fixed and common fixed point theorems. Also, topological vector space-valued version of this concept was defined in [10].

In 1996, Kada et al. [15] introduced the concept of w-distance in metric spaces, where nonconvex minimization problems were treated. Further, Cho et al. [4] defined the concept of c-distance which is a cone version of the w-distance. Then some fixed point results under w-distance in metric spaces and under c-distance in cone metric spaces and tvs-cone metric spaces were proved in [8, 16, 17, 20] (see also the references cited therein). Recently, Hussain et al. [12] defined the concept of wt-distance on a b-metric space and proved some fixed point theorems under wt-distance in a partially ordered b-metric space. Also, very recently, Bao et al. [3] defined generalized c-distance in cone b-metric spaces and obtained several fixed point results in ordered cone b-metric spaces.

In the present work, generalized c-distance in the framework of tvs-cone b-metric spaces is introduced and fixed point and common fixed point results for mappings in tvs-cone b-metric spaces are proved under contractive conditions expressed in the terms of generalized c-distance with the underlying cone which may be not normal. Respective results concerning mappings without periodic points are also deduced. Examples are given to distinguish these results from the known ones.

As an application, sufficient conditions are obtained for the existence of solution for a system of Fredholm integral equations.

2. Preliminaries

Let E be a real Hausdorff topological vector space (*tvs* for short) with the zero vector θ. A proper nonempty and closed subset P of E is called a cone if $P+P\subset P$, $\lambda P\subset P$ for $\lambda\geqslant 0$ and $P\cap(-P)=\{\theta\}$. Given a cone $P\subset E$, we define a partial ordering $\preceq$ with respect to P by $x\preceq y$ if and only if $y-x\in P$. We shall write $x\prec y$ if $x\preceq y$ and $x\neq y$. Also, we write $x\ll y$ if and only if $y-x\in \operatorname{int} P$ where $\operatorname{int} P$ is the interior of P. If $\operatorname{int} P\neq\emptyset$, then the cone P is called solid. The pair (E,P) is an ordered topological vector space.

For a pair of elements $x,y\in E$ such that $x\preceq y$, put $[x,y]=\{z\in E : x\preceq z\preceq y\}$. A subset A of E is said to be order-convex if $[x,y]\subset A$, whenever $x,y\in A$ and $x\preceq y$. Ordered topological vector space (E,P) is order-convex if it has a base of neighborhoods of θ consisting of order-convex subsets. In this case, the cone P is said to be normal. If E is a normed space, this condition means that the unit ball is order-convex, which is equivalent to the condition that there is a number K such that $x,y\in E$ and $\theta\preceq x\preceq y$ imply that $\|x\|\leqslant K\|y\|$.

Theorem 2.1. ([19]) *If the underlying cone of an ordered tvs is solid and normal, then such tvs is an ordered normed space.*

Definition 2.2. *Let X be a nonempty set, (E,P) be an ordered tvs and $s\geqslant 1$ be a real number. A function $d: X\times X\to E$ is called a tvs-cone b-metric and (X,d) is called a tvs-cone b-metric space if the following conditions hold:*

(d_1) *$\theta\preceq d(x,y)$ for all $x,y\in X$ and $d(x,y)=\theta$ if and only if $x=y$;*

(d_2) *$d(x,y)=d(y,x)$ for all $x,y\in X$;*

(d_3) *$d(x,z)\preceq s\big(d(x,y)+d(y,z)\big)$ for all $x,y,z\in X$.*

Obviously, for $s=1$, *tvs*-cone b-metric space is a *tvs*-cone metric space in the sense of [9]. If we replace E by a real Banach space in Definition 2.2, we get the cone b-metric space in the sense of [5]. It is evident that Definition 2.2 coincides with the definition of b-metric spaces if we replace E by the set of real numbers and P by the set of nonnegative real numbers.

In the sequel, E will always denote a topological vector space, with the zero vector θ and with order relation $\preceq$, generated by a solid cone P. For notions such as convergent and Cauchy sequences, completeness, continuity etc. in *tvs*-cone b-metric spaces, we refer to [5, 10]. Also, we shall make use of the following properties when the cone P may be nonnormal.

(p_1) If $u, v, w \in E, u \preceq v$ and $v \ll w$ then $u \ll w$.

(p_2) If $u \in E$ and $\theta \preceq u \ll c$ for each $c \in \operatorname{int} P$ then $u = \theta$.

(p_3) If $u_n, v_n, u, v \in E, \theta \preceq u_n \preceq v_n$ for each $n \in \mathbb{N}$, and $u_n \to u, v_n \to v$ $(n \to \infty)$, then $\theta \preceq u \preceq v$.

(p_4) If $x_n, x \in X, u_n \in E, d(x_n, x) \preceq u_n$ and $u_n \to \theta$ $(n \to \infty)$, then $x_n \to x$ $(n \to \infty)$.

(p_5) If $u \preceq \lambda u$, where $u \in P$ and $0 \leqslant \lambda < 1$, then $u = \theta$.

(p_6) If $c \gg \theta$ and $u_n \in E, u_n \to \theta$ $(n \to \infty)$, then there exists n_0 such that $u_n \ll c$ for each $n > n_0$.

In the following definition, we extend the concept of generalized c-distance in cone b-metric spaces (introduced by Bao et al. [3]) to the setting of *tvs*-cone b-metric spaces.

Definition 2.3. *Let (X, d) be a tvs-cone b-metric space with parameter $s \geqslant 1$. A function $q : X \times X \to E$ is called a generalized c-distance on X if the following properties are satisfied:*

(q_1) *$\theta \preceq q(x, y)$ for all $x, y \in X$;*

(q_2) *$q(x, z) \preceq s[q(x, y) + q(y, z)]$ for all $x, y, z \in X$;*

(q_3) *for $x \in X$ and a sequence $\{y_n\}$ in X, converging to $y \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$ and all $n \geqslant 1$, then $q(x, y) \preceq su$;*

(q_4) *for all* $c \in E$ *with* $\theta \ll c$, *there exists* $e \in E$ *with* $\theta \ll e$ *such that* $q(z, x) \ll e$ *and* $q(z, y) \ll e$ *imply* $d(x, y) \ll c$.

Remark 2.4. *Each wt-distance in a b-metric space (in the sense of Hussain et al. [12]) is a generalized c-distance in the tvs-cone b-metric space* (X, d) *with* $E = \mathbb{R}$ *and* $P = [0, \infty)$. *Indeed, only property* (q_3) *has to be checked. Let* $y_n \in X$, $y_n \to y$ $(n \to \infty)$ *in the tvs-cone b-metric d, and let* $q(x, y_n) \leqslant u_x \in [0, +\infty)$. *Since q is (as a wt-distance) lower semi-continuous, we have that* $q(x, y) \leqslant \liminf_{n\to\infty} sq(x, y_n) \leqslant \liminf_{n\to\infty} su_x = su_x$, *i.e.,* $q(x, y) \leqslant su_x$ *holds true. But the converse does not hold. Thus, generalized c-distance is a generalization of wt-distance. Also, for* $s = 1$, *generalized c-distance is a c-distance of [4]. In this manner, if we consider* $E = \mathbb{R}$ *and* $P = [0, \infty)$, *then we obtain the definition of w-distance introduced by Kada et al. [15].*

Now, we give some examples in the framework of *tvs*-cone *b*-metric spaces.

Example 2.5. Let (X, d) be a *tvs*-cone *b*-metric space such that the metric $d(\cdot, \cdot)$ is a continuous function in second variable. Then, $q(x, y) = d(x, y)$ is a generalized *c*-distance. Indeed, only property (q_3) is nontrivial and it follows from $q(x, y_n) = d(x, y_n) \preceq u$, passing to the limit when $n \to \infty$ and using continuity of d.

The following two examples are variations of the examples from paper [3] adjusted to the case of a *tvs*-cone *b*-metric.

Example 2.6. Let (X, d) be a *tvs*-cone *b*-metric space and let $u \in X$ be fixed. Then $q(x, y) = \frac{1}{s} d(u, y)$ defines a generalized *c*-distance on X. Indeed, (q_1) and (q_3) are clear. Also, (q_2) follows from $sq(x, z) = sd(u, z) \preceq s^2\big(d(u, y) + d(u, z)\big)$, i.e., $q(x, z) \preceq sq(x, y) + sq(y, z)$. Finally, ($q_4$) is obtained by taking $e = \frac{c}{2s^2}$.

Example 2.7. Let $E = C^1_{\mathbb{R}}[0, 1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the nonnormal cone $P = \{x \in E : x(t) \geqslant 0 \text{ on } [0, 1]\}$. Also, let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|^s \psi$ for all $x, y \in X$, where $s \in \{1, 2\}$ and $\psi : [0, 1] \to \mathbb{R}$ is given as $\psi(t) = 2^t$. Then (X, d) is a *tvs*-cone *b*-metric space with $s \in \{1, 2\}$. Define a mapping $q : X \times X \to E$ by $q(x, y) = y^s \psi$ for all $x, y \in X$. Then q is a generalized *c*-distance.

Example 2.8. Consider the Banach space $E = C_{\mathbb{R}}[0,1]$ of real-valued continuous functions with the max-norm and ordered by the cone $P = \{f \in E : f(t) \geqslant 0 \text{ for } t \in [0,1]\}$. This cone is normal in the Banach-space topology on E. Let τ^* be the strongest locally convex topology on the vector space E. Then, the cone P is solid, but it is not normal in the topology τ^*. Indeed, if this were the case, Theorem 2.1 would imply that the topology τ^* is normed, which is impossible since an infinite dimensional space with the strongest locally convex topology cannot be metrizable (see, e.g., [13]).

Let now $X = [0,+\infty)$ and $d : X \times X \to (E,\tau^*)$ be defined by $d(x,y)(t) = |x-y|^s\varphi(t)$ with $s \in \{1,2\}$ for a fixed element $\varphi \in P$. Then (X,d) is a *tvs*-cone b-metric space which is not a cone b-metric space in the sense of [5]. We can introduce two c-distances on this space:

$$q_1(x,y)(t) = y^s\varphi(t), \quad \text{and} \quad q_2(x,y)(t) = (x^s+y^s)\varphi(t).$$

They are examples of generalized c-distances in *tvs*-cone b-metric spaces which are not generalized c-distances in cone b-metric spaces of [3].

These examples show that for a generalized c-distance q in *tvs*-cone b-metric spaces:

- $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$;
- $q(x,y) = \theta$ is not necessarily equivalent to $x = y$.

We will call a sequence $\{u_n\}$ in P a generalized c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geqslant n_0$. It is easy to show that if $\{u_n\}$ and $\{v_n\}$ are c-sequences in E and $\alpha,\beta > 0$ then $\{\alpha u_n + \beta v_n\}$ is a c-sequence. Note that in the case that the cone P is normal, a sequence in E is a c-sequence if and only if it is a θ-sequence. However, when the cone is not normal, a c-sequence need not be a θ-sequence.

Lemma 2.9. *Let (X,d) be a tvs-cone b-metric space and q be a generalized c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x,y,z \in X$, and $\{u_n\}$ and $\{v_n\}$ be two c-sequences in P. Then the following hold:*

(qp_1) *If* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$*, then* $y = z$*. In particular, if* $q(x, y) = \theta$ *and* $q(x, z) = \theta$*, then* $y = z$*.*

(qp_2) *If* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$*, then* $\{y_n\}$ *converges to* z*.*

(qp_3) *If* $q(x_n, x_m) \preceq u_n$ *for* $m > n > n_0$ *(for some* $n_0 \in \mathbb{N}$*), then* $\{x_n\}$ *is a Cauchy sequence in* X*.*

(qp_4) *If* $q(y, x_n) \preceq u_n$ *for* $n \in \mathbb{N}$*, then* $\{x_n\}$ *is a Cauchy sequence in* X*.*

Proof. Since the proof is easy and similar as in the case of c-distance in *tvs*-cone metric spaces in [8], we omit it. □

3. Fixed Point Results

Our first result in this section is the following fixed point theorem of Hardy-Rogers type under generalized c-distance in a *tvs*-cone b-metric space without normality condition on the cone.

Theorem 3.1. *Let* (X, d) *be a complete tvs-cone b-metric space with coefficient* $s \geqslant 1$ *and let* q *be a generalized c-distance on* X*. Suppose that a continuous self-map* $f : X \to X$ *satisfies the following two conditions:*

$$q(fx, fy) \preceq \alpha_1 q(x, y) + \alpha_2 q(x, fx) + \alpha_3 q(y, fy) + \alpha_4 q(x, fy) + \alpha_5 q(y, fx), \quad (1)$$

$$q(fy, fx) \preceq \alpha_1 q(y, x) + \alpha_2 q(fx, x) + \alpha_3 q(fy, y) + \alpha_4 q(fy, x) + \alpha_5 q(fx, y) \quad (2)$$

for all $x, y \in X$*, where* α_i *for* $i = 1, 2, \cdots, 5$ *are nonnegative constants such that*

$$s(\alpha_1 + \alpha_3 + 2\alpha_4) + \alpha_2 + (s^2 + s)\alpha_5 < 1.$$

Then f *has a fixed point in* X*. If* $fv = v$*, then* $q(v, v) = \theta$*.*

Proof. For arbitrary $x_0 \in X$, consider the sequence $\{x_n\}$ with $x_n = f^n x_0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and the proof is finished. Suppose further that $x_n \neq x_{n+1}$ for $n \in \mathbb{N}_0$. Set $x = x_n$ and $y = x_{n-1}$ in (1). Then we have

$$\begin{aligned} q(x_{n+1}, x_n) &= q(fx_n, fx_{n-1}) \\ &\preceq \alpha_1 q(x_n, x_{n-1}) + \alpha_2 q(x_n, fx_n) + \alpha_3 q(x_{n-1}, fx_{n-1}) \\ &\quad + \alpha_4 q(x_n, fx_{n-1}) + \alpha_5 q(x_{n-1}, fx_n) \\ &= \alpha_1 q(x_n, x_{n-1}) + \alpha_2 q(x_n, x_{n+1}) + \alpha_3 q(x_{n-1}, x_n) \\ &\quad + \alpha_4 q(x_n, x_n) + \alpha_5 q(x_{n-1}, x_{n+1}) \\ &\preceq \alpha_1 q(x_n, x_{n-1}) + (\alpha_2 + s\alpha_4 + s\alpha_5) q(x_n, x_{n+1}) \\ &\quad + (\alpha_3 + s\alpha_5) q(x_{n-1}, x_n) + s\alpha_4 q(x_{n+1}, x_n). \end{aligned} \tag{3}$$

Similarly, set $x = x_n$ and $y = x_{n-1}$ in (2). Then we have

$$\begin{aligned} q(x_n, x_{n+1}) \preceq\ & \alpha_1 q(x_{n-1}, x_n) + (\alpha_2 + s\alpha_4 + s\alpha_5) q(x_{n+1}, x_n) \\ & + (\alpha_3 + s\alpha_5) q(x_n, x_{n-1}) + s\alpha_4 q(x_n, x_{n+1}). \end{aligned} \tag{4}$$

Adding up (3) and (4), we obtain

$$\begin{aligned} q(x_{n+1}, x_n) + q(x_n, x_{n+1}) \preceq\ & (\alpha_1 + \alpha_3 + s\alpha_5)[q(x_n, x_{n-1}) + q(x_{n-1}, x_n)] \\ & + (\alpha_2 + 2s\alpha_4 + s\alpha_5)[q(x_{n+1}, x_n) + q(x_n, x_{n+1})]. \end{aligned}$$

Let $u_n = q(x_{n+1}, x_n) + q(x_n, x_{n+1})$. We get that

$$u_n \preceq (\alpha_1 + \alpha_3 + s\alpha_5) u_{n-1} + (\alpha_2 + 2s\alpha_4 + s\alpha_5) u_n,$$

i.e. $u_n \preceq h u_{n-1}$ for all $n \in \mathbb{N}$ with $0 \leqslant h = \dfrac{\alpha_1 + \alpha_3 + s\alpha_5}{1 - (\alpha_2 + 2s\alpha_4 + s\alpha_5)} < \dfrac{1}{s}$, since $s(\alpha_1+\alpha_3+2\alpha_4)+\alpha_2+(s^2+s)\alpha_5 < 1$ and e.g., $s(\alpha_1+\alpha_3)+s^2\alpha_5 > 0$. By repeating the procedure, we get $u_n \preceq h^n u_0$ for all $n \in \mathbb{N}$. Hence,

$$q(x_n, x_{n+1}) \preceq u_n \preceq h^n[q(x_1, x_0) + q(x_0, x_1)]. \tag{5}$$

Let $m > n$. It follows from (5) and $0 \leqslant sh < 1$ that

$$
\begin{aligned}
q(x_n, x_m) &\preceq s[q(x_n, x_{n+1}) + q(x_{n+1}, x_m)] \\
&\preceq sq(x_n, x_{n+1}) + s[sq(x_{n+1}, x_{n+2}) + q(x_{n+2}, x_m)] \\
&\ \ \vdots \\
&\preceq sq(x_n, x_{n+1}) + s^2 q(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} q(x_{m-1}, x_m)] \\
&\preceq (sh^n + s^2 h^{n+1} \cdots + s^{m-n} h^{m-1})[q(x_1, x_0) + q(x_0, x_1)] \\
&\preceq \frac{sh^n}{1 - sh}[q(x_1, x_0) + q(x_0, x_1)].
\end{aligned}
$$

Using Lemma 2.9. (qp_3), $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. By applying continuity of f and since the limit of a sequence is unique, we get $fz = z$. Moreover, let $fv = v$ for $v \in X$. Then (1) implies that

$$
\begin{aligned}
q(v, v) &= q(fv, fv) \\
&\preceq \alpha_1 q(v, v) + \alpha_2 q(v, fv) + \alpha_3 q(v, fv) + \alpha_4 q(v, fv) + \alpha_5 q(v, fv) \\
&= \sum_{i=1}^{5} q(v, v).
\end{aligned}
$$

Since $\sum_{i=1}^{5} \alpha_i < s(\alpha_1 + \alpha_3 + 2\alpha_4) + \alpha_2 + (s^2 + s)\alpha_5 < 1$, we get that $q(v, v) = \theta$ by (p_5). This completes the proof. □

Corollary 3.2. *Let (X, d) be a complete tvs-cone b-metric space, q be a generalized c-distance on X and $f : X \to X$ be a continuous mapping. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $s(\alpha + 2\beta) + (s^2 + s)\gamma < 1$ such that*

$$
\begin{aligned}
q(fx, fy) &\preceq \alpha q(x, y) + \beta q(x, fy) + \gamma q(y, fx), \\
q(fy, fx) &\preceq \alpha q(y, x) + \beta q(fy, x) + \gamma q(fx, y)
\end{aligned}
$$

for all $x, y \in X$. Then f has a fixed point in X. Moreover, if $fv = v$, then $q(v, v) = \theta$.

Proof. We obtain this result by applying Theorem 3.1 with $\alpha_1 = \alpha$, $\alpha_2 = \alpha_3 = 0$, $\alpha_4 = \beta$ and $\alpha_5 = \gamma$. $\square$

Remark 3.3. *For Banach-type fixed point theorem, we need only one condition as follows:*

$$q(fx, fy) \preceq \lambda q(x, y), \qquad \lambda \in \left[0, \frac{1}{s}\right).$$

In the process of proving Theorem 3.1, consider $x = x_{n-1}$ (instead of $x = x_n$) and $y = x_n$ (instead of $y = x_{n-1}$). Then, for Kannan-type and Cho-type [4] fixed point results, we need only one condition:

$$q(fx, fy) \preceq \lambda(q(x, fx) + q(y, fy)), \qquad \lambda \in \left[0, \frac{1}{s+1}\right),$$

and

$$q(fx, fy) \preceq \alpha q(x, y) + \beta q(x, fx) + \gamma q(y, fy), \quad \alpha, \beta, \gamma > 0 \text{ with } s(\alpha+\beta)+\gamma < 1,$$

respectively.

Question 1. Can the continuity condition for mapping f be replaced by another condition in mentioned fixed point results?

Remark 3.4. *In Theorem 3.1, set $s = 1$. Then we obtain Theorem 2 of [8].*

Our second result in this section is a theorem including two mappings and the existence of their common fixed point.

Theorem 3.5. *Let (X, d) be a complete tvs-cone b-metric space with coefficient $s \geqslant 1$ and let q be a generalized c-distance on X. Suppose that continuous self-maps $f, g : X \to X$ satisfy the following two conditions:*

$$q(fx, gy) \preceq \alpha q(x, y) + \beta[q(x, fx) + q(y, gy)] + \gamma[q(x, gy) + q(y, fx)], \tag{6}$$

$$q(gy, fx) \preceq \alpha q(y, x) + \beta[q(fx, x) + q(gy, y)] + \gamma[q(gy, x) + q(fx, y)], \tag{7}$$

for all $x, y \in X$, where α, β, γ are nonnegative constants such that

$$s\alpha + (s+1)\beta + (s^2 + 3s)\gamma < 1.$$

Then f and g have a common fixed point in X. If $fv = gv = v$, then $q(v, v) = \theta$.

Proof. Suppose that x_0 is an arbitrary point in X, and define a sequence $\{x_n\}$ by

$$x_1 = fx_0,\ x_2 = gx_1,\ \cdots,\ x_{2n+1} = fx_{2n},\ x_{2n+2} = gx_{2n+1} \text{ for } n = 0, 1, 2, \cdots.$$

Set $x = x_{2n+2}$ and $y = x_{2n+1}$ in (6). Then we have

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) &= q(fx_{2n+2}, gx_{2n+1}) \\ &= \alpha q(x_{2n+2}, x_{2n+1}) + \beta[q(x_{2n+2}, fx_{2n+2}) + q(x_{2n+1}, gx_{2n+1})] \\ &\quad + \gamma[q(x_{2n+2}, gx_{2n+1}) + q(x_{2n+1}, fx_{2n+2})] \\ &\preceq \alpha q(x_{2n+2}, x_{2n+1}) + \beta[q(x_{2n+2}, x_{2n+3}) + q(x_{2n+1}, x_{2n+2})] \\ &\quad + \gamma[q(x_{2n+2}, x_{2n+2}) + q(x_{2n+1}, x_{2n+3})] \\ &\preceq \alpha q(x_{2n+2}, x_{2n+1}) + (\beta + s\gamma) q(x_{2n+1}, x_{2n+2}) \\ &\quad + (\beta + 2s\gamma) q(x_{2n+2}, x_{2n+3}) + s\gamma q(x_{2n+3}, x_{2n+2}). \end{aligned} \tag{8}$$

Similarly, putting the same values for x, y in (7), we get

$$\begin{aligned} q(x_{2n+2}, x_{2n+3}) \preceq\ & \alpha q(x_{2n+1}, x_{2n+2}) + (\beta + s\gamma) q(x_{2n+2}, x_{2n+1}) \\ & + (\beta + 2s\gamma) q(x_{2n+3}, x_{2n+2}) + s\gamma q(x_{2n+2}, x_{2n+3}). \end{aligned} \tag{9}$$

Adding up (8) and (9), we obtain

$$\begin{aligned} q(x_{2n+3}, x_{2n+2}) + q(x_{2n+2}, x_{2n+3}) \preceq\ & (\alpha + \beta + s\gamma)[q(x_{2n+2}, x_{2n+1}) + q(x_{2n+1}, x_{2n+2}) \\ & + (\beta + 3s\gamma) q(x_{2n+2}, x_{2n+3}) + q(x_{2n+3}, x_{2n+2}). \end{aligned}$$

Let $u_n = q(x_{2n}, x_{2n+1}) + q(x_{2n+1}, x_{2n})$ and $v_n = q(x_{2n+1}, x_{2n+2}) + q(x_{2n+2}, x_{2n+1})$. We get that

$$u_{n+1} \preceq (\alpha + \beta + s\gamma) v_n + (\beta + 3s\gamma) u_{n+1},$$

i.e., $u_{n+1} \preceq hv_n$ for all $n \in \mathbb{N}$ with $0 < h = \dfrac{\alpha+\beta+s\gamma}{1-(\beta+3s\gamma)} < \dfrac{1}{s}$, since $s\alpha+(s+1)\beta+(s^2+3s)\gamma < 1$ and e.g. $s(\alpha+\beta)+s^2\gamma > 0$. By a similar procedure, set $x = x_{2n}$ and $y = x_{2n+1}$ in (6) and (7), one can obtain $v_n \preceq hu_n$ for all $n \in \mathbb{N}$.

Now, it follows from $u_{n+1} \preceq hv_n$ and $v_n \preceq hu_n$ that

$$u_{n+1} \preceq h^2 u_n \quad \text{and} \quad v_n \preceq h^2 v_{n-1}.$$

Thus, $\{u_n\}$ and $\{v_n\}$ are c-sequences. Moreover, we obtain

$$q(x_{2n}, x_{2n+1}) \preceq u_n \quad \text{and} \quad q(x_{2n+1}, x_{2n+2}) \preceq v_n$$

and hence, $q(x_n, x_{n+1}) \preceq s(u_n + v_n)$, where $u_n + v_n$ is a c-sequence. Lemma 2.9. (qp_3) implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. By applying continuity of f and g, and since the limit of a sequence is unique, we get $fz = z = gz$. Thus, z is a common fixed point of f and g. Moreover, let $fv = gv = v$ for $v \in X$. Then (6) implies that

$$\begin{aligned} q(v,v) &= q(fv, gv) \\ &\preceq \alpha q(v,v) + \beta[q(v,fv)+q(v,gv)] + \gamma[q(v,gv)+q(v,fv)] \\ &= (\alpha+2\beta+2\gamma)q(v,v). \end{aligned}$$

Since $\alpha+2\beta+2\gamma < s\alpha+(s+1)\beta+(s^2+3s)\gamma < 1$, we get that $q(v,v) = \theta$ by (p_5). This completes the proof. □

Remark 3.6. *As corollaries, for example, we can obtain the common fixed point result for self-maps f and g satisfying*

$$q(fx, gy) \preceq \alpha q(x,y), \quad q(gy, fx) \preceq \alpha q(y,x), \quad 0 < \alpha < \frac{1}{s}, \tag{10}$$

or for a self-map f satisfying

$$\begin{aligned} q(f^n x, f^m y) &\preceq \alpha q(x,y) + \beta[q(x,f^n x)+q(y,f^m y)] + \gamma[q(x,f^m y)+q(y,f^n x)], \\ q(f^m y, f^n x) &\preceq \alpha q(y,x) + \beta[q(f^n x,x)+q(f^m y,y)] + \gamma[q(f^m y,x)+q(f^n x,y)], \end{aligned}$$

where $m, n \in \mathbb{N}$ and $s\alpha+(s+1)\beta+(s^2+3s)\gamma < 1$.

Example 3.7. Let $E = \mathbb{R}$, $P = [0, +\infty)$, $X = [0, +\infty)$. Also, consider the cone b-metric $d(x, y) = (x - y)^2$ on X with $s = 2$. Take mappings $f, g : X \to X$ defined by $fx = \frac{x}{2}$ and $gx = \frac{x}{4}$. If $x = \frac{15}{2}$ and $y = 5$, then

$$d(fx, gy) = d\left(\frac{15}{4}, \frac{5}{4}\right) = \left(\frac{15}{4} - \frac{5}{4}\right)^2 = \frac{25}{4},$$

and

$$d(x, y) = d\left(\frac{15}{2}, 5\right) = \left(\frac{15}{2} - 5\right)^2 = \frac{25}{4}.$$

Thus, there is no $\alpha \in (0, 1)$ such that $d(fx, gy) \preceq \alpha d(x, y)$ for each $x, y \in [0, +\infty)$, i.e., the existence of a common fixed point of f and g cannot be deduced from the well-known cone b-metric version of Theorem 3.5.

Now, consider the complete *tvs*-cone b-metric d on X, defined as $d(x, y)(t) = (x - y)^2 \varphi(t)$ with fixed $\varphi \in P = \{f \in C[0, 1] : f(t) \geqslant 0 \text{ for } t \in [0, 1]\}$ and take the generalized c-distance $q(x, y)(t) = y^2 \varphi(t)$ (see Example 3.8). Also, select $\frac{1}{4} \leqslant \alpha < \frac{1}{2}$ and $\beta = \gamma = 0$. Then we have

$$\begin{aligned} q(fx, gy)(t) &= (gy)^2 \varphi(t) = \frac{y^2}{16} \varphi(t) \leqslant \alpha y^2 \varphi(t) \\ &= \alpha q(x, y)(t) + \beta[q(x, fx)(t) + q(y, gy)(t)] \\ &\quad + \gamma[q(x, gy)(t) + q(y, fx)(t)], \quad t \in [0, 1], \end{aligned}$$

i.e., $q(fx, gy) \preceq \alpha q(x, y) + \beta[q(x, fx) + q(y, gy)] + \gamma[q(x, gy) + q(y, fx)]$ for all $x, y \in [0, \infty)$. Similarly, we have

$$\begin{aligned} q(gy, fx)(t) &= (fx)^2 \varphi(t) = \frac{x^2}{4} \varphi(t) \leqslant \alpha x^2 \varphi(t) \\ &= \alpha q(y, x)(t) + \beta[q(fx, x)(t) + q(gy, y)(t)] \\ &\quad + \gamma[q(gy, x)(t) + q(fx, y)(t)], \quad t \in [0, 1], \end{aligned}$$

i.e., $q(gy, fx) \preceq \alpha q(y, x) + \beta[q(fx, x) + q(gy, y)] + \gamma[q(gy, x) + q(fx, y)]$ for all $x, y \in [0, \infty)$.

Thus, all conditions of Theorem 3.5 are satisfied. Note that f and g have a (trivial) common fixed point $v = 0$ and that $q(v, v) = 0$.

Question 2. Can the continuity condition for mappings f and g be replaced by another condition in mentioned fixed point results?

Remark 3.8. *In Theorem 3.5, set $s = 1$. Then we obtain Theorem 3 of [8].*

4. Periodic Point Results

Obviously, if f is a map which has a fixed point z, then z is also a fixed point of f^n for each $n \in \mathbb{N}$. However the converse need not be true. If a map $f : X \to X$ satisfies $Fix(f) = Fix(f^n)$ for each $n \in \mathbb{N}$, where $Fix(f)$ stands for the set of fixed points of f [14], then f is said to have property (P). Recall also that two mappings $f, g : X \to X$ are said to have property (Q) if $Fix(f) \bigcap Fix(g) = Fix(f^n) \bigcap Fix(g^n)$ for each $n \in \mathbb{N}$.

Theorem 4.1. *Let (X, d) be a tvs-cone b-metric space and $q : X \times X \to E$ be a generalized c-distance on X. Suppose that a continuous self-map $f : X \to X$ satisfies*

$$q(fx, f^2x) + q(f^2x, fx) \preceq \lambda[q(x, fx) + q(fx, x)] \tag{11}$$

for $x \in X$, where $\lambda \in (0, \frac{1}{s})$. Then f has property (P).

Proof. Since the proof is easy and similar as in the case of c-distance in tvs-cone b-metric spaces in [8], we leave it to the reader. □

Theorem 4.2. *Let q be a generalized c-distance on a tvs-cone b-metric space (X, d) and let $f : X \to X$ be continuous. Suppose that inequalities (1) and (2) hold for all $x, y \in X$, where α_i are nonnegative constants such that $s(\alpha_1 + \alpha_3 + 2\alpha_4) + \alpha_2 + (s^2 + s)\alpha_5 < 1$. Then f has property (P).*

Proof. Putting $x = fx$ and $y = x$ in condition (1), we have

$$\begin{aligned} q(f^2x, fx) &= q(ffx, fx) \\ &\preceq \alpha_1 q(fx, x) + \alpha_2 q(fx, ffx) + \alpha_3 q(x, fx) \\ &\quad + \alpha_4 q(fx, fx) + \alpha_5 q(x, ffx) \end{aligned} \tag{12}$$

$$\begin{aligned}&\preceq \alpha_1 q(fx,x)+\alpha_2 q(fx,f^2x)+\alpha_3 q(x,fx)\\&\qquad+\alpha_4 q(fx,fx)+\alpha_5 q(x,f^2x)\\&\qquad\preceq \alpha_1 q(fx,x)+\alpha_2 q(fx,f^2x)+\alpha_3 q(x,fx)\\&\qquad+s\alpha_4[q(fx,f^2x)+q(f^2x,fx)]+s\alpha_5[q(x,fx)+q(fx,f^2x)].\end{aligned}$$

Similarly, set $x=fx$ and $y=x$ in (2) and we have

$$\begin{aligned}q(fx,f^2x)\preceq{}&\alpha_1 q(x,fx)+\alpha_2 q(f^2x,fx)+\alpha_3 q(fx,x)\\&+s\alpha_4[q(fx,f^2x)+q(f^2x,fx)]+s\alpha_5[q(f^2x,fx)+q(fx,x)].\end{aligned}\tag{13}$$

Adding up (12) and (13) we get

$$\begin{aligned}q(f^2x,fx)+q(fx,f^2x)\preceq{}&(\alpha_1+\alpha_3+s\alpha_5)[q(x,fx)+q(fx,x)]\\&+(\alpha_2+2s\alpha_4+s\alpha_5)[q(f^2x,fx)+q(fx,f^2x)],\end{aligned}$$

i.e. (11) with $0<\lambda=\dfrac{\alpha_1+\alpha_3+s\alpha_5}{1-(\alpha_2+2s\alpha_4+s\alpha_5)}<\dfrac{1}{s}$, since $s(\alpha_1+\alpha_3+2\alpha_4)+\alpha_2+(s^2+s)\alpha_5<1$. □

Remark 4.3. *In Theorem 3.5, set* $s=1$. *Then we obtain Corollary* 3 *of [8].*

Now, one can obtain similar results concerning property (Q) of two self-mappings f and g.

5. An Application

We are going to apply our results to obtain sufficient conditions for existence of solution for a system of Fredholm integral equations.

Theorem 5.1. *Let* $F,G:[0,1]^2\times\mathbb{R}\to\mathbb{R}$ *be two continuous functions and suppose that the following conditions are satisfied for all pairs* $(x,y)\in\big(C_{\mathbb{R}}(I)\big)^2$, $I=[0,1]$*:*

$$\begin{aligned}\max_{t\in I}\bigg(\int_0^1 G(t,u,y(u))\,du\bigg)^2\leqslant{}&\alpha\max_{t\in I}\big(y(t)\big)^2\\&+\beta\bigg[\max_{t\in I}\bigg(\int_0^1 F(t,u,x(u))\,du\bigg)^2+\max_{t\in I}\bigg(\int_{t\in I} G(t,u,y(u))\,du\bigg)^2\bigg],\end{aligned}\tag{14}$$

$$\max_{t\in I}\Big(\int_0^1 F(t,u,x(u))\,du\Big)^2 \leqslant \alpha\max_{t\in I}\big(x(t)\big)^2+\beta\big[\max_{t\in I}\big(x(t)\big)^2+\max_{t\in I}\big(y(t)\big)^2\big], \tag{15}$$

where $\alpha,\beta\geqslant 0$ and $2\alpha+3\beta<1$. Then the system of integral equations

$$\begin{cases} x(t)=\int_0^1 F(t,u,x(u))\,du \\ x(t)=\int_0^1 G(t,u,x(u))\,du \end{cases} \tag{16}$$

has a solution in $C_{\mathbb{R}}(I)$.

Proof. Let, as in Example 2.7, $P=\{x\in E : x(t)\geqslant 0 \text{ for all } t\in I\}$ be a (nonnormal) cone in the Banach space $E=C^1_{\mathbb{R}}(I)$ with the norm $\|x\|=\|x\|_\infty+\|x'\|_\infty$. Further, consider the set $X=C(I)$ equipped with the tvs-cone b-metric $d: X\times X\to P$, given by

$$d(x,y)(v)=e^v\max_{t\in I}\big(x(t)-y(t)\big)^2,\quad v\in I$$

(with $s=2$). Also, define a generalized c-distance $q: X\times X\to P$ by

$$q(x,y)(v)=e^v\max_{t\in I}\big(y(t)\big)^2,\quad v\in I.$$

Further, let $f,g: X\to X$ be defined by

$$fx(t)=\int_0^1 F(t,u,x(u))\,du,$$
$$gx(t)=\int_0^1 G(t,u,x(u))\,du.$$

We have to check that the mappings f,g satisfy conditions (6) and (7) of Theorem 3.5.

For every $x,y\in X$ and $v\in I$, using (14) and (15), we obtain

$$\begin{aligned} q(fx,gy)(v) &= e^v\max_{t\in I}\Big(\int_0^1 G(t,u,y(u))\,du\Big)^2 \\ &\leqslant \alpha\cdot e^v\max_{t\in I}\big(y(t)\big)^2 \\ &+\beta\cdot e^v\Big[\max_{t\in I}\Big(\int_0^1 F(t,u,x(u))\,du\Big)^2+\max_{t\in I}\Big(\int_{t\in I} G(t,u,y(u))\,du\Big)^2\Big] \\ &= \alpha q(x,y)(v)+\beta[q(x,fx)(v)+q(y,gy)(v)], \end{aligned}$$

i.e., $q(fx, gy) \preceq \alpha q(x, y) + \beta[q(x, fx) + q(y, gy)]$, and

$$\begin{aligned} q(gy, fx)(v) &= e^v \max_{t\in I}\left(\int_0^1 F(t, u, x(u))\, du\right)^2 \\ &\leqslant \alpha \cdot e^v \max_{t\in I}\big(x(t)\big)^2 + \beta \cdot e^v\big[\max_{t\in I}\big(x(t)\big)^2 + \max_{t\in I}\big(y(t)\big)^2\big] \\ &= \alpha q(y, x)(v) + \beta[q(fx, x)(v) + q(gy, y)(v)], \end{aligned}$$

i.e., $q(gy, fx) \preceq \alpha q(y, x) + \beta[q(fx, x) + q(gy, y)]$. Thus, the inequalities (6) and (7) are fulfilled with $\gamma = 0$ since $2\alpha + 3\beta = s\alpha + (s+1)\beta < 1$.

Applying Theorem 3.5 we conclude that the mappings f and g have a fixed point $x^* \in X$. It is clear that x^* is a solution of the system (16). □

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