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Some New Generalized Hermite-Hadamard Inequalities for Generalized Convex Functions and Applications

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Abstract. In this paper, some new inequalities for generalized convex functions are obtained. Some applications for some generalized special means are also given.

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1. Introduction

We first give the following important inequality:

Theorem 1.1. [Hermite-Hadamard inequality]*Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [5]*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

In [2], authors gave the following lemma:

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Lemma 1.2. *Let $f : I^{\circ} \subset \mathbb{R} \rightarrow \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$. If $f'' \in L_1[a, b]$, then*

$$\frac{1}{2}\int_0^1 (x-a)(b-x)f''(x)dx = \frac{b-a}{2}(f(a)+f(b)) - \int_a^b f(x)dx. \quad (2)$$

Also,they obtained following inequality:

Theorem 1.3. *With the above assumptions, given that $k \leqslant f''(x) \leqslant K$ on $[a, b]$, we have the inequality*

$$k\frac{(b-a)^2}{12} \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \leqslant K\frac{(b-a)^2}{12}. \quad (3)$$

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [11, 12] and so on.

Recently, the theory of Yang's fractional sets [11] was introduced as follows.

For $0 < \alpha \leqslant 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha}, \pm 1^{\alpha}, \pm 2^{\alpha}, ..., \pm n^{\alpha}, ...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha} = \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, \, q \neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha} \neq \left(\frac{p}{q}\right)^{\alpha} : p, q \in Z, \, q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha} = Q^{\alpha} \cup J^{\alpha}$.

If a^{α}, b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha} + b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
(2) $a^{\alpha} + b^{\alpha} = b^{\alpha} + a^{\alpha} = (a+b)^{\alpha} = (b+a)^{\alpha}$;
(3) $a^{\alpha} + (b^{\alpha} + c^{\alpha}) = (a+b)^{\alpha} + c^{\alpha}$;
(4) $a^{\alpha}b^{\alpha} = b^{\alpha}a^{\alpha} = (ab)^{\alpha} = (ba)^{\alpha}$;
(5) $a^{\alpha}(b^{\alpha}c^{\alpha}) = (a^{\alpha}b^{\alpha})c^{\alpha}$;
(6) $a^{\alpha}(b^{\alpha} + c^{\alpha}) = a^{\alpha}b^{\alpha} + a^{\alpha}c^{\alpha}$;
(7) $a^{\alpha} + 0^{\alpha} = 0^{\alpha} + a^{\alpha} = a^{\alpha}$ and $a^{\alpha}1^{\alpha} = 1^{\alpha}a^{\alpha} = a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. *[11] A non-differentiable function $f : R \to R^{\alpha}$, $x \to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$|f(x) - f(x_0)| < \varepsilon^{\alpha},$$

holds for $|x - x_0| < \delta$, where $\varepsilon, \delta \in R$. If $f(x)$ is local continuous on the interval (a, b), we denote $f(x) \in C_{\alpha}(a, b)$.

Definition 2.2. *[11] The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by*

$$f^{(\alpha)}(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha} \left(f(x) - f(x_0)\right)}{\left(x - x_0\right)^{\alpha}},$$

where $\Delta^{\alpha} \left(f(x) - f(x_0)\right) \cong \Gamma(\alpha + 1) \left(f(x) - f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^{\alpha} ... D_x^{\alpha}}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq R$, then we denoted $f \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, ...$

Definition 2.3. *[3] Let $f(x) \in C_{\alpha}[a, b]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_1, \Delta t_2, ..., \Delta t_{N-1}\}$, where $[t_j, t_{j+1}]$, $j = 0, ..., N - 1$ and $a = t_0 < t_1 < ... < t_{N-1} < t_N = b$ is partition of interval $[a, b]$.

Here, it follows that ${}_aI_b^{\alpha} f(x) = 0$ if $a = b$ and ${}_aI_b^{\alpha} f(x) = -{}_bI_a^{\alpha} f(x)$ if $a < b$. If for any $x \in [a, b]$, there exists ${}_aI_x^{\alpha} f(x)$, then we denoted by $f(x) \in I_x^{\alpha}[a, b]$.

Definition 2.4. [Generalized convex function] *[11] Let $f : I \subseteq R \to R^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0, 1]$, if the following inequality*

$$f(\lambda x_1 + (1 - \lambda)x_2) \leqslant \lambda^{\alpha} f(x_1) + (1 - \lambda)^{\alpha} f(x_2),$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geqslant 0$, $p>1$;

(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

Theorem 2.5. *Let $f\in D_{\alpha}(I)$, then the following conditions are equivalent*

a) *f is a generalized convex function on I*

b) *$f^{(\alpha)}$ is an increasing function on I*

c) *for any $x_1,x_2\in I$,*

$$f(x_2)-f(x_1)\geqslant\frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)}(x_2-x_1)^{\alpha}.$$

Corollary 2.6. *Let $f\in D_{2\alpha}(a,b)$. Then f is a generalized convex function*
(or a generalized concave function) if and only if

$$f^{(2\alpha)}(x)\geqslant 0\left(or\ f^{(2\alpha)}(x)\leqslant 0\right),$$

for all $x\in(a,b)$.

Lemma 2.7. *[11] (1) (Local fractional integration is anti-differentiation) Suppose that $f(x)=g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, then we have*

$${}_aI_b^{\alpha}f(x)=g(b)-g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x),g(x)\in D_{\alpha}[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, then we have

$${}_aI_b^{\alpha}f(x)g^{(\alpha)}(x)=\left.f(x)g(x)\right|_a^b-{}_aI_b^{\alpha}f^{(\alpha)}(x)g(x).$$

Lemma 2.8. *[11]*

$$\frac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b x^{k\alpha}(dx)^{\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right),\ k\in R.$$

Lemma 2.9. [Generalized Hölder's inequality] *[11] Let* $f, g \in C_\alpha[a,b]$, $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha$$

$$\leqslant \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [3], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.10. [Generalized Hermite-Hadamard's inequality] *Let* $f(x) \in I_x^\alpha[a,b]$ *be generalized convex function on* $[a,b]$ *with* $a < b$. *Then*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^\alpha f(x) \leqslant \frac{f(a)+f(b)}{2^\alpha}.$$

In [9], Sarikaya et al. gave the the following generalized Bullen inequality for generalized convex function.

Theorem 2.11. [Generalized Bullen inequality] *Let* $f(x) \in I_x^{(\alpha)}[a,b]$ *be a generalized convex function on* $[a,b]$ *with* $a < b$. *Then we have the inequality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^\alpha f(x) \leqslant \frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha}\right]. \quad (4)$$

For more information and recent developments on local fractional theory, please refer to [1], [3], [4], [6]- [15].

The aim of the paper is to establish some new generalized Hermite-Hadamard inequalities for generalized convex functions and we give some applications for some generalized special means.

3. Main Results

In this section, we establish some new inequalities for generalized convex functions.

Theorem 3.1. *Let* $I \subseteq R$ *be an interval,* $f : I^0 \subseteq R \to R^\alpha$ *(*I^0 *is the interior of* I*) such that* $f \in D_{2\alpha}(I^0)$ *and* $f^{(2\alpha)} \in C_{2\alpha}[a,b]$ *for* $a, b \in I^0$

with $a<b$ Then, we have the identity:

$$\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x) \tag{5}$$

$$=\frac{1}{2^{\alpha}(b-a)^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha}.$$

Proof. Using the local fractional integration by parts twice (Lemma 2.7), we obtain

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)\,(dx)^{\alpha} \tag{6}$$

$$= (x-a)^{\alpha}(b-x)^{\alpha}f^{(\alpha)}(x)\Big|_{a}^{b}$$

$$-\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\Gamma(1+\alpha)\,(-2x+a+b)^{\alpha}f^{(\alpha)}(x)\,(dx)^{\alpha}$$

$$= -\Gamma(1+\alpha)\,(-2x+a+b)^{\alpha}f(x)\big|_{a}^{b}$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left[\Gamma(1+\alpha)\right]^{2}(-2)^{\alpha}f(x)\,(dx)^{\alpha}$$

$$= -\Gamma(1+\alpha)\,(a-b)^{\alpha}f(b)+\Gamma(1+\alpha)\,(b-a)^{\alpha}f(a)$$

$$-\frac{2^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}}{\Gamma(1+\alpha)}\int_{a}^{b}f(x)\,(dx)^{\alpha}$$

$$= \Gamma(1+\alpha)\,(b-a)^{\alpha}\left[f(a)+f(b)\right]-2^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}\ {}_{a}I_{b}^{\alpha}f(x).$$

If we divide the resulting equality (6) by $2^{\alpha}(b-a)^{2\alpha}\Gamma(1+\alpha)$, then we obtain the desired result. □

Theorem 3.2. *With the above assumptions, given that* $k^{\alpha} \leqslant f^{(2\alpha)}(x) \leqslant K^{\alpha}$ *for all* $x \in [a,b]$, $k, K \in R$, *we have the following inequality*

$$k^{\alpha}.\frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right] \tag{7}$$

$$\leqslant \frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)$$

$$\leqslant K^{\alpha}.\frac{(b-a)^{2\alpha}}{2^{\alpha}\Gamma(1+\alpha)}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

Proof. From assumptions, we have

$$k^{\alpha}(x-a)^{\alpha}(b-x)^{\alpha} \leqslant (x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x) \leqslant K^{\alpha}(x-a)^{\alpha}(b-x)^{\alpha} \tag{8}$$

for all $x \in [a,b]$. Dividing (8) by $2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^{2}$, then integrating the resulting inequality with respect to x over $[a,b]$, we get

$$\frac{k^{\alpha}}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}(dx)^{\alpha}$$

$$\leqslant \frac{1}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)(dx)^{\alpha}$$

$$\leqslant \frac{K^{\alpha}}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}K^{\alpha}(x)(dx)^{\alpha}.$$

From Theorem 3.1, we have

$$\frac{1}{2^{\alpha}(b-a)^{\alpha}[\Gamma(1+\alpha)]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}f^{(2\alpha)}(x)(dx)^{\alpha}.$$

$$= \frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)$$

and a simple calculating shows that

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^\alpha (b-x)^\alpha (dx)^\alpha = (b-a)^{3\alpha}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right].$$

This completes the proof. □

Corollary 3.3. *With the assumptions of Theorem 3.2, given that* $\left\| f^{(2\alpha)} \right\|_\infty := \sup_{x\in[a,b]} \left| f^{(2\alpha)}(x) \right|$, *then we have the following inequality .*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(x) \right| \\ \leqslant\ & \frac{(b-a)^{2\alpha}}{2^\alpha \Gamma(1+\alpha)}\left[\frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right]\left\| f^{(2\alpha)} \right\|_\infty . \end{aligned}$$

Theorem 3.4. *The assumptions of Theorem 3.1 are satisfied. If we assume that the new mapping* $\varphi : I^0 \subseteq R \to R^\alpha$, $\varphi(x) = (x-a)^\alpha (b-x)^\alpha f^{(2\alpha)}(x)$ *is generalized convex on* $[a,b]$, *then we have the inequality*

$$\begin{aligned} &\frac{(b-a)^{2\alpha}}{8^\alpha [\Gamma(1+\alpha)]^2} f^{(2\alpha)}\left(\frac{a+b}{2}\right) && (9) \\ \leqslant\ & \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^\alpha f(x) \\ \leqslant\ & \frac{(b-a)^{2\alpha}}{16^\alpha [\Gamma(1+\alpha)]^2} f^{(2\alpha)}\left(\frac{a+b}{2}\right). \end{aligned}$$

Proof. Because of the generalized convexity of φ, applying the first inequality of generalized Hermite-Hadamard (Theorem 2.10) we can state:

$$\frac{1}{(b-a)^\alpha}\int_a^b \varphi(x)\,(dx)^\alpha \geqslant \varphi\left(\frac{a+b}{2}\right) = \frac{(b-a)^{2\alpha}}{4^\alpha} f^{(2\alpha)}\left(\frac{a+b}{2}\right). \quad (10)$$

Similarliy, applying the generalized generalized Bullen inequality (Theorem 2.11) for φ, we get

$$\begin{aligned}\frac{1}{(b-a)^{\alpha}}\int_a^b \varphi(x)\,(dx)^{\alpha} &\leqslant \frac{1}{2^{\alpha}}\left[\varphi\left(\frac{a+b}{2}\right)+\frac{\varphi(a)+\varphi(b)}{2^{\alpha}}\right] \\ &= \frac{(b-a)^{2\alpha}}{8^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right).\end{aligned} \tag{11}$$

Combining (10) and (11), we have

$$\begin{aligned}\frac{(b-a)^{2\alpha}}{4^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right) &\leqslant \frac{1}{(b-a)^{\alpha}}\int_a^b \varphi(x)\,(dx)^{\alpha} \\ &\leqslant \frac{(b-a)^{2\alpha}}{8^{\alpha}} f^{(2\alpha)}\left(\frac{a+b}{2}\right).\end{aligned} \tag{12}$$

If we divide the inequalities (12) by $2^{\alpha}\left[\Gamma(1+\alpha)\right]^2$, we obtain desired result, which completes the proof. □

Theorem 3.5. *The assumptions of Theorem 3.1 are satisfied. Then we have the inequality*

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\;{}_aI_b^{\alpha} f(x)\right| \tag{13}$$

$$\leqslant \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^{\alpha}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f^{(2\alpha)}\right\|_q,$$

where, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\left\|f^{(2\alpha)}\right\|_q$ *is defined by*

$$\left\|f^{(2\alpha)}\right\|_q=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f^{(2\alpha)}(x)\right|^q(dx)^{\alpha}\right)^{\frac{1}{q}}$$

and $B(x,y)$ *is defined by*

$$B(x,y)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^{\alpha}.$$

Proof. Taking modulus in (5) and using the generalized Holder's inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right| \\
\leqslant\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}\left|(x-a)^{\alpha}(b-x)^{\alpha}\right|\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha} \\
\leqslant\ & \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{0}^{b}\left|f^{(2\alpha)}(x)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}} \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}. \\
=\ & \frac{\left\|f^{(2\alpha)}\right\|_{q}}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the changing variable $x=(1-t)\,a+tb$, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{p\alpha}(b-x)^{p\alpha}(dx)^{\alpha}=(b-a)^{(2p+1)\alpha}B(p+1,p+1),$$

which completes the proof. □

Theorem 3.6 The assumptions of Theorem **??** are satisfied. If $\left|f^{(2\alpha)}\right|$ is generalized convex, then we have the inequality

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right| \qquad (14)\\
\leqslant\ & \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right)\left[\frac{\left|f^{(2\alpha)}(a)\right|+\left|f^{(2\alpha)}(b)\right|}{2^{\alpha}}\right].
\end{aligned}$$

Proof. Taking modulus in (5) we get

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right|$$

$$\leqslant \frac{1}{2^{\alpha}(b-a)^{\alpha}\left[\Gamma(1+\alpha)\right]^{2}}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{\alpha}\left|f^{(2\alpha)}(x)\right|(dx)^{\alpha}.$$

Since generalized convexity of $\left|f^{(2\alpha)}\right|$, we have

$$\begin{aligned}\left|f^{(2\alpha)}(x)\right| &= \left|f^{(2\alpha)}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)\right| \\ &\leqslant \left(\frac{x-a}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(b)\right|+\left(\frac{b-x}{b-a}\right)^{\alpha}\left|f^{(2\alpha)}(a)\right|.\end{aligned}$$

Then, it follows that

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right| \tag{15}$$

$$\leqslant \frac{1}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left[\frac{\left|f^{(2\alpha)}(b)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{2\alpha}(b-x)^{\alpha}(dx)^{\alpha}\right.$$

$$\left.+\frac{\left|f^{(2\alpha)}(a)\right|}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{\alpha}(b-x)^{2\alpha}(dx)^{\alpha}\right].$$

Using the changing variable $x=(1-t)\,a+tb$, we have

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}(x-a)^{2\alpha}(b-x)^{\alpha}(dx)^{\alpha} \qquad (16)\\ &= (b-a)^{4\alpha}\int_{a}^{b}t^{2\alpha}(1-t)^{\alpha}(dt)^{\alpha} \\ &= (b-a)^{4\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right),\end{aligned}$$

and similarly

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b (x-a)^{\alpha}(b-x)^{2\alpha}(dx)^{\alpha}=(b-a)^{4\alpha}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}-\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\right). \tag{17}$$

Putting (16) and (17) in (15), we obtain required result. □

4. Applications to Some Special Means

We consider some generalized means as in [7]:

$$A(a,b)=\frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

$$L_n(a,b)=\left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right]\right]^{\frac{1}{n}},$$

where $n\in Z\backslash\{-1,0\}$, $a,b\in R$, $a\neq b$.

Proposition 4.1. Let $a,b\in R$, $0<a<b$, $0\notin[a,b]$ and $n\in Z$, $|n(n-1)|\geqslant 3$. Then, we have the inequality

$$\begin{aligned}&\left|A(a^n,b^n)-\Gamma(1+\alpha)\left[L_n(a,b)\right]^n\right|\\ \leqslant\ &\frac{(b-a)^{\left(1+q+\frac{1}{p}\right)\alpha}}{2^{\alpha}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\\ &\times\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[L_{q(n-2)}^{n-2}(a,b)\right].\end{aligned}$$

Proof. Let us reconsider the inequality (13):

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right|\\ \leqslant\ &\frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^{\alpha}}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left\|f^{(2\alpha)}\right\|_q.\end{aligned}$$

Consider the mapping $f:(0,\infty)\to R^{\alpha},\ f(x)=x^{n\alpha},\ n\in Z\backslash\{-1,0\}.$ Then, $0<a<b$, we have

$$\frac{f(a)+f(b)}{2^{\alpha}} = A(a^{n},b^{n}),\frac{{}_{a}I_{b}^{\alpha}f(x)}{(b-a)^{\alpha}}=[L_{n}(a,b)]^{n},$$

$$\left|f^{(2\alpha)}(x)\right| = \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|x^{(n-2)\alpha}$$

and

$$\begin{aligned}
&\left\|f^{(2\alpha)}\right\|_{q}\\
&= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}x^{q(n-2)\alpha}(dx)^{\alpha}\right)^{\frac{1}{q}}\\
&= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\\
&\quad\times\left(\frac{\Gamma(1+q(n-2)\alpha)}{\Gamma(1+(q(n-2)+1)\alpha)}\left(b^{(q(n-2)+1)\alpha}-a^{(q(n-2)+1)\alpha}\right)\right)^{\frac{1}{q}}\\
&= \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}(b-a)^{q\alpha}\left[L_{q(n-2)}(a,b)\right]^{n-2}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\left|A(a^{n},b^{n})-\Gamma(1+\alpha)[L_{n}(a,b)]^{n}\right|\\
&\leqslant \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^{\alpha}}[B(p+1,p+1)]^{\frac{1}{p}}\\
&\quad\times\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}(b-a)^{q\alpha}\left[L_{q(n-2)}(a,b)\right]^{n-2}\\
&= \frac{(b-a)^{\left(1+q+\frac{1}{p}\right)\alpha}}{2^{\alpha}}[B(p+1,p+1)]^{\frac{1}{p}}\\
&\quad\times\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\right|^{\frac{1}{q}}\left[L_{q(n-2)}^{n-2}(a,b)\right].
\end{aligned}$$

This completes the proof. □

Proposition 4.2. Let $a, b \in R$, $0 < a < b$, $0 \notin [a, b]$ and $n \in Z$, $|n(n-1)| \geqslant 3$. Then, we have the inequality

$$
\begin{aligned}
& |A(a^n, b^n) - \Gamma(1+\alpha)[L_n(a,b)]^n| \\
= & \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)} \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)} \right| \\
& \times \left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \right) A(a^{n-2}, b^{n-2}).
\end{aligned}
$$

Proof. The proof is obvious from Theorem 3. applied $f(x) = x^{n\alpha}$, $n \in Z \backslash \{-1, 0\}$. □

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