Some New Generalized Hermite-Hadamard Inequalities for Generalized Convex Functions and Applications

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Abstract. In this paper, some new inequalities for generalized convex functions are obtained. Some applications for some generalized special means are also given.

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1. Introduction

We first give the following important inequality:

Theorem 1.1. [Hermite-Hadamard inequality] Let \( f : I \subseteq R \rightarrow R \) be a convex function on the interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). If \( f \) is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds [5]

\[
\frac{f\left(\frac{a+b}{2}\right)}{2} \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}.
\]

(1)

In [2], authors gave the following lemma:
Lemma 1.2. Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a, b]$, then
\[
\frac{1}{2} \int_0^1 (x-a)(b-x)f''(x)dx = \frac{b-a}{2} (f(a) + f(b)) - \int_a^b f(x)dx. \tag{2}
\]
Also, they obtained following inequality:

Theorem 1.3. With the above assumptions, given that $k \leq f''(x) \leq K$ on $[a, b]$, we have the inequality
\[
k \frac{(b-a)^2}{12} \leq \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \leq K \frac{(b-a)^2}{12}. \tag{3}
\]

2. Preliminaries

Recall the set $R^\alpha$ of real line numbers and use the Gao-Yang-Kang’s idea to describe the definition of the local fractional derivative and local fractional integral, see [11, 12] and so on. Recently, the theory of Yang’s fractional sets [11] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following $\alpha$-type set of element sets:

$Z^\alpha$: The $\alpha$-type set of integer is defined as the set
\[
\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \ldots, \pm n^\alpha, \ldots\}.
\]

$Q^\alpha$: The $\alpha$-type set of the rational numbers is defined as the set
\[
\left\{\frac{m^\alpha}{n^\alpha} : p, q \in \mathbb{Z}, q \neq 0\right\}.
\]

$J^\alpha$: The $\alpha$-type set of the irrational numbers is defined as the set
\[
\left\{\frac{m^\alpha}{n^\alpha} : p, q \in \mathbb{Z}, q \neq 0\right\}.
\]

$R^\alpha$: The $\alpha$-type set of the real line numbers is defined as the set $R^\alpha = Q^\alpha \cup J^\alpha$.

If $a^\alpha, b^\alpha$ and $c^\alpha$ belongs the set $R^\alpha$ of real line numbers, then

\begin{align*}
(1) \quad & a^\alpha + b^\alpha \text{ and } a^\alpha b^\alpha \text{ belongs the set } R^\alpha; \\
(2) \quad & a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a + b)^\alpha = (b + a)^\alpha; \\
(3) \quad & a^\alpha + (b^\alpha + c^\alpha) = (a + b)^\alpha + c^\alpha; \\
(4) \quad & a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha; \\
(5) \quad & a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha; \\
(6) \quad & a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha; \\
(7) \quad & a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha \text{ and } a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha.
\end{align*}
The definition of the local fractional derivative and local fractional integral can be given as follows.

**Definition 2.1.** [11] A non-differentiable function \( f : \mathbb{R} \to \mathbb{R}^\alpha \), \( x \to f(x) \) is called to be local fractional continuous at \( x_0 \), if for any \( \varepsilon > 0 \), there exists \( \delta > 0 \), such that

\[
|f(x) - f(x_0)| < \varepsilon^\alpha,
\]

holds for \( |x - x_0| < \delta \), where \( \varepsilon, \delta \in \mathbb{R} \). If \( f(x) \) is local continuous on the interval \((a, b)\), we denote \( f(x) \in C_{\alpha}(a, b) \).

**Definition 2.2.** [11] The local fractional derivative of \( f(x) \) of order \( \alpha \) at \( x = x_0 \) is defined by

\[
f^{(\alpha)}(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^\alpha},
\]

where \( \Delta^{\alpha}(f(x) - f(x_0)) \) is the \( \alpha \)-th order difference of \( f(x) \) at \( x_0 \).

If there exists \( f^{(k+1)\alpha}(x) = \underbrace{D^\alpha_x \ldots D^\alpha_x}_\text{k+1 times} f(x) \) for any \( x \in I \subseteq \mathbb{R} \), then we denoted \( f \in D_{(k+1)\alpha}(I) \), where \( k = 0, 1, 2, \ldots \).

**Definition 2.3.** [3] Let \( f(x) \in C_{\alpha}[a, b] \). Then the local fractional integral is defined by,

\[
a^{\alpha}I^b_{\alpha} f(x) = \frac{1}{\Gamma(\alpha + 1)} \int_a^b f(t)(dt)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha,
\]

with \( \Delta t_j = t_{j+1} - t_j \) and \( \Delta t = \max \{\Delta t_1, \Delta t_2, \ldots, \Delta t_{N-1}\} \), where \( [t_j, t_{j+1}] \), \( j = 0, \ldots, N-1 \) and \( a = t_0 < t_1 < \ldots < t_{N-1} < t_N = b \) is partition of interval \([a, b]\).

Here, it follows that \( a^{\alpha}I^b_{\alpha} f(x) = 0 \) if \( a = b \) and \( a^{\alpha}I^b_{\alpha} f(x) = -b^{\alpha}I^a_{\alpha} f(x) \) if \( a < b \). If for any \( x \in [a, b] \), there exists \( a^{\alpha}I^b_{\alpha} f(x) \), then we denoted by \( f(x) \in I_{\alpha}[a, b] \).

**Definition 2.4.** [Generalized convex function] [11] Let \( f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha \). For any \( x_1, x_2 \in I \) and \( \lambda \in [0, 1] \), if the following inequality

\[
f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda^\alpha f(x_1) + (1 - \lambda)^\alpha f(x_2),
\]

holds, then \( f(x) \) is called to be \( \alpha \)-generalized convex.
holds, then \( f \) is called a generalized convex function on \( I \).

Here are two basic examples of generalized convex functions:

1. \( f(x) = x^{\alpha p}, \ x \geq 0, \ p > 1; \)
2. \( f(x) = E_{\alpha}(x^\alpha), \ x \in R \) where \( E_{\alpha}(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)} \) is the Mittag-Leffler function.

**Theorem 2.5.** Let \( f \in D_{\alpha}(I) \), then the following conditions are equivalent

a) \( f \) is a generalized convex function on \( I \)

b) \( f^{(\alpha)} \) is an increasing function on \( I \)

c) for any \( x_1, x_2 \in I \),

\[
f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)} (x_2 - x_1)^{\alpha}.
\]

**Corollary 2.6.** Let \( f \in D_{2\alpha}(a,b) \). Then \( f \) is a generalized convex function (or a generalized concave function) if and only if

\[
f^{(2\alpha)}(x) \geq 0 \quad \text{or} \quad f^{(2\alpha)}(x) \leq 0,
\]

for all \( x \in (a,b) \).

**Lemma 2.7.** [11] (1) (Local fractional integration is anti-differentiation)

Suppose that \( f(x) = g^{(\alpha)}(x) \in C_\alpha [a,b] \), then we have

\[
a I_0^\alpha f(x) = g(b) - g(a).
\]

(2) (Local fractional integration by parts) Suppose that \( f(x), g(x) \in D_\alpha [a,b] \) and \( f^{(\alpha)}(x), g^{(\alpha)}(x) \in C_\alpha [a,b] \), then we have

\[
a I_0^\alpha f(x) g^{(\alpha)}(x) = f(x) g(x)|_a^b - a I_0^\alpha f^{(\alpha)}(x) g(x).
\]

**Lemma 2.8.** [11]

\[
\frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha},
\]

\[
\frac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha} (dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} (b^{(k+1)\alpha} - a^{(k+1)\alpha}), \ k \in R.
\]
**Lemma 2.9.** [Generalized Hölder’s inequality] [11] Let $f, g \in C_\alpha[a, b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\frac{1}{\Gamma(\alpha + 1)} \int_a^b |f(x)g(x)| (dx)^\alpha
\leq \left( \frac{1}{\Gamma(\alpha + 1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left( \frac{1}{\Gamma(\alpha + 1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.
$$

In [3], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

**Theorem 2.10.** [Generalized Hermite-Hadamard’s inequality] Let $f(x) \in I_\alpha^p [a, b]$ be generalized convex function on $[a, b]$ with $a < b$. Then

$$
f\left( \frac{a + b}{2} \right) \leq \frac{\Gamma (1 + \alpha)}{(b - a)^\alpha} \quad aI_\beta^\alpha f(x) \leq \frac{f(a) + f(b)}{2^\alpha}.
$$

In [9], Sarikaya et al. gave the following generalized Bullen inequality for generalized convex function.

**Theorem 2.11.** [Generalized Bullen inequality] Let $f(x) \in I_\beta^\alpha [a, b]$ be a generalized convex function on $[a, b]$ with $a < b$. Then we have the inequality

$$
\frac{\Gamma (1 + \alpha)}{(b - a)^\alpha} aI_\beta^\alpha f(x) \leq \frac{1}{2^\alpha} \left[f\left( \frac{a + b}{2} \right) + \frac{f(a) + f(b)}{2^\alpha} \right]. \tag{4}
$$

For more information and recent developments on local fractional theory, please refer to [1], [3], [4], [6]-[15].

The aim of the paper is to establish some new generalized Hermite-Hadamard inequalities for generalized convex functions and we give some applications for some generalized special means.

### 3. Main Results

In this section, we establish some new inequalities for generalized convex functions.

**Theorem 3.1.** Let $I \subseteq R$ be an interval, $f : I^0 \subseteq R \rightarrow R^\alpha$ ($I^0$ is the interior of $I$) such that $f \in D_{2\alpha}(I^0)$ and $f^{(2\alpha)} \in C_{2\alpha}[a, b]$ for $a, b \in I^0$
with \( a < b \) Then, we have the identity:

\[
\frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) = \frac{1}{2^\alpha (b - a)^\alpha \left[ \Gamma(1 + \alpha) \right]^2} \int_a^b (x - a)^\alpha (b - x)^\alpha f(2\alpha)(x) (dx)^\alpha.
\]

**Proof.** Using the local fractional integration by parts twice (Lemma 2.7), we obtain

\[
\frac{1}{\Gamma(1 + \alpha)} \int_a^b (x - a)^\alpha (b - x)^\alpha f(\alpha)(x) (dx)^\alpha
\]

\[
= (x - a)^\alpha (b - x)^\alpha f(\alpha)(x) \bigg|_a^b
\]

\[
- \frac{1}{\Gamma(1 + \alpha)} \int_a^b \Gamma(1 + \alpha) (-2x + a + b)^\alpha f(\alpha)(x) (dx)^\alpha
\]

\[
= -\Gamma(1 + \alpha) (-2x + a + b)^\alpha f(x) \big|_a^b
\]

\[
+ \frac{1}{\Gamma(1 + \alpha)} \int_a^b \left[ \Gamma(1 + \alpha) \right]^2 (-2)^\alpha f(x) (dx)^\alpha
\]

\[
= -\Gamma(1 + \alpha) (a - b)^\alpha f(b) + \Gamma(1 + \alpha) (b - a)^\alpha f(a)
\]

\[
- \frac{2^\alpha \left[ \Gamma(1 + \alpha) \right]^2}{\Gamma(1 + \alpha)} \int_a^b f(x) (dx)^\alpha
\]

\[
= \Gamma(1 + \alpha) (b - a)^\alpha [f(a) + f(b)] - 2^\alpha \left[ \Gamma(1 + \alpha) \right]^2 a I_b^\alpha f(x).
\]

If we divide the resulting equality (6) by \( 2^\alpha (b - a)^{2\alpha} \Gamma(1 + \alpha) \), then we obtain the desired result. \( \Box \)
Theorem 3.2. With the above assumptions, given that $k^\alpha \leq f^{(2\alpha)}(x) \leq K^\alpha$ for all $x \in [a, b]$, $k, K \in \mathbb{R}$, we have the following inequality

\begin{align*}
  k^\alpha \cdot \frac{(b-a)^{2\alpha}}{2^\alpha \Gamma (1+\alpha)} \left[ \frac{\Gamma (1+\alpha)}{\Gamma (1+2\alpha)} - \frac{\Gamma (1+2\alpha)}{\Gamma (1+3\alpha)} \right] \\
  \leq \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma (1+\alpha)}{(b-a)^\alpha} a I_b^\alpha f(x) \\
  \leq K^\alpha \cdot \frac{(b-a)^{2\alpha}}{2^\alpha \Gamma (1+\alpha)} \left[ \frac{\Gamma (1+\alpha)}{\Gamma (1+2\alpha)} - \frac{\Gamma (1+2\alpha)}{\Gamma (1+3\alpha)} \right].
\end{align*}

Proof. From assumptions, we have

\[ k^\alpha (x-a)^\alpha (b-x)^\alpha \leq (x-a)^\alpha (b-x)^\alpha f^{(2\alpha)}(x) \leq K^\alpha (x-a)^\alpha (b-x)^\alpha \quad (8) \]

for all $x \in [a, b]$. Dividing (8) by $2^\alpha (b-a)^\alpha \Gamma (1+\alpha)^2$, then integrating the resulting inequality with respect to $x$ over $[a, b]$, we get

\begin{align*}
  \frac{k^\alpha}{2^\alpha (b-a)^\alpha \Gamma (1+\alpha)^2} \int_a^b (x-a)^\alpha (b-x)^\alpha (dx)^\alpha \\
  \leq \frac{1}{2^\alpha (b-a)^\alpha \Gamma (1+\alpha)^2} \int_a^b (x-a)^\alpha (b-x)^\alpha f^{(2\alpha)}(x) (dx)^\alpha \\
  \leq \frac{K^\alpha}{2^\alpha (b-a)^\alpha \Gamma (1+\alpha)^2} \int_a^b (x-a)^\alpha (b-x)^\alpha K^\alpha (x) (dx)^\alpha.
\end{align*}

From Theorem 3.1, we have

\begin{align*}
  \frac{1}{2^\alpha (b-a)^\alpha \Gamma (1+\alpha)^2} \int_a^b (x-a)^\alpha (b-x)^\alpha f^{(2\alpha)}(x) (dx)^\alpha \\
  = \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma (1+\alpha)}{(b-a)^\alpha} a I_b^\alpha f(x)
\end{align*}
and a simple calculating shows that

\[
\frac{1}{\Gamma(1+\alpha)} \int_a^b (x-a)^\alpha (b-x)^\alpha \, (dx)^\alpha = (b-a)^{3\alpha} \left[ \frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right].
\]

This completes the proof. \(\Box\)

**Corollary 3.3.** With the assumptions of Theorem 3.2, given that

\[
\left\| f^{(2\alpha)} \right\|_\infty := \sup_{x \in [a,b]} \left| f^{(2\alpha)}(x) \right|,
\]

then we have the following inequality.

\[
\left| \frac{f(a) + f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)\alpha} a I_b^\alpha f(x) \right|
\leq \frac{(b-a)^{2\alpha}}{2^\alpha \Gamma(1+\alpha)} \left[ \frac{\Gamma(1+1\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \right] \left\| f^{(2\alpha)} \right\|_\infty.
\]

**Theorem 3.4.** The assumptions of Theorem 3.1 are satisfied. If we assume that the new mapping \(\varphi : I^0 \subseteq \mathbb{R} \to \mathbb{R}^\alpha, \varphi(x) = (x-a)^\alpha (b-x)^\alpha f^{(2\alpha)}(x)\) is generalized convex on \([a,b]\), then we have the inequality

\[
\frac{(b-a)^{2\alpha}}{8^\alpha [\Gamma(1+\alpha)]^2} f^{(2\alpha)} \left( \frac{a+b}{2} \right)
\leq \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)\alpha} a I_b^\alpha f(x)
\leq \frac{(b-a)^{2\alpha}}{16^\alpha [\Gamma(1+\alpha)]^2} f^{(2\alpha)} \left( \frac{a+b}{2} \right).
\]

**Proof.** Because of the generalized convexity of \(\varphi\), applying the first inequality of generalized Hermite-Hadamard (Theorem 2.10) we can state:

\[
\frac{1}{(b-a)^\alpha} \int_a^b \varphi(x) \, (dx)^\alpha \geq \varphi \left( \frac{a+b}{2} \right) = \frac{(b-a)^{2\alpha}}{4^\alpha} f^{(2\alpha)} \left( \frac{a+b}{2} \right).
\]
Similarly, applying the generalized generalized Bullen inequality (Theorem 2.11) for $\varphi$, we get

$$\frac{1}{(b-a)^{\alpha}} \int_{a}^{b} \varphi(x) (dx)^{\alpha} \leq \frac{1}{2^{\alpha}} \left[ \varphi \left( \frac{a+b}{2} \right) + \varphi(a) + \varphi(b) \right]$$

(11)

Combining (10) and (11), we have

$$\frac{(b-a)^{2\alpha}}{4^{\alpha}} f(2\alpha) \left( \frac{a+b}{2} \right) \leq \frac{1}{(b-a)^{\alpha}} \int_{a}^{b} \varphi(x) (dx)^{\alpha} \leq \frac{(b-a)^{2\alpha}}{8^{\alpha}} f(2\alpha) \left( \frac{a+b}{2} \right).$$

(12)

If we divide the inequalities (12) by $2^{\alpha} [\Gamma(1+\alpha)]^{2}$, we obtain desired result, which completes the proof. □

**Theorem 3.5.** The assumptions of Theorem 3.1 are satisfied. Then we have the inequality

$$\left\| f(a) + f(b) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} a I_{b}^{\alpha} f(x) \right\| \leq \frac{(b-a)^{(1+\frac{1}{p})\alpha}}{2^{\alpha}} \left[ B(p+1, p+1) \right]^{\frac{1}{p}} \left\| f(2\alpha) \right\|_{q},$$

(13)

where, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\left\| f(2\alpha) \right\|_{q}$ is defined by

$$\left\| f(2\alpha) \right\|_{q} = \left( \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} \left| f(2\alpha)(x) \right|^{q} (dx)^{\alpha} \right)^{\frac{1}{q}}$$

and $B(x, y)$ is defined by

$$B(x, y) = \frac{1}{\Gamma(1+\alpha)} \int_{0}^{1} t^{(x-1)\alpha} (1-t)^{(y-1)\alpha} (dt)^{\alpha}.$$
Proof. Taking modulus in (5) and using the generalized Holder’s inequality, we have

\[ \left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) \right| \]

\[ \leq \frac{1}{2^\alpha (b - a)^\alpha \Gamma(1 + \alpha)} \int_a^b |(x - a)^\alpha (b - x)^\alpha| \left| f(2\alpha)(x) \right| (dx)^\alpha \]

\[ \leq \frac{1}{2^\alpha (b - a)^\alpha \Gamma(1 + \alpha)} \left( \frac{1}{\Gamma(1 + \alpha)} \int_0^b \left| f(2\alpha)(x) \right|^q (dx)^\alpha \right)^{\frac{1}{q}} \]

\[ \times \left( \frac{1}{\Gamma(1 + \alpha)} \int_a^b (x - a)^{p\alpha} (b - x)^{p\alpha} (dx)^\alpha \right)^{\frac{1}{p}} \]

\[ = \frac{\|f(2\alpha)\|_q}{2^\alpha (b - a)^\alpha \Gamma(1 + \alpha)} \left( \frac{1}{\Gamma(1 + \alpha)} \int_a^b (x - a)^{p\alpha} (b - x)^{p\alpha} (dx)^\alpha \right)^{\frac{1}{p}}. \]

Using the changing variable \( x = (1 - t)a + tb \), we have

\[ \frac{1}{\Gamma(1 + \alpha)} \int_a^b (x - a)^{p\alpha} (b - x)^{p\alpha} (dx)^\alpha = (b - a)^{2p+1} \alpha B(p + 1, p + 1), \]

which completes the proof. □

Theorem 3.6 The assumptions of Theorem ?? are satisfied. If \( |f(2\alpha)| \) is generalized convex, then we have the inequality

\[ \left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) \right| \leq \left( \frac{(b - a)^2}{\Gamma(1 + \alpha)} \right)^\alpha \left( \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)} - \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 4\alpha)} \right) \left[ \frac{\|f(2\alpha)(a)\| + \|f(2\alpha)(b)\|}{2^\alpha} \right]. \]
Proof. Taking modulus in (5) we get

$$\left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) \right|$$

$$\leq \frac{1}{2^\alpha (b - a)^\alpha \Gamma(1 + \alpha)^2} \int_a^b (x - a)^\alpha (b - x)^\alpha \left| f^{(2\alpha)}(x) \right| (dx)^\alpha.$$  

Since generalized convexity of $|f^{(2\alpha)}|$, we have

$$\left| f^{(2\alpha)}(x) \right| = \left| f^{(2\alpha)} \left( \frac{x - a}{b - a} b + \frac{b - x}{b - a} a \right) \right|$$

$$\leq \left( \frac{x - a}{b - a} \right)^\alpha \left| f^{(2\alpha)}(b) \right| + \left( \frac{b - x}{b - a} \right)^\alpha \left| f^{(2\alpha)}(a) \right|.$$  

Then, it follows that

$$\left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) \right|$$

$$\leq \frac{1}{2^\alpha (b - a)^\alpha \Gamma(1 + \alpha)} \left[ \left| f^{(2\alpha)}(b) \right| \frac{(b - a)^\alpha \Gamma(1 + \alpha)}{(b - a)^\alpha \Gamma(1 + \alpha)} \int_a^b (x - a)^{2\alpha}(b - x)^\alpha (dx)^\alpha$$

$$+ \frac{\left| f^{(2\alpha)}(a) \right|}{(b - a)^\alpha \Gamma(1 + \alpha)} \int_a^b (x - a)^\alpha(b - x)^{2\alpha} (dx)^\alpha \right].$$

Using the changing variable $x = (1 - t) a + t b$, we have

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b (x - a)^{2\alpha}(b - x)^\alpha (dx)^\alpha$$

$$= (b - a)^{4\alpha} \int_a^b t^{2\alpha}(1 - t)^\alpha (dt)^\alpha$$

$$= (b - a)^{4\alpha} \left( \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)} - \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 4\alpha)} \right).$$
and similarly
\[ \frac{1}{\Gamma(1 + \alpha)} \int_a^b (x-a)^\alpha (b-x)^{2\alpha} (dx)^\alpha = (b-a)^{4\alpha} \left( \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + 3\alpha)} - \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 4\alpha)} \right). \]

Putting (16) and (17) in (15), we obtain required result. □

4. Applications to Some Special Means

We consider some generalized means as in [7]:
\[ A(a, b) = \frac{a^\alpha + b^\alpha}{2^\alpha}; \]
\[ L_n(a, b) = \left[ \frac{\Gamma(1 + n\alpha)}{\Gamma(1 + (n + 1)\alpha)} \left[ \frac{b^{(n+1)\alpha} - a^{(n+1)\alpha}}{(b - a)^\alpha} \right] \right]^{\frac{1}{n}}, \]
where \( n \in \mathbb{Z} \setminus \{-1, 0\} \), \( a, b \in R, a \neq b \).

**Proposition 4.1.** Let \( a, b \in R, 0 < a < b, 0 \notin [a, b] \) and \( n \in Z, |n(n-1)| \geq 3 \). Then, we have the inequality
\[ |A(a^n, b^n) - \Gamma(1 + \alpha) [L_n(a, b)]^n| \]
\[ \leq \frac{(b-a)^{\left(1+\frac{1}{p}+\frac{1}{q}\right)\alpha}}{2^\alpha} [B(p + 1, p + 1)]^{\frac{1}{p}} \]
\[ \times \left| \frac{\Gamma(1 + n\alpha)}{\Gamma(1 + (n - 2)\alpha)} \right|^{\frac{1}{q}} \left[ L_{q(n-2)}^{n-2}(a, b) \right]. \]

**Proof.** Let us reconsider the inequality (13):
\[ \left| \frac{f(a) + f(b)}{2^\alpha} - \frac{\Gamma(1 + \alpha)}{(b - a)^\alpha} a I_b^\alpha f(x) \right| \]
\[ \leq \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{2^\alpha} [B(p + 1, p + 1)]^{\frac{1}{p}} \left\| f(2\alpha) \right\|_q. \]
Consider the mapping $f : (0, \infty) \rightarrow \mathbb{R}^\alpha$, $f(x) = x^n\alpha$, $n \in \mathbb{Z} \setminus \{-1, 0\}$. Then, $0 < a < b$, we have

$$\frac{f(a) + f(b)}{2^\alpha} = A(a^n, b^n), \quad \frac{a^\alpha f(x)}{(b-a)^\alpha} = [L_n(a, b)]^n,$$

$$|f^{(2\alpha)}(x)| = \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)} \right| x^{(n-2)\alpha}$$

and

$$\left\| f^{(2\alpha)} \right\|_q = \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)} \right|^\frac{1}{q} \left( \frac{1}{\Gamma(1+\alpha)} \int_a^b x^{q(n-2)\alpha} (dx)^\alpha \right)^\frac{1}{q}$$

$$= \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)} \right|^\frac{1}{q} \left( \frac{\Gamma(1+q(n-2)\alpha)}{\Gamma(1+(q(n-2)+1)\alpha)} \left( b^{q(n-2)+1)\alpha} - a^{q(n-2)+1)\alpha} \right) \right)^\frac{1}{q}$$

$$= \left| \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)} \right|^\frac{1}{q} (b-a)^{q\alpha} [L_{q(n-2)}(a, b)]^{n-2}.$$
This completes the proof. □

**Proposition 4.2.** Let \( a, b \in R, 0 < a < b, 0 \not\in [a, b] \) and \( n \in Z, |n(n-1)| \geq 3 \). Then, we have the inequality

\[
|A(a^n, b^n) - \Gamma (1 + \alpha) [L_n(a, b)]^n| = \frac{(b - a)^{2\alpha}}{\Gamma(1+\alpha)} \left| \frac{\Gamma (1 + n\alpha)}{\Gamma (1 + (n-2)\alpha)} \right| \times \left( \frac{\Gamma (1 + 2\alpha)}{\Gamma (1 + 3\alpha)} - \frac{\Gamma (1 + 3\alpha)}{\Gamma (1 + 4\alpha)} \right) A(a^{n-2}, b^{n-2}).
\]

**Proof.** The proof is obvious from Theorem 3. applied \( f(x) = x^{n\alpha}, n \in Z \setminus \{-1, 0\} \). □

**References**


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