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New Subclass of Univalent Function Involving the Modified Sigmoid Function Using Subordination Principle

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Abstract. In this work, a new subclass of function, $G_\gamma(n,\mu,\lambda)$: $n \in N_0, \mu \geqslant 1, 0 \leqslant \lambda \leqslant 1$, was defined using the Sălăgean differential operator involving the modified sigmoid function and subordination principle. The initial coefficient bounds and the Fekete-Szego functional of this class were obtained.

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1. Introduction

Let the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are analytic in the unit disk $\mathbf{D} = \{z : z \in \mathbb{C} : |z| < 1\}$ be denoted by $\mathbf{A}$.

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Sălăgean [7] introduced the differential operator $D^n f, n \in N_0 = 0, 1, 2, \cdots$ for functions $f(z)$ belonging to class $\mathbf{A}$ of analytic functions in the unit disk $\mathbf{D}$;

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k.$$

The sigmoid function

$$h(z) = \frac{1}{1+e^{-z}}$$

is differentiable and has the following properties:

1. It outputs real numbers between 0 and 1.
2. It maps a very large input domain to a small range of outputs.
3. It never loses information because it is a one-to-one function.
4. It increases monotonically.

Fadipe-Joseph et al. [4] studied the modified sigmoid function

$$G(z) = \frac{2}{1+e^{-z}},$$

and obtained another series of the modified sigmoid function

$$G(z) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \right).$$

Definition 1.1. (Duren [3], Goodman [6]) *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be analytic and univalent in a domain $\boldsymbol{D}$ and suppose $g(z) \in \boldsymbol{D}$. If $g(z)$ is analytic in $\boldsymbol{D}$, $f(0) = g(0)$ and $f(z) \subset g(z)$, then we say that $f(z)$ is subordinate to $g(z)$ in $\boldsymbol{D}$ and it is denoted by*

$$f(z) \prec g(z).$$

Definition 1.2. *A function $f \in \mathbf{A}$ is said to be in the class $G_\gamma(n, \mu, \lambda) : n \in N_0$,*
$\mu \geqslant 1, 0 \leqslant \lambda \leqslant 1$, if the following subordination holds

$$(1-\lambda)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu} + \lambda f'_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} \prec G(z). \qquad (1)$$

Altinkaya and Yalcin [1] defined a subclass of univalent functions and obtained coefficients expansion using Chebyshev polynomial. Also, Altinkaya and Yalcin [2] obtained the Faber polynomial coefficient bound for a subclass of bi-univalent function. The definition of this class was motivated by their work.

Lemma 1.3. *Let $f(z)$ and $g(z)$ be analytic in* $\mathbf{D}$ *and suppose that $g(z)$ is univalent in* $\mathbf{D}$. *Then $f(z) \prec g(z)$ if and only if there exists a Schwarz function $\omega(z)$ with the property $|\omega(z)| \leqslant |z|$ such that*

$$f(z) = g(\omega(z)).$$

1.1 Sălăgean differential operator involving modified sigmoid function

Consider the function

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{2}$$

where,

$$\gamma(s) = \frac{2}{1+e^{-s}},$$

which are analytic and univalent in the unit disk. Then functions of the form (2) belong to the class $\mathbf{A}_\gamma$.

Let $D^n f_\gamma(z); n \in N_0 = \{0, 1, 2, \cdots\}$ denote the Sălăgean differential operator involving modified sigmoid function, then

$$D^0 f_\gamma(z) = f_\gamma(z),$$
$$D^1 f_\gamma(z) = \gamma(s) z f'_\gamma(z),$$
$$\vdots$$
$$D^n f_\gamma(z) = \gamma(s) z (D^{n-1} f_\gamma(z))'. \tag{3}$$

Taking $\lim_{s\to 0} \gamma(s) = 1$, we have the Sălăgean differential operator. (Fadipe-Joseph et al. [5])

2. Main Results

Theorem 2.1. *If $f(z)$ belongs to the class $G_\gamma(n, \mu, \lambda) : n \in N_0, \mu \geqslant 1, 0 \leqslant \lambda \leqslant 1$, then*

$$|a_2| \leqslant \frac{1}{2(A+B)},$$

$$|a_3| \leqslant \frac{1}{2(A'+B')} + \frac{D+E}{4(A+B)^2(A'+B')},$$

$$|a_4| \leqslant \frac{11}{24(D'+E')} + \frac{F+G}{4(A+B)(A'+B')} + \frac{(D+E)(F+G)}{8(A+B)^3(A'+B')} + \frac{F'+G'}{8(A+B)^3}.$$

In particular, when

$$G(z) = 1 + \left(\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m \right).$$

Taking $m = 1$*, we have*

$$G(z) = 1 + \sum_{n=1}^{\infty} c_n z^n; \qquad c_n = \frac{(-1)^{n+1}}{2n!}.$$

If $\omega(z) = G(z) - 1$*, then*

$$a_2 = \frac{1}{4(A+B)},$$

$$a_3 = \frac{-1}{8(A'+B')} - \frac{D+E}{16(A+B)^2(A'+B')},$$

$$a_4 = \frac{7}{192(D'+E')} + \frac{F+G}{32(A+B)(A'+B')(D'+E')},$$

$$+ \frac{(D+E)(F+G)}{64(A+B)^3(A'+B')(D'+E')} - \frac{F'+G'}{64(A+B)^3(D'+E')}$$

where,

$$A = (1-\lambda)2^n \mu \gamma^{n\mu+1}(s),$$

$$B = \lambda \gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2),$$

$$A' = (1-\lambda)3^n \mu \gamma^{n\mu+1}(s),$$

$$B' = \lambda \gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3),$$

$$D = (1-\lambda)2^{2n} \frac{\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s),$$

$$E = \lambda \gamma^{n(\mu-1)+2}(s) \left(\frac{(\mu-1)(\mu-2)}{2!} 2^{2n} + 2^{n+1}(\mu-1) \right),$$

$$D' = (1-\lambda)4^n \mu \gamma^{n\mu+1}(s),$$

$$E' = \lambda \gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4),$$

$$F=(1-\lambda)2^{n+1}\cdot 3^n\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s),$$
$$G=\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n+(\mu-1)(2\cdot 3^n+3\cdot 2^n)\right),$$
$$F'=(1-\lambda)2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s),$$
$$G'=\lambda\gamma^{n(\mu-1)+3}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n}+\frac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right).$$

Proof. If $f(z)\in G_\gamma(n,\mu,\lambda)$, then from (1)

$$(1-\lambda)\left(\frac{D^n f_\gamma(z)}{z}\right)^\mu+\lambda f'_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1}\prec G(z),$$

$$f_\gamma(z)=z+\sum_{k=2}^{\infty}\gamma(s)a_k z^k, \tag{4}$$

where,

$$\gamma(s)=\frac{2}{1+e^{-s}};\qquad s\geqslant 0,$$
$$f_\gamma(z)=z+\gamma(s)(a_2z^2+a_3z^3+a_4z^4+a_5z^5+\cdots). \tag{5}$$

From (3), we have

$$\frac{D^n f_\gamma(z)}{z}=\frac{\gamma^n(s)z+\sum_{k=2}^{\infty}k^n\gamma^{n+1}(s)a_kz^k}{z},$$
$$\frac{D^n f_\gamma(z)}{z}=\gamma^n(s)+\sum_{k=2}^{\infty}k^n\gamma^{n+1}(s)a_kz^{k-1}, \tag{6}$$

$$\frac{D^n f_\gamma(z)}{z}=\gamma^n(s)+2^n\gamma^{n+1}(s)a_2z+3^n\gamma^{n+2}(s)a_3z^2+4^n\gamma^{n+3}(s)a_4z^3+5^n\gamma^{n+4}(s)a_5z^4+\cdots$$
$$\left(\frac{D^n f_\gamma(z)}{z}\right)^\mu=\gamma^{n\mu}(s)(1+2^n\gamma(s)a_2z+3^n\gamma^2(s)a_3z^2+4^n\gamma^3(s)a_4z^3+5^n\gamma^4(s)a_5z^4+\cdots)^\mu,$$

$$\left(\frac{D^n f_\gamma(z)}{z}\right)^\mu=\gamma^{n\mu}(s)\left(\begin{array}{c}1+\mu\gamma(s)\left(\begin{array}{c}2^na_2z+3^na_3z^2\\+4^na_4z^3+5^na_5z^4+\cdots\end{array}\right)\\+\frac{\mu(\mu-1)}{2!}\gamma^2(s)\left(\begin{array}{c}2^na_2z+3^na_3z^2\\+4^na_4z^3+5^na_5z^4+\cdots\end{array}\right)^2\\+\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^3(s)\left(\begin{array}{c}2^na_2z+3^na_3z^2\\+4^na_4z^3+5^na_5z^4+\cdots\end{array}\right)^3+\cdots\end{array}\right), \tag{7}$$

$$\left(\frac{D^n f_\gamma(z)}{z}\right)^\mu = \gamma^{n\mu}(s) + 2^n \mu \gamma^{n\mu+1}(s) a_2 z$$
$$+ \left(3^n \mu \gamma^{n\mu+1}(s) a_3 + 2^{2n} \frac{\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s) a_2^2\right) z^2$$
$$+ \left(\begin{array}{c} 4^n \mu \gamma^{n\mu+1}(s) a_4 + 2^n \cdot 3^n \frac{2\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s) a_2 a_3 \\ +2^{3n} \frac{\mu(\mu-1)(\mu-2)}{3!} \gamma^{n\mu+3}(s) a_2^3 \end{array}\right) z^3.$$
$$+\cdots \tag{8}$$

Similarly,

$$\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} = \gamma^{n(\mu-1)}(s) + 2^n(\mu-1)\gamma^{n(\mu-1)+1}(s) a_2 z$$
$$+ \left(3^n(\mu-1)\gamma^{n(\mu-1)+1}(s) a_3 + 2^{2n} \frac{(\mu-1)(\mu-2)}{2!} \gamma^{n(\mu-1)+2}(s) a_2^2\right) z^2$$
$$+ \left(\begin{array}{c} 4^n(\mu-1)\gamma^{n(\mu-1)+1}(s) a_4 \\ +2^n \cdot 3^n \frac{2(\mu-1)(\mu-2)}{2!} \gamma^{n(\mu-1)+2}(s) a_2 a_3 \\ +2^{3n} \frac{(\mu-1)(\mu-2)(\mu-3)}{3!} \gamma^{n(\mu-1)+3}(s) a_2^3 \end{array}\right) z^3.$$
$$+\cdots \tag{9}$$

Differentiating (5) with respect to z, we have

$$f'_\gamma(z) = 1 + \gamma(s)(2a_2 z + 3a_3 z^2 + 4a_4 z^3 + 5a_5 z^4 + \cdots)\cdot \tag{10}$$

Multiplying (9) and (10), we have

$$f'_\gamma(z)\left(\frac{D^n f_\gamma(z)}{z}\right)^{\mu-1} = \gamma^{n(\mu-1)}(s) + \gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2) a_2 z$$
$$+ \left(\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3) a_3 + \gamma^{n(\mu-1)+2}\left(2^{2n}\frac{(\mu-1)(\mu-2)}{2!} + 2^{n+1}(\mu-1)\right) a_2^2 \right) z^2$$
$$+ \left(\begin{array}{c} \gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4) a_4 \\ +\gamma^{n(\mu-1)+2}(s)\left(2^{n+1}\cdot 3^n \frac{(\mu-1)(\mu-2)}{2!} + (\mu-1)(2^n\cdot 3 + 2\cdot 3^n)\right) a_2 a_3 \\ +\gamma^{n(\mu-1)+3}(s)\left(2^{3n}\frac{(\mu-1)(\mu-2)(\mu-3)}{3!} + 2^{2n+1}\frac{(\mu-1)(\mu-2)}{2!}\right) a_2^3 \end{array}\right) z^3 +$$
$$\cdots$$

$$
\begin{aligned}
&(1-\lambda)\left(\tfrac{D^n f_\gamma(z)}{z}\right)^{\mu} + \lambda f'_\gamma(z)\left(\tfrac{D^n f_\gamma(z)}{z}\right)^{\mu-1} = (1-\lambda)\gamma^{n\mu}(s) + \lambda\gamma^{n(\mu-1)}(s)\\
&+((1-\lambda)2^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2))a_2 z\\
&+\begin{pmatrix} ((1-\lambda)3^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3))a_3 \\ \begin{pmatrix} (1-\lambda)2^{2n}\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) \\ +\lambda\gamma^{n(\mu-1)+2}(s)\left(\frac{(\mu-1)(\mu-2)}{2!}2^{2n} + 2^{n+1}(\mu-1)\right)\end{pmatrix} a_2^2 \end{pmatrix} z^2\\
&+\begin{pmatrix} ((1-\lambda)4^n\mu\gamma^{n\mu+1}(s) + \lambda\gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4))a_4 \\ +\begin{pmatrix} (1-\lambda)2^{n+1}\cdot 3^n\frac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s) \\ +\lambda\gamma^{n(\mu-1)+2}(s)\begin{pmatrix}\frac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n \\ +(\mu-1)(2\cdot 3^n + 3\cdot 2^n)\end{pmatrix}\end{pmatrix} a_2 a_3 \\ +\begin{pmatrix} (1-\lambda)2^{3n}\frac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s) \\ +\lambda\gamma^{n(\mu-1)+3}(s)\left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n} + \frac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right)\end{pmatrix} a_2^3 \end{pmatrix} z^3,\\
&+\cdots
\end{aligned}
\tag{11}
$$

$$G(z) = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \cdots, \tag{12}$$

$$G(\omega(z)) = 1 + \frac{1}{2}\omega(z) - \frac{1}{24}\omega(z)^3 + \frac{1}{240}\omega(z)^5 - \cdots, \tag{13}$$

$$\omega(z) = c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots, \tag{14}$$

$$\omega^3(z) = c_1^3 z^3 + 3c_1^2 z^4 + (3c_1^2 c_3 + 3c_1 c_2^2)z^5 + \cdots, \tag{15}$$

$$\omega^5(z) = c_1^5 z^5 + \cdots. \tag{16}$$

Substituting (14), (15) and (16) into (13), we have

$$G(\omega(z)) = 1 + \frac{c_1}{2}z + \frac{c_2}{2}z^2 + \left(\frac{c_3}{2} - \frac{c_1^3}{24}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1^2 c_2}{8}\right)z^4 + \cdots. \tag{17}$$

It is well known that if $|\omega(z)| = |c_1 z + c_2 z^2 + \cdots| < 1$, then

$$|c_j| \leqslant 1, \tag{18}$$

for all $j \in \mathbf{N}$ and

$$|c_2 - \rho c_1^2| \leqslant max\{1, |\rho|\}. \tag{19}$$

Equating (11) and (17) and comparing the coefficients, we have

$$((1-\lambda)2^n\mu\gamma^{n\mu+1}(s)+\lambda\gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2))a_2z=\tfrac{c_1}{2}$$

$$((1-\lambda)3^n\mu\gamma^{n\mu+1}(s)+\lambda\gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3))a_3$$
$$+\left((1-\lambda)2^{2n}\tfrac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)+\lambda\gamma^{n(\mu-1)+2}(s)\left(\tfrac{(\mu-1)(\mu-2)}{2!}2^{2n}+2^{n+1}(\mu-1)\right)\right)a_2^2=\tfrac{c_2}{2}$$

$$((1-\lambda)4^n\mu\gamma^{n\mu+1}(s)+\lambda\gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4))a_4$$
$$+\left(\begin{array}{c}(1-\lambda)2^{n+1}\cdot 3^n\tfrac{\mu(\mu-1)}{2!}\gamma^{n\mu+2}(s)+\\ \lambda\gamma^{n(\mu-1)+2}(s)\left(\tfrac{(\mu-1)(\mu-2)}{2!}2^{n+1}\cdot 3^n+(\mu-1)(2\cdot 3^n+3\cdot 2^n)\right)\end{array}\right)a_2a_3$$
$$+\left(\begin{array}{c}(1-\lambda)2^{3n}\tfrac{\mu(\mu-1)(\mu-2)}{3!}\gamma^{n\mu+3}(s)+\\ \lambda\gamma^{n(\mu-1)+3}(s)\left(\tfrac{(\mu-1)(\mu-2)(\mu-3)}{3!}2^{3n}+\tfrac{(\mu-1)(\mu-2)}{2!}2^{2n+1}\right)\end{array}\right)a_2^3=\tfrac{c_3}{2}-\tfrac{c_1^3}{24}.$$

Simplifying the above, we have

$$\begin{array}{rl} a_2 = & \frac{c_1}{2(A+B)}, \\ a_3 = & \frac{c_2}{2(A'+B')}-\frac{c_1^2(D+E)}{4(A+B)^2(A'+B')} \\ a_4 = & \frac{1}{D'+E'}\left\{\left(\frac{c_3}{2}-\frac{c_1^3}{24}\right)-\frac{c_1c_2(F+G)}{4(A+B)(A'+B')}\right. \\ & \left.+\frac{c_1^3(D+E)(F+G)}{8(A+B)^3(A'+B')}-\frac{c_1^3(F'+G')}{8(A+B)^3}\right\}. \end{array} \tag{20}$$

Then, from the inequality (18), we have

$$|a_2|=\left|\frac{c_1}{2(A+B)}\right|\leqslant\frac{1}{2(A+B)},$$

$$|a_3|=\left|\frac{c_2}{2(A'+B')}-\frac{c_1^2(D+E)}{4(A+B)^2(A'+B')}\right|\leqslant\frac{1}{2(A'+B')}+\frac{(D+E)}{4(A+B)^2(A'+B')},$$

$$|a_4|=\left|\begin{array}{c}\frac{1}{D'+E'}\left(\frac{c_1}{2}-\frac{c_1^3}{24}\right)-\frac{c_1c_2(F+G)}{4(A+B)(A'+B')}\\ +\frac{c_1^3(D+E)(F+G)}{8(A+B)^3(A'+B')}-\frac{c_1^3(F'+G')}{8(A+B)^3}\end{array}\right|\leqslant\begin{array}{c}\frac{11}{24(D'+E')}+\frac{(F+G)}{4(A+B)(A'+B')}\\ +\frac{(D+E)(F+G)}{8(A+B)^3(A'+B')}+\frac{(F'+G')}{8(A+B)^3},\end{array} \tag{21}$$

Also, when

$$G(z)=1+\left(\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}z^n\right)^m\right),$$

for $m=1$,

$$G(z)-1=\frac{1}{2}z-\frac{1}{4}z^2+\frac{1}{12}z^3-\cdots.$$

Comparing with (14), we have

$$c_1=\tfrac{1}{2};\quad c_2=-\tfrac{1}{4};\quad c_3=\tfrac{1}{12};\quad\cdots. \tag{22}$$

Substituting (22) into (20), we have

$$a_2=\frac{1}{4(A+B)},$$

$$a_3 = \frac{-1}{8(A'+B')} - \frac{D+E}{16(A+B)^2(A'+B')} \ ,$$

$$a_4 = \frac{7}{192(D'+E')} + \frac{F+G}{32(A+B)(A'+B')(D'+E')} \ ,$$

$$+ \frac{(D+E)(F+G)}{64(A+B)^3(A'+B')(D'+E')} - \frac{F'+G'}{64(A+B)^3(D'+E')},$$

where,

$$A = (1-\lambda)2^n \mu \gamma^{n\mu+1}(s),$$

$$B = \lambda \gamma^{n(\mu-1)+1}(s)(2^n(\mu-1)+2),$$

$$A' = (1-\lambda)3^n \mu \gamma^{n\mu+1}(s),$$

$$B' = \lambda \gamma^{n(\mu-1)+1}(s)(3^n(\mu-1)+3),$$

$$D = (1-\lambda)2^{2n} \frac{\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s),$$

$$E = \lambda \gamma^{n(\mu-1)+2}(s) \left(\frac{(\mu-1)(\mu-2)}{2!} 2^{2n} + 2^{n+1}(\mu-1) \right),$$

$$D' = (1-\lambda)4^n \mu \gamma^{n\mu+1}(s),$$

$$E' = \lambda \gamma^{n(\mu-1)+1}(s)(4^n(\mu-1)+4),$$

$$F = (1-\lambda)2^{n+1} \cdot 3^n \frac{\mu(\mu-1)}{2!} \gamma^{n\mu+2}(s),$$

$$G = \lambda \gamma^{n(\mu-1)+2}(s) \left(\frac{(\mu-1)(\mu-2)}{2!} 2^{n+1} \cdot 3^n + (\mu-1)(2 \cdot 3^n + 3 \cdot 2^n) \right),$$

$$F' = (1-\lambda)2^{3n} \frac{\mu(\mu-1)(\mu-2)}{3!} \gamma^{n\mu+3}(s),$$

$$G' = \lambda \gamma^{n(\mu-1)+3}(s) \left(\frac{(\mu-1)(\mu-2)(\mu-3)}{3!} 2^{3n} + \frac{(\mu-1)(\mu-2)}{2!} 2^{2n+1} \right). \quad \square$$

Corollary 2.2. *If $f(z)$ belongs to the class $G_\gamma(0,1,0)$, then*

$$|a_2| \leqslant \frac{1}{2\gamma(s)} \ ,$$

$$|a_3| \leqslant \frac{1}{2\gamma(s)} \ ,$$

$$|a_4| \leqslant \frac{11}{24\gamma(s)} \ .$$

Proof. Substituting $n = 0$, $\mu = 1$ and $\lambda = 0$ into the inequalities (21), we have the above result. □

Corollary 2.3. *If $f(z)$ belongs to the class $G_\gamma(0, 1, 1)$, then*

$$|a_2| \leqslant \frac{1}{4\gamma(s)},$$

$$|a_3| \leqslant \frac{1}{6\gamma(s)},$$

$$|a_4| \leqslant \frac{11}{96\gamma(s)}.$$

Proof. Substituting $n = 0$, $\mu = 1$ and $\lambda = 1$ into the inequalities (21), we have the above result. □

Corollary 2.4. *If $f(z)$ belongs to the class $G_\gamma(1, 1, 0)$, then*

$$|a_2| \leqslant \frac{1}{4\gamma^2(s)},$$

$$|a_3| \leqslant \frac{1}{6\gamma^2(s)},$$

$$|a_4| \leqslant \frac{11}{96\gamma^2(s)}.$$

Proof. Substituting $n = 1$, $\mu = 1$ and $\lambda = 0$ into the inequalities (21), we have the above result. □

Corollary 2.5. *If $f(z)$ belongs to the class $G_\gamma(1, 1, 1)$, then*

$$|a_2| \leqslant \frac{1}{4\gamma(s)},$$

$$|a_3| \leqslant \frac{1}{6\gamma(s)},$$

$$|a_4| \leqslant \frac{11}{96\gamma(s)}.$$

Proof. Substituting $n = 1$, $\mu = 1$ and $\lambda = 1$ into the inequalities (21), we have the above result. □

2.1 Fekete-szego inequality

The Fekete-Szego functional $|a_3 - \rho a_2^2|$ for functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

that are normalized and univalent in the unit disk is given as

$$|a_3 - \rho a_2^2| \leqslant 1 + 2exp(-2\rho/(1-\rho)); \qquad 0 < \rho < 1.$$

Theorem 2.6. *If $f(z)$ belongs to the class $G_\gamma(n, \mu, \lambda) : n \in N_0, \mu \geqslant 1, 0 \leqslant \lambda \leqslant 1$, then*

$$|a_3 - \rho a_2^2| \leqslant \frac{1}{2(A'+B')} max \left\{ 1, \left| \frac{D+E}{2(A+B)^2} + \frac{\rho(A'+B')}{2(A+B)^2} \right| \right\}.$$

Proof. From (20),

$$a_2 = \frac{c_1}{2(A+B)},$$

$$a_3 = \frac{c_2}{2(A'+B')} - \frac{c_1^2(D+E)}{4(A+B)^2(A'+B').}$$

Substituting these into $|a_3 - \rho a_2^2|$, we have

$$|a_3 - \rho a_2^2| = \left| \frac{c_2}{2(A'+B')} - \frac{c_1^2(D+E)}{4(A+B)^2(A'+B')} - \frac{\rho c_1^2}{4(A+B)^2} \right|$$

$$= \left| \frac{c_2}{2(A'+B')} - c_1^2 \left(\frac{D+E}{4(A+B)^2(A'+B')} + \frac{\rho}{4(A+B)^2} \right) \right|$$

$$= \frac{1}{2(A'+B')} \left| c_2 - c_1^2 \left(\frac{D+E}{2(A+B)^2} + \frac{\rho(A'+B')}{2(A+B)^2} \right) \right|.$$

Then, from the inequality (19), we have

$$|a_3 - \rho a_2^2| \leqslant \frac{1}{2(A'+B')} max \left\{ 1, \left| \frac{D+E}{2(A+B)^2} + \frac{\rho(A'+B')}{2(A+B)^2} \right| \right\}. \quad \square$$

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References

[1] S. Altinkaya and S. Yalcin, On the Chebyshev polynomial bounds for classes of univalent functions, *Khayyam J. Math.*, 2 (1) (2016), 1-5.

[2] S. Altinkaya and S. Yalcin, Faber polynomial coefficient bounds for a subclass of bi-univalent functions, *Studia Universitatis Babes-Bolyai Mathematica*, 61 (1) (2016), 37-44.

[3] P. L. Duren,*Univalent Functions*, Springer-Verlag, New York Inc., (1983).

[4] O. A. Fadipe-Joseph, A. T. Oladipo, and A. U. Ezeafulukwe, Modified sigmoid function in univalent theory,*International Journal of Mathematical Sciences and Engineering Applications (IJMSEA)*, 7 (v) (2013), 313-317.

[5] O. A. Fadipe-Joseph, B. O. Moses, and M. O. Oluwayemi, Certain new classes of analytic functions defined by using sigmoid function, *Advances in Mathematics: Scientific Journal* , 5 (1) (2016), 83-89.

[6] A. W. Goodman, *Univalent Functions*, Mariner, Tampa, Florida, 1-2, (1983).

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