Journal of Mathematical Extension
Vol. 12, No. 3, (2018), 21-31
ISSN: 1735-8299
URL: http://www.ijmex.com

Weak amenability of semigroup algebras modulo an ideal

A. Mehrabi
Central Tehran Branch, Islamic Azad University

H. Rahimi*
Central Tehran Branch, Islamic Azad University

Abstract. The main purpose of this paper is to investigate weak amenability of semigroup algebras. We relate this to a new notion of weak amenability modulo an ideal of Banach algebras. As an important result, we show that $l^1(S)$ is weakly amenable modulo I_σ, where I_σ is the corresponding ideal of the group congruence σ.

AMS Subject Classification: 43A07; 46H25.
Keywords and Phrases: Weak amenability, weak amenability modulo an ideal, semigroup algebra, group congruence.

1 Introduction

The notion of amenability for Banach algebras started in 1972, when B.E. Johnson proved the remarkable result that amenability of a locally compact group G (discrete group G) is equivalent to amenability of the group algebra $L^1(G)$ [16]. Over the years, many different variations of amenability have been introduced, among which one can refer

Received: June 2017; Accepted: October 2017
*Corresponding Author

to [19] that is a good survey of various types of amenability. The concept of weak amenability for commutative Banach algebras was introduced by W.G. Bade, P.C. Curtis and H.G. Dales in 1987 [3]. Indeed, by definition, a Banach algebra is weakly amenable if $H^1(A; A^*) = 0$. It is known that for a locally compact group (discrete group) G, the group algebra $L^1(G)$ $(l^1(G))$ is always weakly amenable [17], but it is not known when $l^1(S)$ is weakly amenable (see [4, 5, 13]). Recently amenability modulo an ideal was introduced by the first author and M. Amini in [2]. They showed that the amenability of semigroup S is equivalent to the amenability of semigroup algebra $l^1(S)$ modulo I, where I is a closed ideal corresponding to the least group congruence σ on S. The main purpose of this paper is to characterize the weak amenability modulo an ideal of semigroup algebras. We firstly introduce the concept of the weak amenability modulo an ideal of Banach algebras and we present some useful results, then using the obtained results we consider weak amenability modulo an ideal of semigroup algebras. We show that when S is an eventually inverse semigroup or S is an $E-$inversive $E-$semigroup with commuting idempotents, then the semigroup algebra $l^1(S)$ is weakly amenable modulo I_σ, where I_σ is the corresponding ideal of the least group congruence σ. Finally, we present some interesting examples in order to compare weak amenability and weak amenability modulo an ideal of semigroup algebras.

2 Weak amenability modulo an ideal

Let A be a Banach algebra and X be a Banach A-bimodule. A linear mapping $D : A \to X$ is said to be a derivation if $D(ab) = a.D(b) + D(a)b\,(a, b \in A)$. A derivation D is said to be inner if there exists $x \in X$ such that $D = ad_x$ where $ad_x : A \to X$ defined by $ad_x(a) = a.x. - x.a$. We denote the space of continuous derivations from A to X by $\mathcal{Z}^1(A, X)$, the space of inner derivations from A to X by $\mathcal{N}^1(A, X)$; and the first (continuous) cohomology group of A with coefficients in X by $\mathcal{H}^1(A, X) := \frac{\mathcal{Z}^1(A,X)}{\mathcal{N}^1(A,X)}$. A Banach algebra A is called amenable if and only if $\mathcal{H}^1(A, X^*) = 0$ for each Banach A-bimodule X and is called weakly amenable if $\mathcal{H}^1(A, A^*) = 0$.

Let A be a Banach algebra and I be a closed ideal of A. Then $\frac{A}{I}$ can be made as an A-bimodule where the module actions are defined by

$$(a+I).b = ab + I\,,\; b.(a+I) = ba + I\;(a, b \in A).$$

Definition 2.1. Let I be a closed ideal of A. A Banach algebra A is weakly amenable modulo I if for every derivation D from A into $(\frac{A}{I})^*$, there is $\phi \in (\frac{A}{I})^*$ such that $D = ad_\phi$.

All over this paper we fix A, I and $\frac{A}{I}$ as above, unless they are otherwise specified.

Theorem 2.2. *Let I be a closed ideal of A. Then the following assertions hold;*

(i) if $\frac{A}{I}$ is weakly amenable and $I^2 = I$, then A is weakly amenable modulo I,

(ii) if A is weakly amenable modulo I, then $\frac{A}{I}$ is weakly amenable.

Proof. (i) Let $D : A \to (\frac{A}{I})^*$ be a bounded derivation. Consider $\frac{A}{I}$ as an $\frac{A}{I}$-bimodule where the module actions are just its multiplications. Define $\widetilde{D} : \frac{A}{I} \to (\frac{A}{I})^*$ by $\widetilde{D}(a+I) = D(a)\,(a \in A)$. Since $I^2 = I$, $D(x) = 0\,(x \in I)$, hence $\widetilde{D}$ is a well-defined bounded derivation. Weak amenability of $\frac{A}{I}$ implies that there exists $\phi \in (\frac{A}{I})^*$ such that $D(a) = \widetilde{D}(a+I) = ad_\phi(a+I)$.

(ii) Let $D : \frac{A}{I} \to (\frac{A}{I})^*$ be a bounded derivation. Now $D \circ \pi : A \to (\frac{A}{I})^*$ is a bounded derivation where $\pi : A \to \frac{A}{I}$ is the canonical quotient map. Weak amenability modulo I of A implies that there exists $\psi \in (\frac{A}{I})^*$ such that $D \circ \pi = ad_\psi$. Hence $D(a+I) = D \circ \pi(a) = ad_\psi(a+I)$. $\square$

Remark 2.3. In example 3.6(i), we show that weak amenability of $\frac{A}{I}$ does not imply weak amenability modulo an ideal of A if $I^2 \neq I$. Thus the condition $I^2 = I$ of Theorem 2.2(i), is really necessary.

For commutative Banach algebras, the definition used for weak amenability was that a commutative Banach algebra A is weakly amenable if and only if each continuous derivation from A into a commutative Banach A-module is necessarily zero, i.e. $H^1(A, X) = 0$ [3]. Also it is shown that this is actually equivalent to $H^1(A, A^*) = 0$. In what follows we demonstrate that the same satisfactorily result is reached for weak amenability modulo an ideal of commutative Banach algebras.

Theorem 2.4. *Let A be a commutative Banach algebra.*

(I) If $H^1(A, X) = 0$ for each commutative Banach A-bimodule X with $I \cdot X = 0$, then A is weakly amenable modulo I.

(II) If A is weakly amenable modulo I and $I^2 = I$, then $H^1(A, X) = 0$ for each commutative Banach A-bimodule X with $I \cdot X = 0$.

Proof. (I) Clearly $(\frac{A}{I})^*$ is a commutative A-bimodule Banach algebra and $(\frac{A}{I})^*.I = 0$, so $H^1(A, (\frac{A}{I})^*) = 0$.

(II) Let X be a commutative Banach A-bimodule which $I \cdot X = 0$. Then X can be consider as an Banach $\frac{A}{I}$-bimodule where the module actions are defined by $(a + I).x = ax$. As A is weakly amenable modulo I and $I^2 = I$, $\frac{A}{I}$ is commutative weakly amenable due to Theorem 2.2. Thus $H^1(\frac{A}{I}, X) = 0$, for $\frac{A}{I}$-bimodule X. Since every derivation from A into X induces a derivation from $\frac{A}{I}$ into X when $I^2 = I$, we must have $H^1(A, X) = 0$. □

Recall that for commutative Banach algebras, the continuous homomorphic image of a weakly amenable Banach algebra is weakly amenable. The similar result holds for weakly amenable modulo an ideal.

Theorem 2.5. *Let A and B be commutative Banach algebras, A be weakly amenable modulo I and J be a closed ideal of B. Let $\theta : A \to B$ be a continuous homomorphism with dense range such that $\theta(I) \subseteq J$. Then B is weakly amenable modulo J.*

Proof. Let $D : B \to (\frac{B}{J})^*$ be a bounded derivation. Set $\phi : \frac{A}{I} \to \frac{B}{J}$ by $\phi(a + I) = \theta(a) + J$. Since $\theta(I) \subseteq J$, ϕ is well-defined. Suppose that $\phi^* : (\frac{B}{J})^* \to (\frac{A}{I})^*$ by $\phi^*(F) = F \circ \phi$ $(F \in (\frac{B}{J}^*))$. Then $\phi^* \circ D \circ \theta : A \to (\frac{A}{I})^*$ is a bounded derivation. Weak amenability modulo I of A implies that $\phi^* \circ D \circ \theta$ is inner, i.e. there exists $f \in (\frac{A}{I})^*$ such that $\phi^* \circ D \circ \theta(a) = a.f - f.a$ $(a \in A)$. Thus for an arbitrary $a' \in A$,

$$\begin{aligned} \langle D(\theta(a)), \phi(a' + I)\rangle &= \langle \phi^* \circ D \circ \theta(a), a' + I\rangle \\ &= \langle a.f - f.a, a' + I\rangle \\ &= \langle f, (a' + I).a - a.(a' + I)\rangle = 0 \\ &= \langle f, (a'a + I) - (aa' + I)\rangle = 0. \end{aligned}$$

Hence by the density of $\theta(A)$ in B and continuity of D, we have $D = 0$. □

Theorem 2.6. *If A is weakly amenable modulo I, then $\frac{A}{I} = \overline{(\frac{A}{I})^2}$, where $(\frac{A}{I})^2 = span\{ab + I : a, b \in A\}$.*

Proof. It is enough to show that for each element $\phi \in (\frac{A}{I})^*$, such that $\phi|_{(\frac{A}{I})^2} = 0$, ϕ is identically 0. Consider $\frac{A}{I}$ as a Banach A-bimodule where the module actions are defined by $a.\bar{b} = \overline{ab}$, $\bar{b}.a = \overline{ba}\,(a, b \in A)$, and set $D : A \to (\frac{A}{I})^*$ by $D(a) = \langle \phi, \bar{a} \rangle \phi$. As $\phi|_{(\frac{A}{I})^2} = 0$, D is a bounded derivation on A. Since A is weakly amenable modulo I, D should be inner. Therefore there exists an element ψ in $(\frac{A}{I})^*$ such that for every $a \in A$, $D(a) = \langle \phi, \bar{a} \rangle \phi = \bar{a}.\psi - \psi.\bar{a}$. Then,

$$\langle \langle \phi, \bar{a} \rangle \phi, \bar{a} \rangle = \langle \bar{a}.\psi - \psi.\bar{a}, \bar{a} \rangle = \langle \psi, (\bar{a})^2 - (\bar{a})^2 \rangle = 0$$

Hence ϕ is identically zero. □

Let A be a Banach algebra and $\phi : A \to \mathbb{C}$ be a character on A. A linear operator $d : A \to \mathbb{C}$ by $d(ab) = d(a)\langle \phi, b \rangle + \langle \phi, a \rangle d(b)$, is called a point derivation at the character ϕ. Following Dales, Ghahramani and Gronbaek [7, Prop.1.3(ii)], if $\overline{A^2} = A$, then there are no non-zero continuous point derivations on A. Using Theorem 2.6, we have the following Corollary;

Corollary 2.7. *Let A be a Banach algebra and I be a closed ideal of A. If A is weakly amenable modulo I, then there are no non-zero, continuous point derivations on $\frac{A}{I}$.*

Let A be a Banach algebra and A^{**} be the second dual of A. Then A^{**} is a Banach algebra under two Arens products, of which as usual, we will take the first product. For more details we refer the reader to [1, 8, 14]. For a subspace I of a normed space A, we set $I^{\perp} = \{f \in A^* : \langle F, i \rangle = 0, \text{for each}\, i \in I\}$, and $I^{\perp\perp} = (I^{\perp})^{\perp} \subseteq I^{**}$. It is shown that $I^{\perp\perp} = \overline{(\widehat{I})}^{w*} \simeq I^{**}$ [18, Lemma 3.1]. If I is a closed ideal of Banach algebra A, then I^{**} is a closed ideal of Banach algebra A^{**} [15, Sec5. Theorem 2.1(a)]. Let A be a Banach algebra and I be a closed ideal I of A, then the canonical quotient map $\pi : A \to \frac{A}{I}$ induced a canonical epimorphism $\widetilde{\pi} : A^{**} \to (\frac{A}{I})^{**}$ such that $ker(\widetilde{\pi}) = I^{\perp\perp}$. Thus $(\frac{A}{I})^{**} \simeq \frac{A^{**}}{I^{\perp\perp}} \simeq \frac{A^{**}}{I^{**}}$.

We recall that for a Banach algebra A which A^{**} is weakly amenable, if $\hat{A}$ (the canonical embedding of A into its second dual) is a left ideal in A^{**}, then A is weakly amenable [8, Theorem 2.3]. In the following we have the same result for weak amenability modulo an ideal.

Theorem 2.8. *Let A be a Banach algebra and I be closed ideal of A with $I^2 = I$. If A^{**} is weakly amenable modulo I^{**} and $\hat{A}$ is a left ideal in A^{**}, then A is weakly amenable modulo I.*

Proof. Since A^{**} is weakly amenable modulo I^{**}, $\frac{A^{**}}{I^{**}} \simeq (\frac{A}{I})^{**}$ is weakly amenable. As $\hat{A}$ is a left ideal of A^{**}, $\widehat{(\frac{A}{I})}$ is a left ideal of $(\frac{A}{I})^{**}$. Thus $\frac{A}{I}$ is weakly amenable (by [8, Theorem 2.3]). As $I^2 = I$, using Theorem 2.2, A is weakly amenable modulo I. $\square$

3 Weak amenability of semigroup algebra modulo an ideal

Following B. E. Johnson [17], for a locally compact group (resp. discrete group) G, the group algebra $L^1(G)$ (resp. $l^1(G)$) is always weakly amenable. For a discrete semigroup S, weak amenability of the semigroup algebra $l^1(S)$ is rather more complicated. In this section, we investigate the weak amenability of the semigroup algebra $l^1(S)$ modulo the closed ideal I_σ which is corresponding to the least group congruence σ.

A congruence ρ on semigroup S is called a group congruence if S/ρ is a group. It is shown that $l^1(S/\rho) \simeq l^1(S)/I_\rho$, where I_ρ is the closed ideal in $l^1(S)$ generated by the set $\{\delta_s - \delta_t : \ s, t \in S \ \text{with}\, (s,t) \in \rho\}$ [2, Lemma 1]. Also, if J is an ideal of $l^1(S)$ and ρ_J is the congruence on S defined by $\rho_J = \{(s,t) : s, t \in S, \delta_s - \delta_t \in J\}$, then $I_{\rho_J} \subseteq J$ [2].

Theorem 3.1. *Let S be a semigroup, ρ be a group congruence on S such that I_ρ has an approximate identity, then $l^1(S)$ is weakly amenable modulo I_ρ.*

Proof. Since S/ρ is a group, $l^1(S/\rho) \simeq \frac{l^1(S)}{I_\rho}$ is weakly amenable. Clearly $I_\rho^2 = I_\rho$ (because I_ρ has an approximate identity). Using Theorem 2.2(i), $l^1(S)$ is weakly amenable modulo I_ρ. $\square$

Let S be a semigroup and $E = E(S)$ be the set of idempotents of S. A semigroup S is called an *E-semigroup* if the set of all idempotents of S, $E(S)$ forms a sub-semigroup of S, *E-inversive* if for each $x \in S$, there exists $y \in S$ such that $xy \in E(S)$, *eventually inverse* if every element of S has some power that is regular and $E(S)$ is a semilattice. For more details and some examples, see [10, 11, 12]. It is shown that if S is an E-inversive E-semigroup with commuting idempotents or S is an eventually inverse semigroup, then $\sigma = \{(a,b) \in S \times S \,|\, ea = fb \text{ for some } e, f \in E(S)\}$ is the least group congruence on S [11]. We recall the following result of [2].

Proposition 3.2. *Let S be an eventually inverse semigroup or S be an E-inversive E-semigroup with commuting idempotents. Then $l^1(S/\sigma) \simeq l^1(S)/I_\sigma$ where I_σ is a closed ideal of $l^1(S)$ and $I_\sigma^2 = I_\sigma$.*

Theorem 3.3. *Let S be an eventually inverse semigroup or S be an E-inversive E-semigroup with commuting idempotents, then $l^1(S)$ is weakly amenable modulo I_σ.*

Proof. As S is an eventually inverse semigroup or S is an E-inversive E-semigroup with commuting idempotents, there exist the least group congruence σ on S. Now S/σ is a group, so $l^1(S/\sigma) \simeq \frac{l^1(S)}{I_\sigma}$ is weakly amenable. Using Theorem 2.2, $l^1(S)$ is weakly amenable modulo I_σ. $\square$

It is shown that if A is a Banach algebra such that $\overline{A^2} \neq A$, then A is not weakly amenable. In particular if $S^2 \neq S$, then $l^1(S)$ is not weakly amenable [4, Prop. 4.2].

Theorem 3.4. *Let S be semigroup and ρ be a congruence on S which is not group congruence. If $(S/\rho)^2 \neq S/\rho$, then $l^1(S)$ is not weakly amenable modulo I_ρ.*

Proof. By contradiction assume that $l^1(S)$ is weakly amenable modulo I_ρ. Then $\frac{l^1(S)}{I_\rho} \simeq l^1(S/\rho)$ is weakly amenable (by Theorem 2.2). As ρ is not group congruence, S/ρ is a not group and is just a semigroup and since $(S/\rho)^2 \neq S/\rho$, $l^1(S/\rho)$ is not weakly amenable which is contradiction. $\square$

In [8, Theorem 2.1], Ghahramani, Loy and Willis have shown that for a locally compact group G, if $L^1(G)^{**}$ is weakly amenable, then $M(G)$ is weakly amenable. In the following we present the same result for semigroup algebras.

Theorem 3.5. *Let S be an eventually inverse semigroup or S be an E-inversive E-semigroup with commuting idempotents, then if $l^1(S)^{**}$ is weakly amenable modulo I_σ^{**}, then $l^1(S)$ is weakly amenable modulo I_σ*

Proof. Let σ be the least group congruence on S and I_σ be the it's corresponding closed ideal in $l^1(S)$. Then I_σ^{**} is a closed ideal of $l^1(S)^{**}$. Now, if $l^1(S)^{**}$ is weakly amenable modulo I_σ^{**}, so $\frac{l^1(S)^{**}}{I_\sigma^{**}} \simeq (\frac{l^1(S)}{I_\sigma})^{**} \simeq l^1(S/\sigma)^{**}$ is weakly amenable. Therefore $l^1(S/\sigma)$ is weakly amenable (by [8, Theorem 2.1]). Using Theorem 2.2, $l^1(S)$ is weakly amenable modulo I_σ. □

Example 3.6. (i) Let $S = (\mathbb{N}, +)$ (with respect to addition) be the semigroup of positive integers. It is shown that $\rho_n = \{(k,l) \in \mathbb{N} \times \mathbb{N} : n \mid (k-l)\}\,(n > 0)$ is a group congruence on S [11]. It is known that $l^1(S/\rho_n) \simeq \frac{l^1(S)}{I_{\rho_n}}$ where I_{ρ_n} is the closed ideal of $l^1(S)$ corresponding to ρ_n generated by $\{\delta_k - \delta_l : (k,l) \in \rho_n\}$[2]. Clearly $l^1(S/\rho_n)$ is weakly amenable. We show that $l^1(S)$ is not weakly amenable modulo I_{ρ_n}. Suppose by contradiction that $l^1(S)$ is weakly amenable modulo I_{ρ_n}. Let $D : l^1(S) \to (\frac{l^1(S)}{I_{\rho_n}})^* \simeq I_{\rho_n}^{\perp}$ be a nonzero bounded derivation, so there exists $\phi \in I_{\rho_n}^{\perp}$ such that $D(\delta_k) = \delta_k.\phi - \phi.\delta_k$. Now

$$\begin{aligned} \langle \delta_l, \delta_k.\phi - \phi.\delta_k \rangle &= \langle \delta_l * \delta_k, \phi \rangle - \langle \delta_k * \delta_l, \phi \rangle \\ &= \langle \delta_{l+k}, \phi \rangle - \langle \delta_{k+l}, \phi \rangle = 0 \end{aligned}$$

Therefore D vanishes on $l^1(S) \backslash I_{\rho_n}$, which is a contradiction. We note that the congruence $\rho_n\,(n \in \mathbb{N})$ is not least group congruence and $I_{\rho_n}^2 \neq I_{\rho_n}$.

(ii) Let $S = \{p^m q^n : m, n \geq 0\}$ be the bicyclic semigroup generated by $p\,, q$. Clearly S is an E-unitary semigroup with $E(S) = \{p^n q^n : n = 0, 1, 2, ...\}$. Let $x\sigma y$ if and only if $ex = ey$ for some $e \in E(S)$. Then σ is the least group congruence on S and $S/\sigma = \mathbb{Z}$ [6]. It is shown that $l^1(S)$ is not weakly amenable [5]. It is also not approximately amenable [9]. But $l^1(S)$ is weakly amenable modulo I_σ (by Theorem 3.3).

(iii) Let $S = FI(X)$ be the free inverse semigroup on X where X is a singleton. Then $l^1(G_S) \simeq \frac{l^1(S)}{I_\sigma}$, and $l^1(S)$ is weakly amenable modulo

I_σ where σ is the least group congruence on S. We note that $l^1(S)$ is not weakly amenable [5].

(iv) Let $S = G\times T$ where G is a group and $T = (\mathbb{N}, +)$. Then G is the maximum group homomorphism image of S under the homomorphism $\phi : (g, t) \mapsto g$. Let σ be the congruence on S such that $\frac{S}{\sigma} \simeq G$. Then $l^1(S)$ is weakly amenable modulo I_σ, indeed $l^1(S)/I_\sigma$ is isomorphic to $l^1(G)$, which is weakly amenable. However, since the semigroup algebra $l^1(T)$ is not weakly amenable, $l^1(S)$ could not be weakly amenable [5, Theorem 1.7].

Acknowledgements
The authors would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

[1] R.E. Arens, The adjoint of a bilinear operation. *Am. Math. Soc.* 1951, 2, 839848

[2] M. Amini, H. Rahimi, Amenability of semigroups and their algebras modulo a group congruence, *Acta Mathematica Hungarica*, Vol 144, Issue 2, 407-415, (2014).

[3] W. G. Bade, P. C. Curtis, Jr., and H. G. Dales, Amenability and weak amenability for Beurling and Lipschitz algebras, *Proc. London Math. Soc.* (3) 55, 359-377, (1987).

[4] T. D. Blackmore, Weak amenability of discrete semigroup algebras, *Semigroup Forum* 55 196-205, (1997).

[5] S. Bowling and J. Duncan, First order cohomology of Banach semigroup algebras, *Semigroup Forum* 56 130-145, (1988).

[6] A. H. Clifford and J. B. Preston, The Algebraic Theory of Semigroups I, American Mathematical Society Surveys **7**, American Mathematical Society, Providence (1961).

[7] H.G. Dales, F. Ghahramani, N. Gronbaek, Derivations into iterated duals of Banach algebras, *Studia Math.* 128, 1954, (1998).

[8] F. Ghahramani, R. J. Loy, and G. A. Willis, Amenability and weak amenability of second conjugate Banach algebras,*Proc . Amer. Math. Soc.* 124, 1489-1497, (1996).

[9] F. Gheorge and Y. Zhang, A note on the approximate amenability of semigroup algebras, *Semigroup Forum* 79, 349-354, (2009).

[10] P. M. Edwards, Group congruences on arbitrary semigroups, *International Journal of Algebra*, 6. 531537.(2012).

[11] R. S. Gigon, Congruences and group congruences on a semigroup, *Semigroup Forum* 86, 431-450, (2013).

[12] G. M. S. Gomes, Group congruences on eventually inverse semigroups, *Portugaliae Mathematica*, 49. 417428. (1992).

[13] H. M. Ghlaio and C. J. Read, Irregular commutative semigroups S with weakly amenable semigroup algebra $l^1(S)$, *Semigroup Forum*, 82, 367 - 383, (2011).

[14] F. Gourdeau, Amenablility and the second dual of a Banach algebras. Stud. Math. 1997, 125, 7580. Mathematics 2016, 4, 55 7 of 7.

[15] P. Harmand, D. Werner and Wend Werner, M-Ideals in Banach Spaces and Banach Algebras. Lecture Notes in Mathematics series,Vol 1547 in Springer's. 1993.

[16] B. E. Johnson, Cohomology in Banach Algebras. *American Mathematical Society*, Providence, RI, (1972).

[17] B. E. Johnson. Weak amenability of group algebras. *Bull. London Math. Soc.* 23, 281-284, (1991).

[18] E. Kissin and V. S. Shulman, Dual spaces and isomorphisms of some differetial Banach $*$-Algebra of operators. *Pacific J. Math.* Vol. 190, No. 2, 1999.

[19] V. Runde, *Lectures on Amenability*. Lecture notes in mathematics, 1774, Springer Verlag, (2002).

Ahmad Mehrabi
PhD candidate of Mathematics
Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, P. O. Box 13185/768
Tehran, Iran.
E-mail: ahm.mehrabi.math@iauctb.ac.ir

Hamidreza Rahimi
Professor of Mathematics
Department of Mathematics, Faculty of Science, Central Tehran Branch, Islamic Azad University, P. O. Box 13185/768.
Tehran, Iran.
E-mail: rahimi@iauctb.ac.ir