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Observational Modeling of Logical Entropy of Dynamical Systems with local approach

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Abstract. In this paper, we introduce a new kind of the logical entropy through a local relative approach. The notions of local relative logical entropy and local relative conditional logical entropy from an observer's viewpoint on local relative probability measure space are introduced and some of their ergodic properties are studied. Some properties of the local relative logical entropy of independent partitions are investigated and the concavity property for the local relative logical entropy has been proved. We show that, the basic properties of Shannon entropy of partitions on probability measure spaces, are established for the case of the local relative logical entropy. So the suggested measures can be used besides of the Shannon entropy of partitions. Using the concept of the local relative logical entropy of partitions, we define the local relative logical entropy of a dynamical system and present some of its properties. Finally, it is shown that the local relative logical entropy of dynamical systems is invariant under isomorphism. So the notion of local relative logical entropy of dynamical systems can be a new tool for distinction of non-isomorphic relative dynamical systems.

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1 Introduction

The classical approach in the information theory is based on Shannon entropy [19]. The study of concept entropy in current sciences is very important. The entropy of a system specifies the degree of uncertainty or the degree of chaotic behavior of it. For an information system it determines the amount of information which we can obtain from it. Kolmogorov and Sinai by using the notion of Shannon entropy defined the entropy of measurable partitions. The notion of Kolmogorov- Sinai entropy of finite partitions on probability measure space was studied in [22]. Kolmogorov- Sinai entropy serves as a measure of information of the considered experiment and has an important application in dynamical systems [11, 20, 21]. Some investigations concerning entropy of dynamical systems and related notions were carried in [3, 5, 6, 7, 8]. In [12, 17, 18], Good, Patil, Taillie and Rao defined and studied the concept of logical entropy. Rao introduced precisely this concept as a quadratic entropy [18] and in the years 2009 and 2013, the relation of logical entropy to Shannon entropy was discussed by Ellerman [9, 10, 11]. Let $P = (p_1, \cdots, p_n) \in \mathbb{R}^n$ be a probability distribution. The logical entropy of P was defined in [9] as $h(P) = \sum_{i=1}^{n} p_i(1 - p_i)$. In the paper by Ebrahimzadeh [3] the notion of logical entropy on quantum logic was defined and studied. In [8], Ebrahmzadeh, Eslami Giski and Markechov studied the logical entropy of dynamical systems on an algebraic structure.

One of the main fuzzy objects in physical phenomena is "observer". The amount of information which we can obtain from a system is depend on an observer of it. So the entropy of a dynamical system is depend on an observer of it. Since the view of an observer is different in any point, the study of local entropy of dynamical systems from an observer's viewpoint is important. The notion of observer is very important in physics and has been applied in dynamical systems [2, 16], topology [15, 17], and information theory [4]. If X is a non-empty set, then a mapping $\mu : X \to [0, 1]$ is called a one-dimensional observer of X [13]. Molaei in [13] introduced the notion of relative probability measure by the notion of observer.

In [7], we defined and studied the logical entropy of finite measurable partitions, and using the concept of logical entropy of measurable parti-

tions, we introduced the notion of logical entropy of a dynamical system. In the paper by Asadian and Ebrahimzadeh [1] the notions of relative entropy and relative conditional entropy, by the notion of relative probability measure, were defined and studied. In this paper, we provide analogies of the results on the relative probability measure space for the case of the logical entropy.
We use of the notion of observer to define the local relative logical entropy for dynamical systems. We show that, the basic properties of Shannon entropy of dynamical systems on probability measure spaces, are established for the case of the local relative logical entropy. So the suggested measures can be used besides of the Shannon entropy of dynamical systems as measures of information which we can get from a system. Note that the notion of $L_{\mu}^{f}(x,T)$ denotes the logical entropy of T according to an observer viewpoint when it look at x.
In section 2, we present some basic notions. In section 3, the logical entropy and the conditional logical entropy of finite partitions on the relative probability measure space via a local approach are defined and some of their ergodic properties are investigated. Then we study the local relative logical entropy and the local relative conditional logical entropy of independent partitions. We prove the concavity property for the notion of local relative logical entropy. In section 4, using the suggested concept of local relative logical entropy, we define the local relative logical entropy of a dynamical system and prove some theorems about the measure. Finally, it is shown that isomorphic dynamical systems have the same local relative logical entropy. Accordingly, this concept will be a new tool for distinction of non-isomorphic dynamical systems.

2 Preliminary Facts

In this section, we shall recall some known concepts. Let X be a set. A partition of X is a disjoint collection of subsets of X whose union is X. Let $f : X \to X$ be a mapping and $\mu : X \to [0,1]$ be an observer of X. Moreover let E be a subset of X and let $x \in X$. Then the local relative probability measure of E with respect to an observer μ was defined in

[13] by:

$$m_\mu^f(x,E)=\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_E\left(f^i(x)\right)\mu\left(f^i(x)\right).$$

Note that, $m_\mu^f(x,E)$ denotes the measure of E according to an observer viewpoint when it look at x.
The pair (X,m_μ^f) is called a relative probability measure space.

Remark 2.1. [13] If we restrict our self to a probability measure space (X,β,m) and we take the characteristic function $\mathcal{X}_X$ as an observer, and if we assume that $f:X\to X$ is a measure preserving map, then for given $x\in X$ and $E\in\beta$ the Birkhoff ergodic theorem [10] implies that

$$m_\mu^f(x,E)=\limsup_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\chi_E\left(f^i(x)\right)=m(E)$$

almost every where.

So, relative probability measure is an extension of the notion of probability measure.

An example of this measure is presented in the present paper (see Example 3.2).

Lemma 2.2. *([13], Theorem $2\cdot2$) Let $x\in X$, and let E_1,E_2 be two disjoint subsets of X, then*

$$m_\mu^f(x,E_1\cup E_2)=m_\mu^f(x,E_1)+m_\mu^f(x,E_2).$$

Let $E_1,\cdots,E_n$, be disjoint subsets of X. From Lemma 2.2, by induction we get

$$m_\mu^f\left(x,\bigcup_{i=1}^n E_i\right)=\sum_{i=1}^n m_\mu^f(x,E_i). \tag{1}$$

Let $P=\{A_1,\cdots,A_n\}$ and $Q=\{B_1,\cdots,B_m\}$ be two partitions of X.

The join of P,Q is the partition

$$P\vee Q=\{A_i\cap B_j : A_i\in A, B_j\in B\}.$$

3 Logical Entropy of Partitions via a Local Relative Approach

We shall now define the notion of logical entropy through a local relative approach by the notion of observer.

Definition 3.1. Let $x \in X$ and $P = \{A_1, \cdots, A_n\}$ be a partition of X. The local relative logical entropy of P at x with respect to an observer μ, is defined as follows:

$$L_\mu^f(x, P) := \sum_{i=1}^{n} m_\mu^f(x, A_i)(1 - m_\mu^f(x, A_i)).$$

Remark that we may write

$$\begin{aligned} L_\mu^f(x, P) &= \sum_{i=1}^{n} m_\mu^f(x, A_i)(1 - m_\mu^f(x, A_i)) \\ &= \sum_{i=1}^{n} m_\mu^f(x, A_i) - \sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2 \\ &= m_\mu^f(x, X) - \sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2. \end{aligned}$$

We give an example of this measure.

Example 3.2. Let $X = [0, 1]$ and let $f : X \to X$ be defined by $x \mapsto 1 - x$. Moreover let $\mu : X \to [0, 1]$ be defined by $x \longmapsto \frac{1}{2}x$ and let $P = \{A_1, A_2, A_3, A_4\}$ be a partition of X, where

$$A_1 = \left[0, \frac{1}{4}\right], A_2 = (\frac{1}{4}, \frac{1}{2}], A_3 = (\frac{1}{2}, \frac{3}{4}], A_4 = \left(\frac{3}{4}, 1\right].$$

If $x = \frac{1}{3}$ then we obtain

$$\begin{aligned} m_\mu^f(x, A_1) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{A_1}\left(f^i(x)\right) \mu\left(f^i(x)\right) \\ &= \limsup_{n\to\infty} \frac{1}{n} \left(\chi_{A_1}(x) . \frac{1}{2}x + \chi_{A_1}(1 - x) . \frac{1}{2}(1 - x) + \cdots\right) \end{aligned}$$

$$\begin{aligned}
&= \limsup_{n\to\infty} \frac{1}{n}\,(0+0+0+\cdots) = 0,\\
m_\mu^f\,(x, A_2) &= \limsup_{n\to\infty} \frac{1}{n}\left(\frac{1}{2}\times\frac{1}{3}+0+\frac{1}{2}\times\frac{1}{3}+0+\cdots\right)\\
&= \limsup_{n\to\infty} \frac{1}{n}\left(\frac{n}{2}\times\frac{1}{6}\right) = \frac{1}{12},\\
m_\mu^f\,(x, A_3) &= \limsup_{n\to\infty} \frac{1}{n}\left(0+\frac{1}{2}\times\frac{2}{3}+0+\frac{1}{2}\times\frac{2}{3}+0+\cdots\right)\\
&= \limsup_{n\to\infty} \frac{1}{n}\left(\frac{n}{2}\times\frac{1}{3}\right) = \frac{1}{6},\\
m_\mu^f\,(x, A_4) &= \limsup_{n\to\infty} \frac{1}{n}\,(0+0+0+\cdots) = 0,
\end{aligned}$$

and

$$\begin{aligned}
m_\mu^f\,(x, X) &= \limsup_{n\to\infty} \frac{1}{n}\left(\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}+\cdots\right)\\
&= \limsup_{n\to\infty} \frac{1}{n}\left(\frac{n}{4}(\frac{1}{3}+\frac{2}{3})\right) = \frac{1}{4}.
\end{aligned}$$

So

$$\begin{aligned}
L_\mu^f\,(x, P) &= m_\mu^f(x, X) - \sum_{i=1}^{4}\left(m_\mu^f(x, A_i)\right)^2\\
&= \frac{1}{4} - \left(0^2 + \left(\frac{1}{12}\right)^2 + \left(\frac{1}{6}\right)^2 + 0^2\right)\\
&= \frac{31}{144}.
\end{aligned}$$

Let $x \in X$ and let $P = \{A_1, \cdots, A_n\}$, $Q = \{B_1, \cdots, B_m\}$ be two partitions of X. The local relative conditional logical entropy of P given Q is defined by:

$$L_\mu^f\,(x, P|\,Q) := \sum_{i=1}^{n}\sum_{j=1}^{m} m_\mu^f(x, A_i \cap B_j)(m_\mu^f(x, B_j) - m_\mu^f(x, A_i \cap B_j)).$$

It is clear that $L_\mu^f\,(x, P|\,Q) \geq 0$.

Theorem 3.3. *Let $x \in X$ and P, Q be two partitions of X. Then*

1) $L_\mu^f\left(x, P\middle|\{X,\emptyset\}\right) \leq L_\mu^f(x,P)$ *with the equality if and only if* $m_\mu^f(x,X) = 1$,

2) $L_\mu^f(x, P \vee Q) = L_\mu^f(x,Q) + L_\mu^f\left(x, P\middle|Q\right)$.

Proof. Let $P = \{A_1, \cdots, A_n\}$ and $Q = \{B_1, \cdots, B_m\}$.

1) Since for each $x \in X$, $m_\mu^f(x,X) \leq 1$, by the definition we obtain

$$\begin{aligned} L_\mu^f\left(x, P\middle|\{X,\emptyset\}\right) &= \sum_{i=1}^{n} m_\mu^f(x,A_i)(m_\mu^f(x,X) - m_\mu^f(x,A_i)) \\ &\leq \sum_{i=1}^{n} m_\mu^f(x,A_i)\left(1 - m_\mu^f(x,A_i)\right) = L_\mu^f(x,P). \end{aligned}$$

2) From (1) and Definition 3.1, for each $x \in X$ we have

$$\begin{aligned} L_\mu^f(x, P \vee Q) &= \sum_{i=1}^{n}\sum_{j=1}^{m} m_\mu^f(x, A_i \cap B_j)(1 - m_\mu^f(x, A_i \cap B_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} m_\mu^f(x, A_i \cap B_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} (m_\mu^f(x, A_i \cap B_j))^2 \\ &= m_\mu^f(x,X) - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(m_\mu^f(x, A_i \cap B_j)\right)^2. \end{aligned}$$

On the other hand by (1) we may write

$$\begin{aligned} L_\mu^f(x,Q) &= \sum_{j=1}^{m} m_\mu^f(x,B_j)(1 - m_\mu^f(x,B_j)) \\ &= \sum_{j=1}^{m} m_\mu^f(x,B_j) - \sum_{i=1}^{n} (m_\mu^f(x,B_j))^2 \\ &= m_\mu^f(x,X) - \sum_{j=1}^{m} (m_\mu^f(x,B_j))^2. \end{aligned}$$

Also by (1) we have

$$\begin{aligned} L_{\mu}^{f}\left(x, P\middle|Q\right) &= \sum_{i=1}^{n}\sum_{j=1}^{m} m_{\mu}^{f}(x, A_i \cap B_j)(m_{\mu}^{f}(x, B_j) - m_{\mu}^{f}(x, A_i \cap B_j)) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} m_{\mu}^{f}(x, A_i \cap B_j) m_{\mu}^{f}(x, B_j) - \sum_{i=1}^{n}\sum_{j=1}^{m} (m_{\mu}^{f}(x, A_i \cap B_j))^2 \\ &= \sum_{j=1}^{m} (m_{\mu}^{f}(x, B_j))^2 - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(m_{\mu}^{f}(x, A_i \cap B_j)\right)^2. \end{aligned}$$

By combining the above relations we get

$$L_{\mu}^{f}(x, P \vee Q) = L_{\mu}^{f}(x, Q) + L_{\mu}^{f}\left(x, P\middle|Q\right).$$

□

In the next theorem, it is proved subadditivity of local relative logical entropy of partitions.

Theorem 3.4. *Let $x \in X$ and P, Q be two partitions of X. Then*

1) $L_{\mu}^{f}(x, P) \geq L_{\mu}^{f}\left(x, P\middle|Q\right)$,
2) $L_{\mu}^{f}(x, P \vee Q) \leq L_{\mu}^{f}(x, P) + L_{\mu}^{f}(x, Q)$,
3) $L_{\mu}^{f}(x, P \vee Q) \geq \max\left\{L_{\mu}^{f}(x, P), L_{\mu}^{f}(x, Q)\right\}$.

Proof. Let $P = \{A_1, \cdots, A_n\}$ and $Q = \{B_1, \cdots, B_m\}$.

1) Since $m_{\mu}^{f}(x, X) \leq 1$, by (1) for every $i = 1, \cdots, n$, we obtain

$$\begin{aligned} &\sum_{j=1}^{m} m_{\mu}^{f}(x, A_i \cap B_j)(m_{\mu}^{f}(x, B_j) - m_{\mu}^{f}(x, A_i \cap B_j)) \\ &\leq \sum_{j=1}^{m} m_{\mu}^{f}(x, A_i \cap B_j)(\sum_{j=1}^{m} (m_{\mu}^{f}(x, B_j) - m_{\mu}^{f}(x, A_i \cap B_j))) \\ &= m_{\mu}^{f}(x, A_i)(\sum_{j=1}^{m} (m_{\mu}^{f}(x, B_j) - m_{\mu}^{f}(x, A_i \cap B_j))) \end{aligned}$$

$$= m_\mu^f(x, A_i)\left(m_\mu^f(x, X) - \sum_{j=1}^{m} m_\mu^f(x, A_i \cap B_j)\right)$$
$$\leq m_\mu^f(x, A_i)\left(1 - m_\mu^f(x, A_i)\right).$$

Therefore

$$L_\mu^f\left(x, P\middle|Q\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_\mu^f(x, A_i \cap B_j)(m(x, B_j) - m_\mu^f(x, A_i \cap B_j))$$
$$\leq \sum_{i=1}^{n} m_\mu^f(x, A_i)\left(1 - m_\mu^f(x, A_i)\right) = L_\mu^f(x, P).$$

2) According to the first part of this theorem and Theorem 3.3 we obtain

$$L_\mu^f(x, P \vee Q) = L_\mu^f(x, Q) + L_\mu^f\left(x, P\middle|Q\right) \leq L_\mu^f(x, Q) + L_\mu^f(x, P).$$

3) Since for each $i, j, A_i \cap B_j = B_j \cap A_i$ we get

$$L_\mu^f(x, P \vee Q) = L_\mu^f(x, Q \vee P).$$

Thus from Theorem 3.3, the assertion holds. □

Definition 3.5. Let P and Q be two partitions of X. Then P and Q are called independent if for each $A \in P, C \in Q$ and $x \in X$ we have

$$m_\mu^f(x, A \cap C) = m_\mu^f(x, A) . m_\mu^f(x, C).$$

Theorem 3.6. *Let $x \in X$ and P, Q be two independent partitions of X and let $m_\mu^f(x, X) = 1$. Then*

1) $L_\mu^f(x, P \vee Q) = L_\mu^f(x, P) + L_\mu^f(x, Q) - L_\mu^f(x, P)\, L_\mu^f(x, Q)$,

2) $L_\mu^f\left(x, P\middle|Q\right) = L_\mu^f(x, P)\left(1 - L_\mu^f(x, Q)\right)$.

Proof. Let $P = \{A_1, \cdots, A_n\}$ and $Q = \{B_1, \cdots, B_m\}$.

1) From Definition 3.1, we obtain

$$\begin{aligned}
L_\mu^f(x, P \vee Q) &= m_\mu^f(x, X) - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(m_\mu^f(x, A_i \cap C_j)\right)^2 \\
&= m_\mu^f(x, X) - \sum_{i=1}^{n}\sum_{j=1}^{m} \left(m_\mu^f(x, A_i)\right)^2 \left(m_\mu^f(x, C_j)\right)^2 \\
&= 1 - \Big(\sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2\Big)\Big(\sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2\Big) \\
&= 1 - \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 + \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \\
&\quad - \Big(\sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2\Big)\Big(\sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2\Big) \\
&= 1 - \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \\
&\quad + \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \Big(1 - \sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2\Big) \\
&= m_\mu^f(x, X) - \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \\
&\quad + \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \Big(m_\mu^f(x, X) - \sum_{i=1}^{n} \left(m_\mu^f(x, A_i)\right)^2\Big) \\
&= L_\mu^f(x, Q) + \sum_{j=1}^{m} \left(m_\mu^f(x, C_j)\right)^2 \left(L_\mu^f(x, P)\right) \\
&= L_\mu^f(x, Q) + (1 - L_\mu^f(x, Q)) L_\mu^f(x, P) \\
&= L_\mu^f(x, P) + L_\mu^f(x, Q) - L_\mu^f(x, P)\, L_\mu^f(x, Q)\,.
\end{aligned}$$

2) Based on the first part of this theorem and Theorem 3.3, we obtain

$$\begin{aligned}
L_\mu^f\left(x, P\middle|Q\right) &= L_\mu^f(x, P \vee Q) - L_\mu^f(x, Q) \\
&= L_\mu^f(x, P) - L_\mu^f(x, P)\, L_\mu^f(x, Q)
\end{aligned}$$

$$= L_{\mu}^{f}(x,P)\left(1 - L_{\mu}^{f}(x,Q)\right).$$

□

In the following theorem, we prove the concavity property for the notion of local relative logical entropy.

Theorem 3.7. *Let $x \in X$ and $P = \{A_1, \cdots, A_n\}$ be a partition of X. Moreover let $f : X \to X$ be a mapping, and $\mu_1 : X \to [0,1]$ and $\mu_2 : X \to [0,1]$ be two observers of X, such that for each $\alpha \in [0,1]$ and $A_i \in P$, $m^f_{\alpha\mu_1+(1-\alpha)\mu_2}(x, A_i) = \alpha m^f_{\mu_1}(x, A_i) + (1-\alpha)\, m^f_{\mu_2}(x, A_i)$. Then we have*

$$L^f_{\alpha\mu_1+(1-\alpha)\mu_2}(x,P) \geq \alpha L^f_{\mu_1}(x,P) + (1-\alpha)\, L^f_{\mu_2}(x,P).$$

Proof. Since

$$\sum_{i=1}^{n}\left(m^f_{\mu_1}(x,A_i)\right)^2 + \sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2 \geq 2\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i)\, m^f_{\mu_2}(x,A_i),$$

from Definition 3.1 we obtain

$$\begin{aligned}
L^f_{\alpha\mu_1+(1-\alpha)\mu_2}(x,P) &= \sum_{i=1}^{n} m^f_{\alpha\mu_1+(1-\alpha)\mu_2}(x,A_i) \\
&\quad - \sum_{i=1}^{n}\left(m^f_{\alpha\mu_1+(1-\alpha)\mu_2}(x,A_i)\right)^2 \\
&= \sum_{i=1}^{n}\left(\alpha m^f_{\mu_1} + (1-\alpha)\, m^f_{\mu_2}\right)(x,A_i) \\
&\quad - \sum_{i=1}^{n}\left(\left(\alpha m^f_{\mu_1} + (1-\alpha)\, m^f_{\mu_2}\right)(x,A_i)\right)^2 \\
&= \alpha\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i) - \alpha^2\sum_{i=1}^{n}\left(m^f_{\mu_1}(x,A_i)\right)^2 \\
&\quad + (1-\alpha)\sum_{i=1}^{n} m^f_{\mu_2}(x,A_i) \\
&\quad - (1-\alpha)^2\sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&- 2\alpha(1-\alpha)\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i)m^f_{\mu_2}(x,A_i)\\
\geq\ & \alpha\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i)\\
&- \alpha^2\sum_{i=1}^{n}\left(m^f_{\mu_1}(x,A_i)\right)^2 + (1-\alpha)\sum_{i=1}^{n} m^f_{\mu_2}(x,A_i)\\
&- (1-\alpha)^2\sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2\\
&-\alpha\left(1-\alpha\right)\sum_{qi=1}^{n}\left(\left(m^f_{\mu_1}(x,A_i)\right)^2 + \left(m^f_{\mu_2}(x,A_i)\right)^2\right)\\
=\ & \alpha\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i)\\
&+ \left(-\alpha^2 - \alpha\left(1-\alpha\right)\right)\sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2\\
&+ (1-\alpha)\sum_{i=1}^{n} m^f_{\mu_2}(x,A_i)\\
&+ \left(-(1-\alpha)^2 - \alpha\left(1-\alpha\right)\right)\sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2\\
=&\alpha\sum_{i=1}^{n} m^f_{\mu_1}(x,A_i) - \alpha\sum_{i=1}^{n}\left(m^f_{\mu_1}(x,A_i)\right)^2\\
&+(1-\alpha)\sum_{i=1}^{n} m^f_{\mu_2}(x,A_i)\\
&- (1-\alpha)\sum_{i=1}^{n}\left(m^f_{\mu_2}(x,A_i)\right)^2\\
=&\alpha L^f_{\mu_1}(x,P) + (1-\alpha)\,L^f_{\mu_2}(x,P)\,.
\end{aligned}
$$

□

4 Local relative logical entropy of dynamical systems

If $T : X \to X$ is a mapping, then T is called local relative probability measure-preserving (with respect to an observer μ and $x \in X$) if

$$m_\mu^f\left(x, T^{-1}E\right) = m_\mu^f\left(x, E\right).$$

Let $T : X \to X$ be a local relative probability measure-preserving and P be a finite partition of X. Then $T^{-1}P$ is a finite partition of X, too. In this section, T is called a local relative dynamical system.

Theorem 4.1. *Let $T : X \to X$ be a local relative dynamical system and P be a finite partition of X and $x \in X$. Then*

$$L_\mu^f\left(x, T^{-1}P\right) = L_\mu^f\left(x, P\right).$$

Proof. Let $P = \{A_1, \cdots, A_n\}$. Since for each $A_i \in P$, $m_\mu^f\left(x, T^{-1}A_i\right) = m_\mu^f\left(x, A_i\right)$ we obtain

$$\begin{aligned} L_\mu^f\left(x, T^{-1}P\right) &= \sum_{i=1}^{n} m_\mu^f(x, T^{-1}A_i)(1 - m_\mu^f(x, T^{-1}A_i)) \\ &= \sum_{i=1}^{n} m_\mu^f(x, A_i)(1 - m_\mu^f(x, A_i)) \\ &= L_\mu^f\left(x, P\right). \end{aligned}$$

□

Corollary 4.2. *If T is a local relative dynamical system and P, Q are two finite partitions of X and $x \in X$, then*

$$L_\mu^f\left(x, T^{-1}P \middle| T^{-1}Q\right) = L_\mu^f\left(x, P \middle| Q\right).$$

Proof. Let $P = \{A_1, \cdots, A_n\}$ and $Q = \{C_1, \cdots, C_m\}$. For each $A_i \in P$ and $C_j \in Q$ we have

$$T^{-1}(A_i \cap C_j) = T^{-1}A_i \cap T^{-1}C_j.$$

So $T^{-1}(P \vee Q) = T^{-1}P \ \vee T^{-1}Q$. From Theorems 3.3 and 4.1, we obtain

$$\begin{aligned} L_{\mu}^{f}\left(x, T^{-1}P \middle| T^{-1}Q\right) &= L_{\mu}^{f}\left(x, T^{-1}P \vee T^{-1}Q\ \right) - L_{\mu}^{f}\left(x, T^{-1}Q\ \right) \\ &= L_{\mu}^{f}\left(x, T^{-1}(P \vee Q)\ \right) - L_{\mu}^{f}\left(x, T^{-1}Q\ \right) \\ &= L_{\mu}^{f}\left(x, P \vee Q\ \right) - L_{\mu}^{f}\left(x, Q\right) \\ &= L_{\mu}^{f}\left(x, P \middle| Q\right). \end{aligned}$$

□

Lemma 4.3. *([22], Theorem 4·9) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers such that $a_{k+t} \leq a_k + a_t$ for every $k, t \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists.*

Theorem 4.4. *Let T be a local relative dynamical system and let P be a finite partition of X and $x \in X$. Then $\lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right)$ exists.*

Proof. Put $a_n = L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right)$. According to the subadditivity property of the local relative logical entropy (Theorem 3.4 ii)) and Theorem 4.1, we get

$$\begin{aligned} a_{n+p} &= L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n+p-1} T^{-i}P\right) \\ &\leq L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right) + L_{\mu}^{f}\left(x, \bigvee_{i=n}^{n+p-1} T^{-i}P\right) \\ &= a_n + L_{\mu}^{f}\left(x, \bigvee_{i=0}^{p-1} T^{-i}P\right) = a_n + a_p. \end{aligned}$$

By the preceding lemma $\lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right)$ exists. □

Definition 4.5. Let T be a local relative dynamical system and P be a finite partition of X. The local relative logical entropy of T with respect to P is defined by the formula:

$$L_{\mu}^{f}(x, T, P) := \lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right).$$

The local relative logical entropy of T is defined as:

$$L_{\mu}^{f}(x,T) := \sup_P L_{\mu}^{f}(x,T,P) \ ,$$

where the supremum is taken over all finite partitions P of X.

Remark that $L_{\mu}^{f}(x,T)$ denotes the logical entropy of T according to an observer viewpoint when it look at x.

Observe that $L_{\mu}^{f}(x,T) \geq 0$, and

$$L_{\mu}^{f}(x, id_X) = 0.$$

Theorem 4.6. *Let T be a local relative dynamical system and let P be a finite partition of X and $x \in X$. Then*

$$L_{\mu}^{f}\left(x,T,T^{-1}P\right) = L_{\mu}^{f}(x,T,P) \, .$$

Proof. From Theorem 4.1 we have for each $n \in \mathbb{N}$,

$$L_{\mu}^{f}\left(x, \bigvee_{i=1}^{n} T^{-i}P\right) = L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}P\right).$$

Therefore we obtain

$$\begin{aligned} L_{\mu}^{f}\left(x,T,T^{-1}P\right) &= \lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-i}(T^{-1}P)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=0}^{n-1} T^{-(i+1)}P\right) \\ &= \lim_{n\to\infty} \frac{1}{n} L_{\mu}^{f}\left(x, \bigvee_{i=1}^{n} T^{-i}P\right) \ = L_{\mu}^{f}(x,T,P) \, . \end{aligned}$$

□

Definition 4.7. Let $T_1 : X \to X$ and $T_2 : X \to X$ be two local relative dynamical system with respect to $x \in X$. We say that T_1 and T_2 are isomorphic if there exists a bijective mapping $\varphi : X \to X$ satisfying the following conditions:

1) $\varphi o T_1 (x) = T_2 o \varphi (x)$,

2) $m_\mu^f (x, E) = m_\mu^f (x, \varphi(E))$ for every $E \subseteq X$.

In the following theorem we prove that the local relative logical entropy of local relative dynamical systems is invariant under isomorphism.

Theorem 4.8. *If local relative dynamical systems T_1 and T_2 are isomorphic, then for each $x \in X$,*

$$L_\mu^f (x, T_1) = L_\mu^f (x, T_2) .$$

Proof. Let a mapping $\varphi : X \to X$ represents an isomorphism of local relative dynamical systems T_1 and T_2. Let $P = \{A_1, \cdots, A_n\}$ be an arbitrary partition of X, then $\varphi (P) = \{\varphi (A_1), \cdots, \varphi (A_n)\}$ is also a partition of X. By Definition 3.1, we have

$$\begin{aligned} L_\mu^f (x, \varphi(P)) &= \sum_{i=1}^{n} m_\mu^f(x, \varphi(A_i))(1 - m_\mu^f (x, \varphi(A_i))) \\ &= \sum_{i=1}^{n} m_\mu^f(x, A_i)(1 - m_\mu^f (x, A_i)) = L_\mu^f (x, P) . \end{aligned}$$

Therefore

$$\begin{aligned} L_\mu^f (x, T_2,\ P) &= \lim_{n\to\infty} \frac{1}{n} L_\mu^f(x, \bigvee_{i=0}^{n-1} {T_2}^{-i} P) \\ &= \lim_{n\to\infty} \frac{1}{n} L_\mu^f(x, \varphi^{-1} \left(\bigvee_{i=0}^{n-1} {T_2}^{-i} P \right)) \\ &= \lim_{n\to\infty} \frac{1}{n} L_\mu^f(x, \bigvee_{i=0}^{n-1} \varphi^{-1}({T_2}^{-i} P)) \\ &= \lim_{n\to\infty} \frac{1}{n} L_\mu^f(x, \bigvee_{i=0}^{n-1} {T_1}^{-i}(\varphi^{-1} P)) = L_\mu^f \left(x, T_1,\ \varphi^{-1} P\right) . \end{aligned}$$

Thus

$$L_\mu^f (x, T_2) = \sup_P L_\mu^f (x, T_2,\ P)\ = \sup_P L_\mu^f \left(x, T_1,\ \varphi^{-1} P\right)$$

$$\leq \sup_P L_\mu^f (x, T_1,\ P)\ = L_\mu^f (x, T_1),$$

where the supremum on the left side of the inequality is taken over all finite partitions P of X and the supremum on the right side of the inequality is taken over all finite partitions P of X.

On the other hand, let $Q = \{B_1, \cdots, B_m\}$ be any partition of X, then $\varphi^{-1}(Q)$ is a partition of X and we obtain

$$L_\mu^f \left(x, \varphi^{-1}(Q)\right) = L_\mu^f (x, Q).$$

Therefore similar the above we get

$$L_\mu^f (x, T_1,\ Q) = L_\mu^f (x, T_2,\ \varphi (Q)),$$

and hence

$$L_\mu^f (x, T_1) \leq L_\mu^f (x, T_2).$$

□

5 Conclusion

In this paper we introduced the notions of logical entropy and logical conditional entropy of finite partitions through a local relative approach. We proved some of their ergodic properties. The property of subadditivity for the local relative logical entropy of partitions was proved. Furthermore we studied the notion of local relative logical entropy of independent partitions and we proved the concavity property for this measure. In the final section, using the suggested concept of local relative logical entropy of finite partitions, we defined the local relative logical entropy of a dynamical system. Finally, it was shown that the isomorphic dynamical systems have the same local relative logical entropy. So the notion of local relative logical entropy of dynamical systems can be a new tool for distinction of non-isomorphic relative dynamical systems. The suggested measure can be used besides of the Shannon entropy of dynamical systems especially we mentioned which entropy of a dynamical system is depend on an observer of it and we used the notion of observer in the definition of Local relative logical entropy of a dynamical system.

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References

[1] M. H. Asadian and A. Ebrahimzadeh, Relative entropy and relative conditional entropy with infinite partitions, *Italian Journal of Pure and Applied Mathematics*, 35 (2014), 319-326.

[2] M. H. Asadian and A. Ebrahimzadeh, Entropy of dynamical systems from the observer's viewpoint, with countable sigma-algebras, *Journal of Uncertain Systems*, 11 (2017), 197-204.

[3] A. Ebrahimzadeh, Logical entropy of quantum dynamical systems, *Open Physics*, 14 (2016), 1-5.

[4] A. Ebrahimzadeh, The observational modelling of information function with finite partitions, *Journal of Mathematical Extension*, 10 (2016), 47-57.

[5] A. Ebrahimzadeh and M. Ebrahimi, Entropy of dynamical systems on weights of a graph, *Journal of Mahani Mathematical Research Center*, 2 (2013), 53-63.

[6] A. Ebrahimzadeh and M. Ebrahimi, The entropy of countable dynamical systems, *U.P.B. Sci. Bull. Series A.*, 76 (2014), 107-114.

[7] D. Markechova, A. Ebrahimzadeh and Z. Eslami Giski, Logical entropy of dynamical systems, *Advances in Difference Equations*, 70 (2018), DOI:10.1186/s13662-018-1524-z.

[8] A. Ebrahimzadeh, Z. Eslami Giski and D. Markechova, Logical entropy of dynamical systems - A general model, *Mathematics*, 5 (2017), DOI:10.3390/math5010006.

[9] D. Ellerman, An introduction to logical entropy and its relation to Shannon entropy, *International Journal of Semantic Computing*, 7 (2013), 121-145.

[10] D. Ellerman, Counting Distinctions, On the conceptual foundations of shannon Information Theory, *Synthese*, 168 (2009), 119-149.

[11] D. Ellerman, The logic of partitions: Introduction to the dual of the logic of subsets. *Review of Symbolic Logic*, 3 (2010), 287-350.

[12] I. J. Good, Comment (on Patil and Taillie: Diversity as a Concept and its Measurement), *Journal of the American Statistical Association*, 77 (1982), 561-563.

[13] M. R. Molaei, The concept of synchronization from the observers viewpoint, *Cankaya University Journal of Science and Engineering*, 8 (2011), 255-262.

[14] M. R. Molaei, Observatinal modeling of topological space, *Chaos, Solitons and Fractals*, 42 (2009), 615-619.

[15] M. R. Molaei, Relative semi-dynamical, *International Journal of Uncertainty Fuzziness and Knowledge-based Systems*, 12 (2004), 237-243.

[16] M. R. Molaei, S. H. Salili, A new approach to fuzzy topological space, *Hadronic Journal*, 25 (2002), 81-90.

[17] G. P. Patil, C. Taillie, Diversity as a concept and its measurement, *Journal of the American Statistical Association*, 77 (1982), 548-561.

[18] C. R. Rao, Diversity and dissimilarity coefficients, A unified approach. *Theoretical Population Biology*, 21 (1982), 24-43.

[19] C. E. Shannon, A Mathematical theory of communication, *Bell System Technical Journal*, 27 (1948), 379-423.

[20] Ya. G. Sinai, *Ergodic theory with applications to dynamical systems and statistical Mechanics*, Springer, (1990).

[21] Ya. G. Sinai, *Dynamical systems*, Aarhus University, (1970).

[22] P. Walters, *An introduction to ergodic theory*, Springer, (1982).

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