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H_v-Normal subgroups and H_v-quotient groups

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Abstract. A larger class of algebraic hyperstructures satisfying the group-like axioms is the class of H_v-groups. In this paper, without any condition and in general, we define the H_v-normal subgroup and the H_v-quotient group of an H_v-group. We introduce the fundamental equivalence relation of an H_v-quotient group and prove the first and third isomorphism theorems for H_v-groups.

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1 Introduction

The theory of hyperstructures has been introduced by Marty in 1934 during the 8^{th} Congress of the Scandinavian Mathematics [10]. This theory has been studied in the following decades and nowadays by many mathematicians. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence and probabilities. The theory of H_v-structures introduced and studied by Vougiouklis [15]. The concept of H_v-structures not only constitutes a generalization of the well-known hyperstructures, but it also comprises a very interesting and deep algebraic theory. Fundamental structures appeared in [13] which is the crucial point to define the H_v-structures which first appeared in [14]. Actually some axioms concerning the above

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hyperstructures are replaced by their corresponding weak axioms. This concepts and the basic definitions of which can be found in [15] has been investigated in [1, 13, 16]. P-H_v-structures wich is a subclasses of H_v-structures studied in [13] by S. Spaartalis and T.vougiouhly. The reader will found in [11, 12] a deep discussion on P-hyperstructiones theory. The notion of n-ary hypergroups is defined and considered by Davvas and Vougiouhly in [4], which is a generalization of hypergroups in the sense of Marty and a generalization of n-ary groups. The reader will found the notions of n-ary H_v-groups and n-ary P-H_v-groups in [6, 8]. The concepts of normal subgroups as a fandumental concepts in hyperstructures and H_v-structures only are disscussed in some special subclasses of H_v-structures for example P-H_v-structures, n-ary P-H_v-structures, polygroups and n-ary polygroups see [2, 7, 8]. In this paper it is introduced the H_v-normal subgroups of an H_v-groups, and it is investigated the relative concepts. In section 2, it is introduced the notion of weak homomorphism. Also, we recall basic concepts and modify some definitions of [3] for the sake of completeness. In section 3, H_v-normal subgroups and H_v-quotient groups are defined and then some problems are covered. In section 4, the fundamental relation of an H_v-quotient group is introduced and the first and third isomorphism theorems for H_v-groups are presented and proved.

2 Weak Homomorphism and Basic Concepts

Let H be a non-empty set and $\mathcal{P}^*(H)$ be the family of non-empty subsets of H. Every mapping $* : H \times H \to \mathcal{P}^*(H)$ is called a hyperoperation on H and $(H, *)$ is called a hyperstructure. The hyperstructure $(H, *)$ is called an H_v-group if "$*$" is weak associative: $x*(y*z) \cap (x*y)*z \neq \emptyset$ and the reproduction axiom is hold: $a*H = H*a = H$ for every $a \in H$. The H_v-group H is called weak commutative if for every $x, y \in H$, $x*y \cap y*x \neq \emptyset$. The nonempty subset K of H is called an H_v-subgroup if $(K, *)$ is an H_v-group. A mapping $\phi : H_1 \longrightarrow H_2$ on H_v-groups $(H_1, *)$ and $(H_2, \cdot)$ is called a strong homomorphism if for every $x, y \in H_1$ we have $f(x*y) = f(x) \cdot f(y)$.

Suppose U is the set of all finite sums of elements of H_v-group H.

Then the relation β on H defined by:

$$x\beta y \Leftrightarrow \{x, y\} \subseteq u, \text{ for some } u \in U,$$

is reflexive and symmetric, but not transitive necessary. The transitive closure of β is called β^*. The fundamental relation β^* is the smallest equivalence relation on the H_v-group H such that H/β^* consisting of all equivalence classes is a group. Suppose that $\beta^*(x)$ and $\beta^*(y)$ is the equivalence class of $x, y \in H$ respectively. On H/β^* the operation $\oplus$ is defined as follows that $(H/\beta^*, \oplus)$ is a group:

$$\beta^*(x) \oplus \beta^*(y) = \beta^*(c), \text{ for every c} \in \text{x} + \text{y}.$$

The group $(H/\beta^*, \oplus)$ is called the fundamental group of the H_v-group H. This concept was introduced on hypergroups by Koskas [9] and studied mainly by Corsini [1].

The map ϕ from an H_v-structure onto it's fundamental structure where x maps to $\beta^*(x)$, is called fundamental map. If $\phi : H \longrightarrow H/\beta^*$ is the fundamental mapping of the H_v-group H, then the core of H is defined by $\omega_H = \{x \in H|\ \phi(x) = 0\}$, where 0 denotes the unit of the group H/β^*. One can show that

$$\omega_H \oplus \beta^*(x) = \beta^*(x) \oplus \omega_H = \beta^*(x) \text{ for every } x \in H.$$

The following concepts and statements are modified of [3] with the w-homom-
orphism view that is defined as follows.

Definition 2.1. Let $(G, +)$ and $(H, \oplus)$ be H_v-groups. A mapping $f : G \longrightarrow H$ of H_v-groups is called weak homomorphism (w-hom.) if

$$\beta^*_G(f(g_1 + g_2)) = \beta^*_H(f(g_1)) \oplus \beta^*_H(f(g_2)), \text{ for every } g_1, g_2 \in G,$$

where β^*_G and β^*_H are the fundamental relations of H_v-group G and H respectively. And the w-hom. f is called w-monic if

$$f(g_1) = f(g_2) \Rightarrow \beta^*_G(g_1) = \beta^*_G(g_2), \text{ for every } g_1, g_2 \in G.$$

Also, f is called w-epic if for every $h \in H$ there exists $g \in G$ such that $\beta^*_H(h) = \beta^*_H(f(g))$. Finally the weak homomorphism f is called w-isomorphism if f is w-monic and w-epic, in this case we write $G \overset{W}{\cong} H$. It is clear every strong hom. is w-hom. and so if $G \cong H$, then $G \overset{W}{\cong} H$.

Theorem 2.2. *Let $f : G \longrightarrow H$ be a strong (w-)hom. of H_v-groups. Then ω_G and kernel of f, $Ker(f) = \{g \in G| \ f(a) \in \omega_H\}$, are H_v-subgroups of G and ω_G is contained in $Ker(f)$.*

Lemma 2.3. *Let $f : G \longrightarrow H$ be a strong (w-)hom. of H_v-groups. The mapping $F : G/\beta^*_G \longrightarrow H/\beta^*_H$ defined by $F(\beta^*(g)) = \beta^*_H(f(g))$ is a homomorphism of groups and the following conditions are equivalent:*
(i) F is one to one,
(ii) f is w-monic,
(iii) $Ker(f) = \omega_G$.

Proof. Let f be a w-hom.. We show that F is well-defined. Suppose that $\beta^*_G(a) = \beta^*_G(b)$, then there exist $g_1, \cdots, g_{m+1} \in G$ and $u_1, \cdots, u_m \in U_G$ with $g_1 = a, g_{m+1} = b$ such that $\{g_i, g_{i+1}\} \subseteq u_i$, $i = 1, 2, \cdots, m$.
So $\{\beta^*_H(f(g_i)), \beta^*_H(f(g_{i+1}))\} \subseteq \beta^*_H(f(u_i)) = \beta^*_H(h)$ for some $h \in H$ since f is w-hom.. Thus $\beta^*_H(f(g_i)) = \beta^*_H(f(g_{i+1}))$ for every $i = 1, 2, \cdots, m$ and consequently $\beta^*_H(f(a)) = \beta^*_H(f(b))$. Therefore F is well-defined. Now, we have

$$\begin{aligned} F(\beta^*_G(a) \oplus \beta^*_G(b)) &= F(\beta^*_G(a + b)) = \beta^*_H(f(a + b)) = \beta^*_H(f(a)) \oplus \beta^*_H(f(b)) \\ &= F(\beta^*_G(a)) \oplus F(\beta^*_G(b)). \end{aligned}$$

Remaining of proof is similar as follows in [3]. □

Lemma 2.4. *Let $\varphi : G \longrightarrow H$ be a strong epimorphism of H_v-groups. Then*
(i) for $a, b \in G$, $a \ \beta^*_G \ b$ iff $\varphi(a) \ \beta^*_H \ \varphi(b)$,
(ii) for $N \subseteq G$, $\varphi(\beta^*_G(N)) = \beta^*_H(\varphi(N))$.

Proof. (i) By definition of relations β^*_G and β^*_H, the proof is straightforward.
(ii)

$$\begin{aligned} h \in \varphi(\beta^*_G(N)) &\Leftrightarrow h = \varphi(x), \ x \in \beta^*_G(n), \ for \ some \ n \in N \\ &\Leftrightarrow x \ \beta^*_G(n), \ h = \varphi(x) \\ &\Leftrightarrow \varphi(x) \ \beta^*_H \ \varphi(n), \ h = \varphi(x), \ by \ (i) \\ &\Leftrightarrow h = \varphi(x) \in \beta^*_H(\varphi(N)). \end{aligned}$$

□

Remark 2.5. It is easy to see that, If $G_1, \cdots, G_k$ be H_v-groups with fundamental relations $\beta_1^*, \cdots, \beta_k^*$ respectively, then $G = G_1 \times G_2 \times \cdots G_k$ with hyperoperation induced by hyperoperations of G_i as below:

$$(x_1, \cdots, x_k) \cdot (y_1, \cdots, y_k) = \{(t_1, \cdots, t_k) | t_i \in x_i \cdot y_i, \ i = 1, \cdots, k\}$$

is an H_v-group. Also $(x_1, \cdots, x_k) \ \beta_g^* \ (y_1, \cdots, y_k)$ iff $x_i \ \beta_i^* \ y_i$ for $i = 1, \cdots, k$.

3 H_v-Normal Subgroup & H_v-Quotient Group

In this section G and H are H_v-groups with fundamental relations $\beta_G^*(=\beta^*)$ and β_H^* respectively. The identity and inverse element that is the essential elements for building the quotient (group and ring) there are not in H_v-structures. Since the H_v-structures are introduced up to now, the H_v-normal subgroup and H_v-quotient group were not introduced. In this section it is introduced the normal notion of H_v-groups based on corresponding fundamental group.

Lemma 3.1. *If H is an H_v-subgroup of G, then $\beta^*(H) = \{\beta^*(h) | \ h \in H\}$ is a subgroup of G/β^*.*

Proof. If $\beta^*(x), \beta^*(y) \in \beta^*(H)$ then there exist $h_1, h_2 \in H$ such that

$$\beta^*(x) = \beta^*(h_1), \ \beta^*(y) = \beta^*(h_2).$$

So, $\beta^*(x) \oplus \beta^*(y) = \beta^*(h_1) \oplus \beta^*(h_2) = \beta^*(h)$ for some $h \in h_1 + h_2$. Thus $\beta^*(x) \oplus \beta^*(y) \in \beta^*(H)$.

For associativity law, let $\beta^*(x), \beta^*(y), \beta^*(z) \in \beta^*(H)$. We have:

$$\beta^*(x) \oplus (\beta^*(y) \oplus \beta^*(z)) = \beta^*(x + (y + z)),$$

$$(\beta^*(x) \oplus \beta^*(y)) \oplus \beta^*(z) = \beta^*((x + y) + z).$$

Since H is an H_v-group, we have $x + (y + z) \cap (x + y) + z \neq \emptyset$. On the other hand the left sides of above equations are single-member, so

$$\beta^*(x) \oplus (\beta^*(y) \oplus \beta^*(z)) = (\beta^*(x) \oplus \beta^*(y)) \oplus \beta^*(z).$$

Suppose $\beta^*(x) = \beta^*(h_1) \in \beta^*(H)$, where $h_1 \in H$. By reproduction axiom of H there exists $h \in H$ such that $h_1 \in h_1 + h$. Thus $\beta^*(h_1) = \beta^*(h_1) \oplus \beta^*(h)$ and $\omega_G = \beta^*(h) \in \beta^*(H)$, so $\beta^*(H)$ has zero element.

If $\beta^*(y) = \beta^*(h_2) \in \beta^*(H)$, where $h_2 \in H$, there exists $h_3 \in H$ such that $h \in h_2 + h_3$. So $\omega_G = \beta^*(h) = \beta^*(h_2) \oplus \beta^*(h_3)$ and $\beta^*(h_3)$ is the inverse of $\beta^*(h_2)$ in $\beta^*(H)$. Therefore $\beta^*(H)$ is a subgroup of G/β^*. □

Definition 3.2. Let N be an H_v-subgroup of G. N is called an H_v-normal subgroup of G if $\beta^*(N) \lhd G/\beta^*$.

Example 3.3. Consider the following H_v-group H:

·	a	b	c	d	e	f
a	a	b	c,d	d	f,e	f
b	b	a	e,f	f	c	d,c
c	d,c	f,e	a	e,f	d	b
d	d,c	e	f	a	b	c,d
e	e	d,c	b	c,d	f,e	a
f	f,e	c	d,c	b	a	e,f

We have:

$$H/\beta_H^* = \{\beta_H^*(a), \beta_H^*(b)\},$$

where

$$\beta_H^*(a) = \{a, e, f\},\ \beta_H^*(b) = \{b, c, d\}.$$

Now $(N = \{a, e, f\}, \cdot)$ is an H_v-subgroup of $(H, \cdot)$ and by Lemma 3.1 $\beta_H^*(N) = \{\beta_H^*(a)\}$ is a subgroup of the fundamental group H/β_H^*. We have $\beta_H^*(N) = \omega_H \trianglelefteq H/\beta_H^*$. Therefore N is an H_v-normal subgroup of H.

Lemma 3.4. *Let K be a subgroup of G. If for every g and g' in G, where $\beta^*(g) \oplus \beta^*(g') = \omega_G$, $x \in \beta^*(g)$ and $y \in \beta^*(g')$ implies $y + K + x \subseteq K$, then K is an H_v-normal subgroup of G.*

Proof. Suppose $\beta^*(g) \oplus \beta^*(g') = \omega_G$, $x \in \beta^*(g)$ and $y \in \beta^*(g')$, then

$$\beta^*(x) \oplus \beta^*(K) \oplus \beta^*(y) \subseteq \beta^*(K),$$

$$\beta^*(g') \oplus \beta^*(K) \oplus \beta^*(g) \subseteq \beta^*(K).$$

Therefore $\beta^*(K) \trianglelefteq G/\beta^*$ and so K is an H_v-normal subgroup of G. □

Theorem 3.5. *If $f : G \longrightarrow H$ is a strong (w-)hom., then $K = Ker(f)$ is an H_v-normal subgroup of G.*

Proof. By Theorem 2.2, $K = Ker(f)$ is an H_v-subgroup of G. Suppose $x \in \beta^*(g)$, $y \in \beta^*(g^{'})$, $k \in K$ and $s \in x + k + y$; where

$$\beta^*(g^{'}) \oplus \beta^*(g) = \omega_G.$$

So $\beta^*(x) \oplus \beta^*(y) = \omega_G$. For $s \in x + k + y$, $f(s) \in f(x + k + y)$ and we have

$$\begin{aligned}\beta^*_H(f(s)) \in \beta^*_H(f(x+k+y)) &= \beta^*_H(f(x)) \oplus \beta^*_H(f(k)) \oplus \beta^*_H(f(y)) \\ &= \beta^*_H(f(x)) \oplus \omega_H \oplus \beta^*_H(f(y)) \\ &= F(\beta^*(x)) \oplus F(\beta^*(y)) \\ &= F(\beta^*(x) \oplus \beta^*(y)) \\ &= F(\omega_G) = \omega_H.\end{aligned}$$

Thus $f(s) \in \omega_H$, $s \in Ker(f)$ and $x + k + y \subseteq K$. So by Lemma 3.4 $K = Ker(f)$ is an H_v-normal subgroup of G. $\square$

Corollary 3.6. *ω_H is an H_v-normal subgroup of G.*

Proof. If $\phi : G \longrightarrow G/\beta^*$ be the fundamental mapping, then ϕ is a strong homomorphism of H_v-groups and $Ker(\phi) = \omega_G$. So, by Theorem 3.5 ω_G is an H_v-normal subgroup of G. $\square$

Theorem 3.7. *Let K be an H_v-normal subgroup of G. Define the hyperoperation $\hat{+}$ on $G/K = \{g + K|\ g \in G\}$ by $(g_1 + K)\hat{+}(g_2 + K) = \beta^*(g_1+g_2+K)+K$. Then $(G/K, \hat{+})$ is an H_v-group. $(G/K, \hat{+})$ is called the H_v-quotient group of G on K.*

Proof. Suppose $g_1 + K = g_1^{'} + K$ and $g_2 + K = g_2^{'} + K$. So

$$\beta^*(g_1) \oplus \beta^*(K) \oplus \beta^*(g_2) \oplus \beta^*(K) = \beta^*(g_1^{'}) \oplus \beta^*(K) \oplus \beta^*(g_2^{'}) \oplus \beta^*(K).$$

Since K is an H_v-normal subgroup in G, then $\beta^*(K)$ is normal in $\beta^*(G)$ and we have:

$$\beta^*(g_1) \oplus \beta^*(g_2) \oplus \beta^*(K) = \beta^*(g_1^{'}) \oplus \beta^*(g_2^{'}) \oplus \beta^*(K),$$

$$\beta^*(g_1 + g_2 + K) = \beta^*(g_1^{'} + g_2^{'} + K),$$

$$\beta^*(g_1+g_2+K)+K=\beta^*(g_1^{'}+g_2^{'}+K)+K,$$

$$(g_1+K)\hat{+}(g_2+K)=(g_1^{'}+K)\hat{+}(g_2^{'}+K).$$

Therefore, $\hat{+}$ is a well defined hyperoperation on G/K.

Let $g_1+K, g_2+K, g_3+K \in G/K$ we have:

$$\begin{aligned}[(g_1+K)\hat{+}(g_2+K)]\hat{+}(g_3+K) &= [\beta^*(g_1+g_2+K)+K]\hat{+}(g_3+K)\\ &= \{x+K \mid x\in\beta^*(g_1+g_2+K)\}\hat{+}(g_3+K)\\ &= \cup\beta^*(x+g_3+K)+K;\ x\in\beta^*(g_1+g_2+K)\\ &= \{y+K \mid y\in\beta^*(x+g_3+K),\ x\in\beta^*(g_1+g_2+K)\}\\ &= \{y+K \mid y\in\beta^*(g_1+g_2)\oplus\beta^*(g_3)\oplus\beta^*(K)\}\\ &= \{y+K \mid y\in\beta^*(g_1+g_2+g_3+K)\}\\ &= (g_1+K)\hat{+}[(g_2+K)\hat{+}(g_3+K)].\end{aligned}$$

Therefore $\hat{+}$ is an associative hyperoperation.

Now we prove that $(G/K, \hat{+})$ satisfies the reproduction axiom. We know

$$(g_1+K)\hat{+}G/K=\{x+K \mid x\in\beta^*(g_1+g+K),\ g\in G\}\subseteq G/K.$$

if $g_0+K\in G/K$, by the reproduction axiom of G there exists $g_2\in G$ such that

$$g_0\in\beta^*(g_0)=\beta^*(g_1+g_2),$$

$$\begin{aligned}g_0+K\in\beta^*(g_1+g_2)+K &\subseteq \beta^*(g_1+g_2)\oplus\beta^*(K)\oplus K\\ &= \beta^*(g_1+g_2+K)+K\end{aligned}$$

Thus $g_0+K\in\beta^*(g_1+g_2+K)+K$ and $G/K\subseteq(g_1+K)\hat{+}G/K$. So $(G/K,\hat{+})$ satisfies the reproduction axiom. $\square$

4 Fundamental Relation of H_v-Quotient Group

Let N be an H_v-normal subgroup of G and U be the set of all finite sums of elements of G. One can show that every finite sums of elements of G/N is equal to $\beta^*(u+N)+N$ for some $u\in U$.

Lemma 4.1. *Let β_q^* be the fundamental relation of G/N. Then for $g_1, g_2\in G$, $\beta_q^*(g_1+N)=\beta_q^*(g_2+N)$ if and only if $\beta^*(g_1+N)=\beta^*(g_2+N)$.*

Proof. Suppose that $\beta_q^*(g_1 + N) = \beta_q^*(g_2 + N)$. Then there exist $u_1, u_2, \cdots, u_m \in U$ and $x_1, x_2, \cdots, x_{m+1} \in G$ such that:

$$x_1 + N = g_1 + N,\ x_{m+1} + N = g_2 + N$$

and

$$\{x_i + N,\ x_{i+1} + N\} \subseteq u_i + N \text{ for } i = 1, 2, \cdots, m.$$

Thus

$$\beta^*(x_1) \oplus \beta^*(N) = \beta^*(g_1) \oplus \beta^*(N),\ \beta^*(x_{m+1}) \oplus \beta^*(N) = \beta^*(g_2) \oplus \beta^*(N),$$

$$\{\beta^*(x_i) \oplus \beta^*(N),\ \beta^*(x_{i+1}) \oplus \beta^*(N)\} \subseteq \beta^*(u_i) \oplus \beta^*(N) \text{ for } u_i \in U.$$

We have $u_i = u_{i_1} + u_{i_2} + \cdots + u_{i_{n_i}}$ where $u_{i_j} \in G$ for $j = 1, 2, \cdots, n_i$. Now, by properties of fundamental relation we have

$$\beta^*(u_i) = \beta^*(u_{i_1}) \oplus \cdots \oplus \beta^*(u_{n_i}) = \beta^*(t_i) \text{ for every } t_i \in u_i.$$

Since $\beta^*(N) \triangleleft \beta^*(G)$, then $\beta^*(x_i) \oplus \beta^*(N)$, $\beta^*(u_i) \oplus \beta^*(N)$ and $\beta^*(t_i) \oplus \beta^*(N)$ are cosets of $\beta^*(N)$ in $\beta^*(G)$ and

$$\beta^*(x_i) \oplus \beta^*(N) = \beta^*(x_{i+1}) \oplus \beta^*(N) = \beta^*(u_i) \oplus \beta^*(N) \text{ for } i = 1, 2, \cdots, m.$$

Therefore $\beta^*(g_1) \oplus \beta^*(N) = \beta^*(g_2) \oplus \beta^*(N)$.
Conversly;

$$\begin{aligned}
\beta^*(g_1 + N) = \beta^*(g_2 + N) &\Rightarrow \beta^*(g_1 + N) + \omega_G + N = \beta^*(g_2 + N) + \omega_G + N \\
&\Rightarrow \beta^*(g_1 + N + \omega_G) + N = \beta^*(g_2 + N + \omega_G) + N \\
&\Rightarrow (g_1 + N)\hat{+}(g_0 + N) = (g_2 + N)\hat{+}(g_0 + N), for\ g_0 \in \omega_G \\
&\Rightarrow \beta_q^*(g_1 + N)\hat{+}\beta_q^*(g_0 + N) = \beta_q^*(g_2 + N)\hat{+}\beta_q^*(g_0 + N) \\
&\Rightarrow \beta_q^*(g_1 + N) = \beta_q^*(g_2 + N).
\end{aligned}$$

□

Theorem 4.2. *Let G and H be H_v-groups with fundamental relations β^* and β_H^* respectively.*
(i) If N is an H_v-normal subgroup of G then the map $f : \begin{array}{l} G \longrightarrow G/N \\ x \longmapsto x + N \end{array}$ *is a weak and inclusion epimorphism.*
(ii) Let $\varphi : G \longrightarrow H$ be a strong epimorphism such that $N \subseteq ker\varphi$. Then there exists the strong epimorphism $\bar{\varphi} : \begin{array}{l} G/N \longrightarrow H/\omega_H \\ x + N \longmapsto \varphi(x) + \omega_H \end{array}$, *and $ker\bar{\varphi} = ker\varphi/N$.*

Proof. (i) $x = y \Rightarrow x + N = y + N \Rightarrow f(x) = f(y)$.

$$\begin{aligned} f(x+y) = x+y+N &\subseteq \beta^*(x+y+N)+N \\ &= (x+N)\hat{+}(y+N) \\ &= f(x)\hat{+}f(y). \end{aligned}$$

since $\beta^*((x+y)+N) = \beta^*(\beta^*((x+y)+N)+N)$, by Lemma 4.1, we have $\beta^*_q(x+y+N) = \beta^*_q((x+N)\hat{+}(y+N))$ and so $\beta^*_q(f(x+y)) = \beta^*_q(f(x))\hat{+}\beta^*_q(f(y))$. (ii) For $x+N, y+N \in G/N$,

$$\begin{aligned} x+N = y+N &\Rightarrow \varphi(x+N) = \varphi(y+N) \\ &\Rightarrow \varphi(x)+\varphi(N) = \varphi(y)+\varphi(N); \quad \varphi \textit{ is strong hom.} \\ &\Rightarrow \varphi(x)+\varphi(N)+\omega_H = \varphi(y)+\varphi(N)+\omega_H \\ &\Rightarrow \varphi(x)+\omega_H = \varphi(y)+\omega_H, \quad N \subseteq ker\varphi \subset \omega_H \\ &\Rightarrow \bar{\varphi}(x) = \bar{\varphi}(y). \end{aligned}$$

$$\begin{aligned} \bar{\varphi}((x+N)\hat{+}(y+N)) &= \bar{\varphi}(\beta^*(x+y+N)+N) \\ &= \{\varphi(t)+\omega_H \mid t \in \beta^*(x+y+N)\} \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y+N))\} \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y)\oplus\beta^*(N))\} \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y))\oplus\varphi(\beta^*(N))\}, \ \varphi \textit{ is strong hom.} \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y))\oplus\beta^*_H(\varphi(N))\}, \ \textit{by lemma 2.4 (ii)} \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y))\oplus\beta^*_H(\omega_H)\}, \ \textit{since } N \subseteq ker\varphi \\ &= \{s+\omega_H \mid s \in \varphi(\beta^*(x+y))\}. \qquad (1) \end{aligned}$$

$$\begin{aligned} \bar{\varphi}(x+N)\hat{+}\bar{\varphi}(y+N) &= (\varphi(x)+\omega_H)\hat{+}(\varphi(y)+\omega_H) \\ &= \beta^*_H(\varphi(x)+\varphi(y)+\omega_H)+\omega_H \\ &= \{s+\omega_H \mid s \in \beta^*_H(\varphi(x)+\varphi(y)+\omega_H)\} \\ &= \{s+\omega_H \mid s \in \beta^*_H(\varphi(x+y))\}. \qquad (2) \end{aligned}$$

By (ii) of Lemma 2.4 and (1), (2) the proof is completed. □

Example 4.3. Consider the H_v-group H and it's H_v-normal subgroup N in Example 3.3. We have:

$$H/N = \{a+N, b+N\},$$

and
$(a+N)\hat{+}(a+N) = \beta^*(a+a+N)+N = \beta^*(N)+N = \{a+N\}$,
$(a+N)\hat{+}(b+N) = \beta^*(a+b+N)+N = \beta^*(b)+N = \{b+N\}$,
$(b+N)\hat{+}(a+N) = \beta^*(b+a+N)+N = \beta^*(b)+N = \{b+N\}$,
$(b+N)\hat{+}(b+N) = \beta^*(b+b+N)+N = \beta^*(a)+N = \{a+N\}$.
Also, by Lemma 4.1 we have $\frac{H/N}{\beta^*_q} = \{\beta^*_q(a+N), \beta^*_q(b+N)\}$.
For inclusion canonical epimorphis $f : H \longrightarrow H/N$ we have

$$f(b+c) = (b+c)+N = N,$$

and

$$f(b)+f(c) = (b+N)\hat{+}(c+N) = \beta^*(b+c+N)+N = N,$$

because

$$\beta^*(b+c+N) = \beta^*(\{e,f\}+N) = \beta^*((e+N)\cup(f+N)) = \beta^*(N) = N$$

Theorem 4.4. *If β^*_{qK} and β^* are the fundamental relations of G/K and G respectively, then $\frac{G}{K}/\beta^*_{qK} \cong \frac{\beta^*(G)}{\beta^*(K)}$.*

Proof. Define $\theta : \beta^*_{qK}(G/K) \longrightarrow \beta^*(G)/\beta^*(K)$ by $\theta(\beta^*_K(g+K)) = \beta^*(g)\oplus\beta^*(K)$. By Lemma 4.1, θ is an one-to-one mapping. If $g_1 + K, g_2 + K \in G/K$, we have:

$$\begin{aligned}
\theta(\beta^*_{qK}(g_1+K)\oplus\beta^*_{qK}(g_2+K)) &= \theta(\beta^*_{qK}(\beta^*(g_1+g_2+K)+K) \\
&= \{\theta(\beta^*_{qK}(x+K))|\ x\in\beta^*(g_1+g_2+K)\} \\
&= \{\beta^*(x)\oplus\beta^*(K)|\ x\in\beta^*(g_1+g_2+K)\} \\
&= \beta^*(g_1+g_2)\oplus\beta^*(K) \\
&= \beta^*(g_1)\oplus\beta^*(g_2)\oplus\beta^*(K) \\
&= [\beta^*(g_1)\oplus\beta^*(K)]\oplus[\beta^*(g_2)\oplus\beta^*(K)].
\end{aligned}$$

Therefore θ is a homomorphism and it is clear that θ is epic. □

Corollary 4.5. *If G is an H_v-group and $K \lhd G$, then:*
(i) $\omega_{G/K} = \beta^*(K)+K (= \dfrac{\beta^*(K)}{K} = \{x+K|\ x\in\beta^*(K)\})$,
(ii) $\omega_{\frac{G}{\omega_G}} = \omega_G$.

Proof. (i) By Theorem 4.4

$$\begin{aligned}\omega_{G/K} &= \{g+K \mid \theta(\beta^*_N(g+K)) = \beta^*(K)\}\\ &= \{g+K \mid \beta^*(g) \oplus \beta^*(K) = \beta^*(K)\}\\ &= \{g+K \mid \beta^*(g) \in \beta^*(K)\}\\ &= \beta^*(K) + K.\end{aligned}$$

(ii) By (i) we have $\omega_{\frac{G}{\omega_K}} = \beta^*(\omega_G) + \omega_G = \omega_G + \omega_G = \omega_G$. □

Lemma 4.6. *Let H be an H_v-group, for $a \in H$, if $a + \omega_H = \omega_H$ then $a \in \omega_H$.*

Proof.

$$\begin{aligned}a + \omega_H = \omega_H &\Rightarrow \beta^*(a + \omega_H) = \beta^*(\omega_H)\\ &\Rightarrow \beta^*_H(a) \oplus \beta^*_H(\omega_H) = \beta^*_H(\omega_H)\\ &\Rightarrow \beta^*_H(a) = \omega_H\\ &\Rightarrow a \in \omega_H.\end{aligned}$$

□

Now, we extend the first isomorphism theorem to H_v-groups.

Theorem 4.7. (First isomorphism theorem of H_v-groups) *If $f : G \longrightarrow H$ is a strong epimorphism of H_v-groups and $K = Ker(f)$ then $G/K \overset{w}{\cong} H/\omega_H$.*

Proof. We define $\theta : G/K \longrightarrow H/\omega_H$ by $\theta(g + K) = f(g) + \omega_H$. Then by (ii) of Theorem 4.2 θ is a strong epimorphism.
Now, we show that θ is w-monic:

$$\begin{aligned}\theta(g_1 + K) &= \theta(g_2 + K)\\ &\Rightarrow f(g_1) + \omega_H = f(g_2) + \omega_H\\ &\Rightarrow f(g_1) + \omega_H + f(K) = f(g_2) + \omega_H + f(K)\\ &\Rightarrow \beta^*_H(f(g_1)) \oplus \beta^*_H(\omega_H) \oplus \beta^*_H(f(K))\\ &= \beta^*_H(f(g_2)) \oplus \beta^*_H(\omega_H) \oplus \beta^*_H(f(K))\\ &\Rightarrow \beta^*_H(f(g_1)) = \beta^*_H(f(g_2));\ \text{since } f(K) \subseteq \omega_H\\ &\Rightarrow F(\beta^*(g_1)) = F(\beta^*(g_2)),\ \textit{by Lemma } 2.3\\ &\Rightarrow \beta^*(g_1) \oplus \beta^*(K) = \beta^*(g_2) \oplus \beta^*(K);\ \text{since } \beta^*(K) = Ker(F)\\ &\Rightarrow \beta^*(g_1 + K) = \beta^*(g_2 + K)\\ &\Rightarrow \beta^*_q(g_1 + K) = \beta^*_q(g_2 + K),\ \text{by Lemma 4.1}.\end{aligned}$$

Thus, θ is w-monic, it is straightforward that θ is w-epic. Therefore, θ is w-isomorphism. $\square$

Theorem 4.8. (Third isomorphis theorem) *Let G be an H_v-group and $L \trianglelefteq K \trianglelefteq G$ such that $\beta^*(L) = \beta^*(K)$, then* $\dfrac{G/L}{\beta^*(K)/L} \overset{w}{\cong} \dfrac{G/K}{\omega_{G/K}}$.

Proof. Defin $\varphi : \begin{array}{l} G/L \longrightarrow G/K \\ x+L \longmapsto x+K \end{array}$ and prove that φ is a weak epimorphis with $ker\ \varphi = \beta^*(K)/L$, then by first isomorphism theorem, proof is completed. If $x + L = y + L$, then $x + L + K = y + L + K$ and by reproduction axiom $x + K = y + K$. For $x + L$ and $y + L$ in G/L;

$$\varphi((x+L)\hat{+}(y+L)) = \varphi(\beta^*(x+y+L)+L) = \beta^*(x+y+L)+K,$$
$$\varphi(x+L)\hat{+}\varphi(y+L) = (x+K)\hat{+}(y+K) = \beta^*(x+y+K)+K.$$

Since $\beta^*(L) = \beta^*(K)$, we have $\beta^*(x + y + L) = \beta^*(x + y + K)$ and so φ is strong hom.. Also

$$\begin{aligned} ker\ \varphi &= \{x+L|\ x+K \in \omega_{G/K}\} \\ &= \{x+L|\ x+K \in \beta^*(K)/K\}, \quad \textit{by (i) of corollary 4.5} \\ &= \beta^*(K)/L. \end{aligned}$$

$\square$

Example 4.9. Consider the H_v-groups H and N as in Example 3.3. Set $G = H \times H \times H$, $K = N \times H \times H$, $L = N \times N \times H$. Then $G/\beta^* = H/\beta^*_H \times H/\beta^*_H \times H/\beta^*_H \simeq Z_2 \times Z_2 \times Z_2$. Therefor by definition 3.2 $L \trianglelefteq K \trianglelefteq G$. Also

$$\begin{aligned} G/K &= \{(x,y,z)+(N\times H\times H)|\ x \in \{b,c,d\}\} \\ &= \{(x,y,z)+(N\times H\times H)|\ x,y,z \in \{b,c,d\}\}, \end{aligned}$$

because $b+N = c+N = d+N = \{b, c, d\}$ and $a+N = e+N = f +N = N$. By theorem 4.4 $\frac{G/K}{\beta^*_N} \cong \frac{\beta^*(G)}{\beta^*(K)} \simeq \dfrac{Z_2 \times Z_2 \times Z_2}{\{0\} \times Z_2 \times Z_2} \simeq Z_2$. By corollary 4.5

$$\begin{aligned} \omega_{G/K} &= \frac{\beta^*(K)}{K} = \frac{\beta^*(N)\times\beta^*(H)\times\beta^*(H)}{K} \\ &= \frac{N\times H\times H}{K} = \{(x,y,z)+(N\times H\times H)|\ x\in N\} \\ &= \{(a,a,a)+K, (e,e,e)+K, (f,f,f)+K\}, \end{aligned}$$

because $y + H = H$, for every $y \in H$.

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