Journal of Mathematical Extension
Vol. 12, No. 3, (2018), 147-163
ISSN: 1735-8299
URL: http://www.ijmex.com

L-limited and Almost L-limited Sets in Dual Banach Lattices

H.Ardakani*
Yazd University
Payame Noor University

S.M.S.Modarres Mosadegh
Yazd University

M.Salimi
Farhangian University

S.M.Moshtaghioun
Yazd University

Abstract. Following the concept of L-limited sets in dual Banach spaces introduced by Salimi and Moshtaghioun, we introduce the concept of almost L-limited sets in dual Banach lattices and then by a class of disjoint limited completely continuous operators on Banach lattices, we characterize Banach lattices in which almost L-limited subsets of their dual, coincide with L–limited sets.

AMS Subject Classification: 46A40; 46B40; 46B42
Keywords and Phrases: L-limited set, limited completely continuous operator, Gelfand–Phillips property

Received: June 2017; Accepted: May 2018
*Corresponding Author

1 Introduction and Preliminaries

A subset A of a Banach space X is called limited, if every weak* null sequence (x_n^*) in X^* converges uniformly on A, that is,

$$\lim_{n\to\infty} \sup_{a\in A} |\langle a, x_n^*\rangle| = 0.$$

Also if $B \subseteq X^*$ and every weak null sequence (x_n) in X converges uniformly on B, we say that B is an L-set.

We know that every relatively compact subset of X is limited and every limited subset of a dual Banach space is an L-set, but the converse of these assertions, in general, are false. If every limited subset of a Banach space X is relatively compact, then X has the Gelfand–Phillips (GP) property. For example, the classical Banach spaces c_0 and ℓ_1 have the GP property and every separable Banach sequence space, every Schur space (i.e., weak and norm convergence of sequences in the space coincide), and dual of spaces containing no copy of ℓ_1, such as reflexive spaces, have the same property [4]. A Banach space X is Grothendieck if weak* convergent sequences in X^* are weak convergent. The reader can be find some useful and additional properties of limited and Banach spaces with the GP property in [6] and [7].

Recently, the authors in [12] and [13], introduced the class of L–limited sets and limited completely continuous (lcc) operators on Banach spaces. In fact, a bounded linear operator $T : X \to Y$ between two Banach spaces is lcc if it carries limited and weakly null sequences in X to norm null ones in Y. The class of all lcc operators from X to Y is denoted by $Lcc(X, Y)$. Also, a Banach space X has the L–limited property, if every L-limited set in X^* is relatively weakly compact. The authors in [12] and [13], characterized these concepts with respect to some well known geometric properties of Banach spaces, such as, GP, reciprocal DP and Grothendieck property.

It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Banach lattice is said to be σ–

Dedekind complete if for its countable subset that is bounded above has a supremum. A Riesz space (or a vector lattice) is an ordered vector space E with the additional property that for each pair of vectors $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ both exist in E.
Here, by the definition of almost L– sets in [9] and L–limited sets in dual Banach spaces, we introduce the concept of almost L–limited sets in Banach lattices and then we obtain Banach lattices in which this class of sets coincide with the class of L–limited sets. We will give some equivalent condition for $T^*(B)$ to be an L-limited set (resp. almost L-limited set), where B is a norm bounded solid subset of E^* and T is an operator (resp. order bounded operator) from a Banach space X into a Banach lattice E. Finally by introducing the concept of disjoint limited completely continuous (dlcc) operators between Banach lattices and positive Gelfand–Phillips (positive GP) property, we obtain some characterizations of them. The class of all dlcc operators from X to Y is denoted by $L^dcc(X, Y)$.
Throughout this article, X and Y denote the arbitrary Banach spaces and X^* refers to the dual of the Banach space X. Also E and F denote arbitrary Banach lattices and $E^+ = \{x \in E : x \geq 0\}$ refers to the positive cone of the Banach lattice E and B_E is the closed unit ball of E. If x is an elemeny of a Banach lattice E, then positive part, negative part and absolute value of x is represented by x^+, x^- and $|x|$, respectively. A subset A of E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. If a, b belong to a Banach lattice E and $a \leq b$, the interval $[a, b]$ is the set of all $x \in E$ such that $a \leq x \leq b$. A subset of a Banach lattice is called order bounded if it is contained in an order interval. A Banach lattice E has the positive Schur property, if $(x_n) \in E^+$ and $x_n \to 0$ weakly, imply $\|x_n\| \to 0$. We refer the reader to [1] and [11] for unexplained terminologies on Banach lattice theory and positive operators.

2 L-limited Sets in Banach Lattices

Following the introducing of the concept of L–limited sets in [12], we give some additional properties of them in Banach spaces and specially in Banach lattices.

Proposition 2.1. *Let X be a Banach space and B be a bounded subset of X^*. Then the following are equivalent:*

(a) B is an L-limited set,

(b) for each sequence (f_n) in B, $f_n(x_n) \to 0$, for every weakly null and limited sequence (x_n) of X.

Proof. $(a) \Rightarrow (b)$: This follows from the inequality

$$|f_n(x_n)| \leq \sup_{f \in B} |f(x_n)|,$$

that is consistant for each sequence (f_n) in B and for every weakly null and limited sequence (x_n) of X.
$(b) \Rightarrow (a)$: Assume that B is not an L-limited set in X^*. So there exsits an $\epsilon > 0$ and a weakly null and limited sequence (x_n) in X such that $\sup_{f \in B} |f(x_n)| > \epsilon$ for all n. This implies the existence of a sequence (f_n) in B such that $|f_n(x_n)| > \epsilon$, for all n. □

Corollary 2.2. *Let X be a Banach space and (f_n) be a norm bounded sequence of X^*. Then the following are equivalent:*

(a) The subset $\{f_n : n \in N\}$ is an L-limited set,

(b) $f_n(x_n) \to 0$, for every weakly null and limited sequence (x_n) of X.

From [11], a Banach space X has the L–limited property, if every L–limited subset of X^* is relatively weakly compact. The following evident proposition gives a characterization of the L–limited property by L–limited setes.

Proposition 2.3. *Let X be a Banach space. Then the following are equivalent:*

(a) X has the L–limited property,

(b) every L–limited sequence in X^* is relatively weakly compact.

Proposition 2.4. *Let T be an operator from a Banach space X into a Banach lattice E and $f \in (E^*)^+$. Then the following are equivalent:*

(a) $T^*[-f,f]$ *is an L–limited set,*

(b) for every weakly null and limited sequence (x_n) *of* X*,* $f(|T(x_n)|) \to 0$.

Proof. It follows immediately from the equality

$$f(|T(x_n)|) = \sup_{g \in T^*[-f,f]} |g(x_n)|.$$

□

Corollary 2.5. *For a Banach lattice E, the following are equivalent:*

(a) For each $f \in (E^*)^+, [-f,f]$ *is an L–limited set,*

(b) for every weakly null and limited sequence (x_n) *of E,* $(|x_n|)$ *is weakly null.*

Proof. $(a) \Rightarrow (b)$: In Proposition 2.4, take $T = Id_E$.
$(b) \Rightarrow (a)$: For every weakly null and limited sequence (x_n) of E and $f \in (E^*)^+$, we have $\sup_{g\in[-f,f]} |g(x_n)| = f(|x_n|)$. Also by (b), $[-f,f]$ is an L-limited set. This completes the proof. □

Recall that the lattice operations in a Banach lattice E are weakly sequentially continuous if for every weakly null sequence (x_n) in E, $|x_n| \to 0$ for $\sigma(E,E^*)$. The lattice operations in the dual Banach lattice E^* are weak* sequentially continuous if for every weak* null sequence (f_n) in E^*, $|f_n| \to 0$ for $\sigma(E^*,E)$. The following theorem shows that generally, the absolute value of a limited set is not limited.

Theorem 2.6. *Suppose thst A is a limited subset of a Banach lattice E and* E^* *has the weak* sequentially continuous lattice operations. Then* $|A| = \{|a| : a \in A\}$ *is limited.*

Proof. We show that every weak* null sequence (x_n^*) in E^* converges uniformly on $|A|$, that is, $\lim_{n\to\infty} \sup_{x\in A} |\langle x_n^*, |x|\rangle| = 0$.
From [10, Lemma 1.4.4], $\langle |x_n^*|, |x|\rangle = max\{\langle z_n^*, x\rangle : |z_n^*| \leq |x_n^*|\}$ for all n. So, there exists $z_n^* \in E^*$, such that $|z_n^*| \leq |x_n^*|$ and $\langle |x_n^*|, |x|\rangle = \langle z_n^*, x\rangle$. Since E^* has the weak* sequentially continuous lattice operations, the sequences $(|x_n^*|)$ and so (z_n^*) are weak* null. But the set A is limited. So, $\sup_{x\in A} |\langle z_n^*, x\rangle| \to 0$. Now from the inequality $sup_{x\in A}|\langle x_n^*, |x|\rangle| \leq \sup_{x\in A}\langle |x_n^*|, |x|\rangle$, we have $\sup_{x\in A} |\langle x_n^*, |x|\rangle| \to 0$ and so the set $|A|$ is limited. □

Theorem 2.7. *Let E be a Banach lattice such that E^* has an order continuous norm. Then for each $f \in (E^*)^+, [-f, f]$ is an L–limited set.*

Proof. If E^* has an order continuous norm, then from [1, Theorem 4.9], for each $f \in (E^*)^+, [-f, f]$ is relatively weakly compact and by [12, Lemma 2.2], it is an L–limited set. □

Theorem 2.8. *Let E be a Banach lattice with the L–limited property. Then E^* has an order continuous norm.*

Proof. It is evident that every L-set in E^* is an L–limited set. If E is a Banach lattice E with the L–limited property, then every L–set in E^* is relatively weakly compact and so by [2, Theorem 3.1], E^* has an order continuous norm. □

The converse of Theorem 2.8, is false, in general. For example c_0 does not have L–limited property, but its dual ℓ_1 has an order continuous norm. The next main result, gives an equivalent condition for $T^*(B)$ being L–limited set, where B is a norm bounded solid subset of E^* and T is an operator from a Banach space X into a Banach lattice E.

Theorem 2.9. *Let T be an operator from a Banach space X into a Banach lattice E and B be a norm bounded solid subset of E^*. Then the following are equivalent:*

- *(a) $T^*(B)$ is an L–limited set in X^*,*
- *(b) For each $f \in B^+$ and for each norm bounded disjoint sequence $(f_n) \in B^+$, the sets $T^*[-f, f]$ and $\{T^* f_n : n \in N\}$ are L–limited.*

Proof. $(a) \Rightarrow (b)$: It is obvious, since $[-f, f]$ and B^+ are contained in B.
$(b) \Rightarrow (a)$: To prove that $T^*(B)$ is an L–limited set, it is sufficient to show that $\sup_{h \in B} |T^*(h(x_n))| \to 0$ for every weakly null and limited sequence (x_n) in X. This follows by a similar techniques used in [3, Theorem 2.7]. □

Corollary 2.10. *Let E be a Banach lattice and B be a norm bounded solid subset of E^*. Then the following are equivalent:*

- *(a) B is an L–limited set,*

(b) $[-f, f]$ *and* $\{f_n : n \in N\}$ *are an L–limited sets, for each* $f \in B^+$ *and for each disjoint sequence* $(f_n) \in B^+$.

The next result characterizes lcc operators by L–limited sets.

Corollary 2.11. *For an operator* T *from a Banach space* X *into a Banach lattice* E*, the following are equivalent:*

(a) T *is lcc,*

(b) $T^*(B_{E^*})$ *is an L– limited set, where* B_{E^*} *is the closed unit ball of* E^**,*

(c) $T^*[-f, f]$ *and* $\{T^* f_n : n \in N\}$ *are L–limited sets, for each* $f \in (B_{E^*})^+$ *and for each norm bounded disjoint sequence* $(f_n) \in (B_{E^*})^+$*,*

(d) $|T(x_n)| \to 0$ *for* $\sigma(E, E^*)$ *and* $f_n(Tx_n) \to 0$*, for every weakly null and limited sequence* (x_n) *in* X *and for each disjoint sequence* (f_n) *in* $(B_{E^*})^+$.

Proof. $(a) \Leftrightarrow (b)$: By the equality $\sup_{f \in T^*(B_{E^*})} |f(x_n)| = \|Tx_n\|_E$, the set $T^*(B_{E^*})$ is an L–limited set in X^*, ,if and only if, T is an lcc operator.
By Theorem 2.9, the statements (b) and (c) are equivalent and the equivalence of $(c) \Leftrightarrow (d)$ is a direct consequence of Propositions 2.2 and 2.4.
□

3 Almost L-limited Sets in Banach Lattices

In this section we introduce a new class of sets and operators. Recall that a sequence (x_n) in a Banach lattice E is (pairwise) disjoint, if for each $i \neq j$, $|x_i| \wedge |x_j| = 0$.

Definition 3.1. Let E be a Banach lattice and X be a Banach space. Then

(a) A norm bounded subset B of a dual Banach lattice E^* is said to be an almost L–limited set if every disjoint weakly null and limited sequence (x_n) of E converges uniformly to zero on the set B, that is $\sup_{f \in B} |f(x_n)| \to 0$.

(b) An operator T from a Banach lattice E into a Banach space X is a disjoint limited completely continuous (dlcc) operator if the sequence $(\|Tx_n\|)$ converges to zero for every weakly null and limited sequence of pairwise disjoint elements in E.

Note that every L–limited set of a dual Banach lattice, is an almost L–limited set, but the converse is false, in general. In fact for many Banach lattices E with the positive GP property and without the GP property, the closed unit ball of the dual Banach lattice E^* is an almost L–limited set, but it is not L–limited set. As an example, the closed unit ball B_{ℓ_∞} of ℓ_∞ is an almost L–limited set in ℓ_∞, but the closed unit ball $B_{(\ell_\infty)^*}$ is not an almost L–limited set in $(\ell_\infty)^*$. In the following, we give a usefull chracterization of almost L-limited sets, that can be proved by the similar method of Proposition 2.1.

Proposition 3.2. *Let E be a Banach lattice and B be a norm bounded set in E^*. Then the following are equivalent:*

(a) B is an almost L–limited set,

(b) for each sequence (f_n) in B, $f_n(x_n) \to 0$, for every disjoint weakly null and limited sequence (x_n) of E.

In particular, we obtain:

Proposition 3.3. *Let E be a Banach lattice and (f_n) be a norm bounded sequence in E^*. Then the following are equivalent:*

(a) The subset $\{f_n : n \in N\}$ is an almost L–limited set,

(b) $f_n(x_n) \to 0$, for every disjoint weakly null and limited sequence (x_n) of E.

Recall that, an operator T from a Banach lattice E into another F is said to be order bounded if for each $x \in E^+$, the subset $T([-x, x])$ is order bounded in F. From [3, Proposition 3.3], for each $f \in (F^*)^+$ and for each order bounded operator from E into F, the subsets $[-f, f]$ and $T^*([-f, f])$ are almost L–limited sets.

Theorem 3.4. *Let T be an order bounded operator from a Banach lattice E into a Banach lattice F and B be a norm bounded solid subset of F^*. Then the following are equivalent:*

(a) $T^*(B)$ *is an almost L–limited set in* E^*,

(b) $\{T^* f_n : n \in N\}$ *is an almost L–limited set, for each* $f \in B^+$ *and for each disjoint sequence* (f_n) *in* B^+.

Proof. The proof is the same as the proof of Theorem 2.9 and we omit it. □

As some consequences, we deduce:

Corollary 3.5. *Let* T *be an order bounded operator from a Banach lattice* E *into another Banach lattice* F *and* B *be a norm bounded solid subset of* F^*. *Then the following are equivalent:*

(a) $T^*(B)$ *is an almost L–limited set in* E^*,

(b) $f_n(Tx_n) \to 0$, *for every disjoint weakly null and limited sequence* (x_n) *of* E^+ *and for each disjoint sequence* (f_n) *in* B^+.

Corollary 3.6. *Let* E *be a Banach lattice and* B *be a norm bounded solid subset of* E^*. *Then the following are equivalent:*

(a) B *is an almost L–limited set,*

(b) $\{f_n : n \in N\}$ *is an almost L–limited set for each disjoint sequence* (f_n) *in* B^+.

The next result characterizes the class of dlcc operators by almost L–limited sets.

Corollary 3.7. *For an order bounded operator* T *from a Banach lattice* E *into another Banach lattice* F, *the following are equivalent:*

(a) T *is dlcc,*

(b) $T^*(B_{F^*})$ *is an almost L– limited set, where* B_{F^*} *is the closed unit ball of* F^*,

(c) $\{T^*(f_n) : n \in N\}$ *is an almost L–limited set for each disjoint sequence* (f_n) *in* $(B_{F^*})^+$,

(d) $f_n(T(x_n)) \to 0$, *for every disjoint weakly null and limited sequence* (x_n) *of* E^+ *and for each disjoint sequence* (f_n) *in* $(B_{F^*})^+$.

Proof. $(a) \Leftrightarrow (b)$: By the equality, $\sup_{f\in T^*(B_{F^*})} |f(x_n)| = \|Tx_n\|_F$, for every sequence (x_n) in E, it follows easily that, $T^*(B_{F^*})$ is an almost L-limited set in E^* if and only if T is an dlcc operator.
By Theorem 3.4, the statements (b) and (c) are equivalent and the equivalence $(c) \Leftrightarrow (d)$ is a direct consequence of Proposition 3.3.
□

Definition 3.8. A Banach lattice E has the positive GP property if each weakly null and limited sequence with the positive terms is norm null.

It is clear that the GP property implies the positive GP property.

Theorem 3.9. *Let E be a Banach lattice. Then the following are equivalent:*

(a) E has the positive GP property,

(b) every weakly null and disjoint limited sequence in E converges to zero in norm.

Proof. $(a) \Rightarrow (b)$: Let (x_n) be a weakly null and disjoint limited sequence in E. From [14, Proposition 1.3], the sequence $(|x_n|)$ is weakly null and by [8, Lemma 3.7], it is limited in E. From (a), the sequence $(|x_n|)$ and so (x_n) converges to zero in norm.
$(b) \Rightarrow (a)$: Suppose that $\inf_n \|x_n\| = c > 0$ for some weakly null and limited sequence $(x_n) \subset E^+$. Putting $y_n = c^{-1}x_n$ and using [10, Corollary 5] we find a subsequence (y_{n_k}), a constant $d > 0$, and a disjoint sequence (z_k) of E^+ such that $0 < z_k \leq y_{n_k}$ and $\|z_k\| \geq d$. It is clear that disjoint limited sequence (z_k) tends weakly to zero, but $\|z_k\| \geq d$. This fact contradicts the assumption. □

Theorem 3.10. *Let E be a Banach lattice. Then the following are equivalent:*

(a) E has the positive GP property,

(b) B_{E^*} is an almost L–limited set.

Proof. $(a) \Rightarrow (b)$: If E has the positive GP property and (x_n) is a weakly null and disjoint limited sequence in E, then by Theorem 3.9, (x_n) is norm null. Now the equality $\|x_n\| = \sup_{f\in(B_{E^*})} |f(x_n)|$ implies that B_{E^*} is an almost L–limited set.
$(b) \Rightarrow (a)$. If B_{E^*} is an almost L–limited set, then for every weakly null and disjoint limited sequence (x_n) in E, $\|x_n\| = \sup_{f\in(B_{E^*})} |f(x_n)| \to 0$. Now apply Theorem 3.9. □

Theorem 3.11. *Let E be a Banach lattice. Then the following are equivalent:*

(a) E has the positive GP property,

(b) for each Banach lattice F, $L^d cc(E, F) = L(E, F)$.

Proof. $(a) \Rightarrow (b)$: If E has the positive GP property and (x_n) is a weakly null and disjoint limited sequence in E, then by Theorem 3.9, (x_n) is norm null and for each bounded operator T on E, $\|Tx_n\| \to 0$; that is, $L^d cc(E, F) = L(E, F)$.
$(b) \Rightarrow (a)$: If $F = E$, then (b) implies that the identity operator on E is dlcc and so for every weakly null and disjoint limited sequence (x_n) in E, $\|x_n\| \to 0$. Now apply Theorem 3.9. □
In the following Theorem 3.13, we show that the positive GP property and the GP property coincide in the class of σ-Dedekind complete Banach lattices. Let us start with the following two lemmas. The first lemma shows that every weakly null sequence in ℓ_∞ is limited.

Lemma 3.12. *Let X be a Grothendieck space which has the DP property. Then every weakly null sequence in X is limited.*

Proof. Suppose $x_n \to 0$ weakly in X and (f_n) is a weak* null sequence in X^*, but $sup_n|f_k(x_n)| > \eta$ for some $\eta > 0$ and all k. Let $k_1 = 1$ and find n_1 with $|f_1(x_{n_1})| > \eta$. Since the sequence (f_k) is a weak* null sequence, we can indicate $k_2 > k_1$ satisfying the condition $|f_{k_2}(x_m)| < \eta$ for $m = 1, ..., n_1$. Hence $|f_{k_2}(x_{n_2})| > \eta$ for some $n_2 > n_1$. Following by induction we can find subsequences $k_1 = 1 < k_2 < k_3 < ...$ and $n_1 < n_2 < n_3 < ...$ such that $|f_{k_m}(x_{n_m})| > \eta$. Put $g_m = f_{k_m}$ and $y_m = x_{n_m}$. Clearly $y_m \to 0$ weakly in X and $g_m \to 0$ weak* in X^*. Since X is a Grothendieck space, then $g_m \to 0$ weakly in X^*, and so $g_m(y_m) \to 0$ by the DP property of X, a contradiction because $|g_m(y_m)| > \eta$. □

Example 3.13. ℓ_∞ dose not have the positive GP property.

Proof. It is enough to remember that ℓ_∞ dose not have the positive Schur property and use Lemma 3.12. By [12], a Banach lattice E has the positive Schur property, whenever $0 \leq x_n \to 0$ weakly implies $\|x_n\| \to 0$. □

Now we are able to formulate the following equivalence condition.

Theorem 3.14. *Let E be a σ–Dedekind complete Banach lattice. Then E has the positive GP property, if and only if, it has the GP property.*

Proof. Since the positive GP property is inherited by closed Riesz subspaces and ℓ_∞ does not have the positive GP property, then E does not contain any order copy of ℓ_∞. According to [11, Corollary 2.4.3], E has order continuous norm, and so it possesses the GP property by the Bukhvalov's Theorem in [5]. □

Corollary 3.15. *The dual Banach lattice $C(K)^*$ has the GP property, where K is a compact Hausdorff space.* ***Proof.*** *For each positive and weakly null sequence (f_n) in $C(K)^*$, $\|f_n\| = f_n(1_K) \to 0$, where 1_K denotes the constant function 1 on K. That is $C(K)^*$ has the positive GP property. On the other hands from [1], the Banach lattice $C(K)^*$ is σ–Dedekind complete and by Theorem 3.14, it has the GP property.* □

It should be noted that, a σ–Dedekind complete Banach lattice E, has the GP property, if and only if, the norm of E is order continuous (cf.[5]). So, by Theorem 3.14, a σ–Dedekind complete Banach lattice E is a positive GP space, if and only if, the norm of E is order continuous. Now, we want to give a relation between order weakly compact and dlcc of operators. Recall that a continuous operator $T : E \to X$ from a Banach lattice E to a Banach space X is order weakly compact (o-weakly compact), whenever $T[0, x]$ is a relatively weakly compact subset of X for each $x \in E^+$. We know that by [1, Theorem 5.57], an operator T from a Banach lattice E into a Banach space X is order weakly compact if and only if $\|Tx_n\| \to 0$ for every disjoint order bounded sequence (x_n) in E.

Lemma 3.16. *[9] Every order bounded disjoint sequence in a σ-Dedekind complete Banach lattice is limited.*

It should be noted that every order bounded disjoint sequence (x_n) in an arbitrary Banach lattice E is not limited, in general. Consider the space c of all convergent sequences with the sup norm. Clearly the sequence (e_n) of unit vectors in c is disjoint and order bounded from below by 0 and from above by the vector $e = (1, 1, 1, ...)$. Define f_k on c by $f_k(x_n) = x_k - lim_{n\to\infty} x_n$. Then $f_k(x) \to 0$ as $k \to \infty$ for every $(x_n) \in c$, but $\sup_n |f_k(e_n)| \geq f_k(e_k) = 1$. We have just shown that $\{e_n : n \in N\}$ is not limited.

Theorem 3.17. *Let E be a σ-Dedekind complete Banach lattice. Then each dlcc operator on E is order weakly compact.*

Proof. Let (x_n) be an order bounded disjoint sequence of E. It follows from [1] and Lemma 3.16 that (x_n) is a limited weakly null sequence. Since T is dlcc then, $\|Tx_n\| \to 0$. Hence Dodds's theorem ([1, Theorem 5.57]) implies that T is order weakly compact. □

Theorem 3.18. *Let $\mathcal{M} \subset L(X, Y)$ be a Banach lattice. If $\mathcal{M}$ has the positive GP property, then all of the evaluation operators ϕ_x and ψ_{y^*} are dlcc operators, where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$.*

Proof. See Theorem 3.11. □

Now, we show that the *dlcc*ness of evaluation operators is a sufficient condition for the positive GP property of their domain.

Theorem 3.19. *Let Y has the Schur property and $\mathcal{M} \subset L(X, Y)$ be a Banach lattice. If for every $y^* \in Y^*$, the evaluation operator ψ_{y^*} on $\mathcal{M}$ is dlcc, then $\mathcal{M}$ has the positive GP property.*

Proof. If $\mathcal{M}$ does not have the positive GP property, by Theorem 3.9, there exists a weakly null and disjoint limited sequence (T_n) in $\mathcal{M}$ and some $\epsilon > 0$ such that $\|T_n\| > \epsilon$, for all n. So there exists a sequence (x_n) in B_X such that $\|T_n(x_n)\| > \epsilon$, for all n. On the other hand, the evaluation operator ψ_{y^*} on $\mathcal{M}$ is $L^d cc$ for all $y^* \in Y^*$ and so $\|T_n^*(y^*)\| = \|\psi_{y^*}(T_n)\| \to 0$. Hence $|\langle T_n x_n, y^*\rangle| \leq \|T_n^*(y^*)\| \to 0$. So the sequence $(T_n x_n)$ is weakly null and it is norm null by the Schur property, a fact that is impossible. □

Theorem 3.20. *Let X has the Schur property and $\mathcal{M} \subset L_{w^*}(X^*, Y)$ be a Banach lattice. If for every $x^* \in X^*$, the evaluation operator ϕ_{x^*} on $\mathcal{M}$ is dlcc, then $\mathcal{M}$ has the positive GP property.*

Proof. If $\mathcal{M}$ does not have the positive GP property, by Theorem 3.9, there exists a weakly null and disjoint limited sequence (T_n) in $\mathcal{M}$ and some $\epsilon > 0$ such that $\|T_n\| > \epsilon$, for all n. On the other hand, the evaluation operator ϕ_{x^*} on $\mathcal{M}$ is *dlcc* for all $x^* \in X^*$ and so $\|T_n(x^*)\| = \|\phi_{x^*}(T_n)\| \to 0$. Since $\|T_n^*\| > \epsilon$, there exists a sequence (y_n^*) in B_{Y^*} such that $\|T_n^* y_n^*\| > \epsilon$, for all n. But the Schur property of X shows that the weakly null sequence $(T_n^* y_n^*)$ is norm null, which is a contradiction. □

4 Almost L-limited Sets Which Are L-limited Sets

As we noted in the beginning of section 3, every L-limited set in the dual Banach lattice E^*, is an almost L-limited set, but the converse is false in general. In this section we characterize Banach lattices in which the class of almost L-limited sets and that of L-limited sets coincide in their dual.

Theorem 4.1. *For a Banach lattice E, the following are equivalent:*

(a) Each almost L-limited set in E^* is an L-limited set,

(b) for each Banach space Y, $L^d cc(E, Y) = Lcc(E, Y)$,

(c) $L^d cc(E, \ell_\infty) = Lcc(E, \ell_\infty)$.

Proof. $(a) \Rightarrow (b)$: Let $T : E \to Y$ be an operator. From $\sup_{f \in T^*(B_{Y^*})} |f(x_n)| = \|Tx_n\|_Y$, for every sequence (x_n) in E, it follows easily that, $T^*(B_{Y^*})$ is an almost L-limited (respectively, L-limited) set in E^*, if and only if, T is an dlcc (respectively, lcc) operator. Now, let T be an dlcc operator. Then $T^*(B_{Y^*})$ is an almost L-limited set in E^* and from the hypothesis (a), it is an L-limited set in E^*. Hence T is an lcc operator.
$(b) \Rightarrow (c)$: It is clear.
$(c) \Rightarrow (a)$:. Let B be an almost L-limited set in E^*. To prove that

B is an L-limited set, it sufficies to show that $f_n(x_n) \to 0$ for each sequence (f_n) in B and for every weakly null and limited sequence (x_n) in E (Proposition 2.1). Consider the operator $S : E \to \ell_\infty$ defined by $S(x) = (f_n(x))_{n=1}^\infty$, for each $x \in E$. Since B is almost L-limited, S is an dlcc operator. In fact, for every weakly null and disjoint limited sequence (z_i) in E, we have

$$\|Sz_i\|_\infty = \|f_n(z_i)_{n=0}^\infty\|_\infty \leq \sup_{f \in B} |f(z_i)| \to 0,$$

as $i \to \infty$. Which implies that S is an dlcc operator and so from our hypothesis, S is lcc. Thus $\|Sx_n\|_\infty \to 0$ and now the desired conclusion follows from the inequality $|f_n(x_n)| \leq \|Sx_n\|_\infty$ for each n. □

Recall that, an operator T from a Banach space X into a Banach lattice E is said to be semicompact if for each $\epsilon > 0$ there exists some $u \in E^+$ satisfying $T(B_X) \subset [-u, u] + \epsilon B_E$. According to [3, Theorem 4.3], each operator $T : E \to X$ is dlcc, whenever its adjoint $T^* : X^* \to E^*$ is semicompact.

At the end of this section, it should be noted that the adjoint of a dlcc operator is not necessary dlcc and vice versa. For example, the identity operator on the Banach lattice ℓ_1 is dlcc (because ℓ_1 has the GP-property, [13, Theorem 2.2]) but its adjoint, $Id_{\ell_\infty} : \ell_\infty \to \ell_\infty$, is not dlcc. In fact, if $e_n = (0, 0, ..., 1, 0, ...)$ with n'th entry equals to 1 and all others zero, then (e_n) is an order bounded disjoint sequence of ℓ_∞. Hence (e_n) is weakly null and by Lemma 3.16, it is limited, but $\|Id_{\ell_\infty}(e_n)\| = \|e_n\|_\infty = 1$ for all n. Also the identity operator on ℓ_∞ is not dlcc but its adjoint is dlcc, because $(\ell_\infty)^*$ has order continuous norm and so has GP property. Also by Corollary 3.7, $B_{(\ell_\infty)^*}$ is not an almost L–limited set in $(\ell_\infty)^*$, as noted that in the begining of section 3.

References

[1] C. D. Aliprantis and O. Burkishaw, *Positive Operators*, Academic Press, New York and London, 1978.

[2] B. Aqzzouz and K. Bouras, Dunford-Pettis sets in Banach lattices, *Acta Math. Univ. Comenianae*, 81(2012), 185–196.

[3] B. Aqzzouz and K. Bouras, L-sets and almost L-sets in Banach lattices, *Quaestiones Mathematicae*, 36(2013), 107–118.

[4] J. Bourgain and J. Diestel, Limited operators and strict cosingularity, *Math. Nachr.*, 119 (1984), 55–58.

[5] A.V. Buhvalov, Locally convex spaces that are generated by weakly compact sets, (Russian), *Vestnik Leningrad Univ. Mat. Meh. Astronom,* 7 (1973), 11–17.

[6] L. Drewnowski, On Banach spaces with the Gelfand-Phillips property, *Math. Z.,* 193 (1986), 405–411.

[7] G. Emmanuele, On Banach spaces with the Gelfand–Phillips property, III, *J. Math. Pures Appl.*, 72 (1993), 327–333.

[8] K.E. Fahri, N. Machrafi and M. Moussa,Banach Lattices with the Positive Dunford-Pettis Relatively Compact Property, *Extracta Math.*, 80 (2015), 161–179.

[9] A. El. Kaddouri, M. Moussa, About the class of ordered limited operators, *Acta Universitatis Carolinae. Mathematica et Physica*, 54 (2013), 37–43.

[10] G. Groenewegen and P. Meyer-Nieberg, An elementary and unified approach to disjoint sequence theorems, *Indag. Math.* , 48 (1986), 313–317.

[11] P. Meyer- Nieberg, *Banach attices*, Universitext, springer- Verlag, Berlin, 1991.

[12] M. Salimi and S. M. Moshtaghioun, A new class of Banach spaces and its relation with some geometric properties of Bancah spaces, *Abstract and Applied Analysis*, ID 212957, 2012.

[13] M. Salimi and S. M. Moshtaghioun, The Gelfand-Phillips property in closed subspaces of some operator spaces, *Banach J. Math. Anal.*, 2 (2011), 84–92.

[14] W. Wnuk, On the dual positive Schur property in Banach lattices, *J. Positivity*,17 (2013), 759–773.

H. Ardakani
Assistant Professor of Mathematics
Department of Mathematics
Yazd University
Yazd, Iran
Department of Mathematics
Payame Noor University
Yazd, Iran
E-mail: halimeh_ardakani@yahoo.com

S. M. S. Modarres Mosadegh
Professor of Mathematics
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: smodarres@yazd.ac.ir

M. Salimi
Assistant Professor of Mathematics
Department of Mathematics
Farhangian University
Isfahan, Iran
E-mail: m_salimi52@yahoo.com

S. M. Moshtaghioun
Associate Professor of Mathematics
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: moshtagh@yazd.ac.ir