Journal of Mathematical Extension
Vol. 12, No. 3, (2018), 105-129
ISSN: 1735-8299
URL: http://www.ijmex.com

On the q-parametric Stancu type Dunkl generalization of the Kantorovich-Szász-Mirakjan operators

Mohammad Mursaleen *
Aligarh Muslim University

Taqseer Khan
Jamia Millia Islamia

Md Nasiruzzaman
Jamia Hamdard

Abstract. In this paper we construct the Stancu type q-Kantrovich-Szász-Mirakjan operators generated by Dunkl generalization of the exponential function. We obtain some approximation results using the Korovkin approximation theorem and the weighted Korovkin-type theorem for these operators. We also study convergence properties by using the modulus of continuity and the rate of convergence for functions belonging to the Lipschitz class. Furthermore, we obtain the rate of convergence in terms of the classical, the second order, and the weighted modulus of continuity..

AMS Subject Classification: 41A25; 41A36; 33C45
Keywords and Phrases: q-integers, Dunkl analogue, generalization of exponential function, Szász operator, modulus of continuity, weighted modulus of continuity

Received: June 2017; Accepted: September 2017

*Corresponding Author

1 Introduction

In 1912, Bernstein [3] introduced the following sequence of operators $B_n : C[0,1] \to C[0,1]$ defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad x \in [0,1], \tag{1}$$

for $n \in \mathbb{N}$ and $f \in C[0,1]$.

In 1950, Szász [24] introduced the following operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad f \in C[0,\infty), \tag{2}$$

for $x \geq 0$.

The introduction of q-calculus has emerged as a new venue of research in the field of approximation theory. Lupaş [10] introduced the first q-analogue of the well-known Bernstein polynomials in 1987. Another q-analogue of the classical Bernstein polynomials was given by Phillips [19] in the year of 1997. This intrigued many researchers to introduce the q-generalizations of various operators and to study their approximation properties [18, 11, 20, 12, 14].

Also, recently, Mursaleen et al. have introduced the (p,q)-calculus, which is a generalization of the q-calculus. They have introduced the (p,q)-analogues of several well known operators and have studied their approximation properties. Some of them are in [15], [16], [17].

For $p = 1$, the (p,q)-integers, $0 < q < p \leq 1$ reduce to the q-integers.

Below we present some basics of the q-calculus.
The q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q-binomial coefficient of

$n \in \mathbb{N}$ are defined by (see [2])

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ n, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

respectively.
The q-analogue of $(1+x)^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x) & n = 1,2,3,\cdots \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined by

$$(x;q)_0 = 1, \ (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x), \ (x;q)_\infty = \prod_{j=0}^{\infty}(1-q^j x).$$

The Gauss binomial formula is given by

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k x^{n-k}.$$

The q-analogue of the Bernstein operators [19] is defined as

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) \ \ f\left(\frac{[k]_q}{[m]_q}\right), \quad x \in [0,1], n \in \mathbb{N}. \tag{3}$$

There are two q-companions of the exponential function e^z, defined as

For $\mid z \mid < \frac{1}{1-q}$ and $\mid q \mid < 1$,

$$e(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \frac{1}{1 - ((1-q)z)_q^{\infty}} \tag{4}$$

and for $\mid q \mid < 1$,

$$E(z) = \prod_{j=0}^{\infty} \left(1 + (1-q)q^j z\right)_q^{\infty} = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{z^k}{k!} = (1 + (1-q)z)_q^{\infty}, \tag{5}$$

where $(1-x)_q^{\infty} = \prod_{j=0}^{\infty}(1 - q^j x)$.

Using a generalization of the exponential function [21], Sucu [23] defined a Dunkl analogue of Szász operators as

$$S_n^*(f;x) := \frac{1}{e_\mu(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_\mu(k)} f\left(\frac{k + 2\mu\theta_k}{n}\right), \tag{6}$$

where $x \geq 0, \quad f \in C[0,\infty), \mu \geq 0, \quad n \in \mathbb{N}$
and

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)},$$

where

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma\left(k + \mu + \frac{1}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)}$$

and

$$\gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma\left(k + \mu + \frac{3}{2}\right)}{\Gamma\left(\mu + \frac{1}{2}\right)}.$$

A recursion formula for γ_μ is given by

$$\gamma_\mu(k+1) = (k + 1 + 2\mu\theta_{k+1})\gamma_\mu(k), \qquad k = 0, 1, 2, \cdots,$$

where

$$\theta_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N} \\ 1 & \text{if } k \in 2\mathbb{N} + 1. \end{cases}$$

Cheikh et al. [4] stated the q-Dunkl classical q-Hermite type polynomials and defined the q-Dunkl analogues of exponential functions and recursion relations for $\mu > -\frac{1}{2}$ and $0 < q < 1$.

$$e_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_{\mu,q}(n)}, \quad x \in [0,\infty) \tag{7}$$

$$E_{\mu,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} x^n}{\gamma_{\mu,q}(n)}, \quad x \in [0,\infty) \tag{8}$$

$$\gamma_{\mu,q}(n+1) = \left(\frac{1 - q^{2\mu\theta_{n+1}+n+1}}{1-q} \right) \gamma_{\mu,q}(n), \qquad n \in \mathbb{N}, \tag{9}$$

where

$$\theta_n = \begin{cases} 0 & \text{if } n \in 2\mathbb{N}, \\ 1 & \text{if } n \in 2\mathbb{N}+1. \end{cases}$$

An explicit formula for $\gamma_{\mu,q}(n)$ is the following

$$\gamma_{\mu,q}(n) = \frac{(q^{2\mu+1}, q^2)_{[\frac{n+1}{2}]} (q^2, q^2)_{[\frac{n}{2}]}}{(1-q)^n} \gamma_{\mu,q}(n), \qquad n \in \mathbb{N}.$$

Some particular cases of $\gamma_{\mu,q}(n)$ are

$$\gamma_{\mu,q}(0) = 1, \quad \gamma_{\mu,q}(1) = \frac{1-q^{2\mu+1}}{1-q}, \quad \gamma_{\mu,q}(2) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right),$$

$$\gamma_{\mu,q}(3) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right),$$

$$\gamma_{\mu,q}(4) = \left(\frac{1-q^{2\mu+1}}{1-q}\right)\left(\frac{1-q^2}{1-q}\right)\left(\frac{1-q^{2\mu+3}}{1-q}\right)\left(\frac{1-q^4}{1-q}\right).$$

In [8], Gürhan Içöz gave a Dunkl generalization of the Kantorovich type integral generalization of the Szász operators. In [9], they gave a q-Dunkl generalization of the Szász operators as

$$D_{n,q}(f;x) = \frac{1}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} f\left(\frac{1-q^{2\mu\theta_k+k}}{1-q^n}\right), \tag{10}$$

for $\mu > \frac{1}{2}, \quad x \geq 0, \quad 0 < q < 1$ and $f \in C[0, \infty)$.
We have the following lemma.

Lemma 1.1. *1.* $D_{n,q}(1; x) = 1$,

2. $D_{n,q}(t; x) = x$,

3. $x^2 + [1 - 2\mu]_q q^{2\mu} \frac{e_{\mu,q}(q[n]_q(x))}{e_{\mu,q}([n]_q x)} \frac{x}{[n]_q} \leq D_{n,q}(t^2; x) \leq x^2 + [1 + 2\mu]_q \frac{x}{[n]_q}$,

4. $D_{n,q}(t^3; x) \geq x^3 + (2q+1)[1-2\mu]_q \frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)} \frac{x^2}{[n]_q} + q^{4\mu}[1-2\mu]_q^2 \frac{e_{\mu,q}(q^2[n]_q x)}{e_{\mu,q}([n]_q x)} \frac{x}{[n]_q^2}$,

5. $D_{n,q}(t^4; x) \leq x^4 + 6[1 + 2\mu]_q \frac{x^3}{[n]_q} + 7[1 + 2\mu]_q^2 \frac{x^2}{[n]_q^2} + [1 + 2\mu]_q^3 \frac{x}{[n]_q^3}$.

2 Construction of Operators and Auxiliary Results

Mursaleen et al. gave the following Dunkl generalisation of the Kantorovich type Szász-Mirakjan operators via q-calculus as follows:
For any $x \in [0, \infty), \quad n \in \mathbb{N}, \quad 0 < q < 1$, and $\mu > \frac{1}{2}$

$$K_{n,q}^*(f; x) = \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} f(t) d_q t, \tag{11}$$

where f is a continuous and nondecreasing function on $[0, \infty)$. They obtained approximation results and studied rate of convergence for these operators in [22]. Inspired by their work, we introduce the Stancu type Dunkl generalization of the Kantorovich-Szász-Mirakjan operators via q-calculus as follows:
For any $x \in [0, \infty), \quad n \in \mathbb{N}, \quad 0 < q < 1$, and $\mu > \frac{1}{2}$, we define

$$T_{n,q}^*(f; x) = \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} f\left(\frac{nt + \alpha}{n + \beta}\right) d_q t, \tag{12}$$

where α, β are such that $0 < \alpha \leq \beta$ and f is a continuous and nondecreasing function on $[0, \infty)$. If we take $\alpha = 0 = \beta$ in the above operators, they reduce to the operators (11). Therefore, the operators in (12) are generalised form of the operators in (11).

Lemma 2.1. *Let $T^*_{n,q}(.\ ;\ .)$ be the operators given by (12). Then we have the following identities and inequalities:*

1. $T^*_{n,q}(1;x)=1$,

2. $T^*_{n,q}(t;x)=\frac{2qn}{(n+\beta)[2]_q}x+\frac{n}{(n+\beta)[2]_q[n]_q}+\frac{\alpha}{(n+\beta)}$,

3. $\frac{n^2}{(n+\beta)^2}\frac{1}{[3]_q[n]_q^2}+\frac{2n\alpha}{(n+\beta)^2[2]_q[n]_q}+\frac{\alpha^2}{(n+\beta)^2}+\Big(\frac{n^2}{(n+\beta)^2}\frac{3q}{[3]_q[n]_q}+\frac{2n\alpha}{(n+\beta)^2}\frac{2q}{[2]_q}+\frac{3q^{2(\mu+1)}}{[3]_q[n]_q}$ $\times[1-2\mu]_q\frac{e_{\mu,q}(q[n]_q(x))}{e_{\mu,q}([n]_qx)}\Big)x+\frac{n^2}{(n+\beta)^2}\frac{3q^2}{[3]_q}x^2\leq T^*_{n,q}(t^2;x)\leq\frac{n^2}{(n+\beta)^2}\frac{1}{[3]_q[n]_q^2}+\frac{2n\alpha}{(n+\beta)^2[2]_q[n]_q}+\frac{\alpha^2}{(n+\beta)^2}+\Big(\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q}+\frac{2n\alpha}{(n+\beta)^2}\frac{2}{[2]_q}+\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q}[1+2\mu]_q\Big)x$ $+\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q}x^2$,

4. $\frac{n}{(n+\beta)^3[n]_q}\Big(\frac{n^2}{[4]_q[n]_q^2}+\frac{3n\alpha}{[3]_q[n]_q}+\frac{3\alpha^2}{[2]_q}\Big)+\frac{\alpha^3}{(n+\beta)^3}+\frac{1}{(n+\beta)^3}\Big(n\big(\frac{4n^2q}{[4]_q[n]_q^2}+\frac{3n\alpha}{[n]_q}\frac{3q}{[3]_q}+\frac{6\alpha^2q}{[2]_q}\big)+\big(\frac{2n}{[4]_q[n]_q}+\frac{3\alpha}{[3]_q}\big)\frac{3n^2q^{2(\mu+1)}}{[n]_q}[1-2\mu]_q\frac{e_{\mu,q}(q[n]_q(x))}{e_{\mu,q}([n]_qx)}+\frac{4n^3}{[4]_q[n]_q^2}q^{4\mu+3}[1-2\mu]_q^2\frac{e_{\mu,q}(q^2[n]_q(x))}{e_{\mu,q}([n]_q(x))}\Big)x+\frac{n^2}{(n+\beta)^3}\Big(\frac{6nq^2}{[4]_q[n_q]}+\frac{9\alpha q^2}{[3]_q}+\frac{4nq^3}{[4]_q[n]_q}(2q+1)[1-2\mu]_q\frac{e_{\mu,q}(q[n]_qx)}{e_{\mu,q}([n]_qx)}\Big)x^2+\frac{4n^3q^3}{(n+\beta)^3[4]_q}x^3\leq T^*_{n,q}(t^3;x)\leq\frac{n}{(n+\beta)^3[n]_q}\Big(\frac{n^2}{[4]_q[n]_q^2}+\frac{3n\alpha}{[3]_q[n]_q}+\frac{3\alpha^2}{[2]_q}\Big)+\frac{\alpha^3}{(n+\beta)^3}+\frac{1}{(n+\beta)^3}\Big(\frac{4n^3}{[4]_q[n]_q^2}(1+[1+2\mu]_q^2)+\frac{9n^2\alpha}{[3]_q[n]_q}(1+[1+2\mu]_q)+6n\big(\frac{\alpha^2}{[2]_q}+\frac{n^2[1+2\mu]_q}{[4]_q[n]_q}\big)\Big)x+\frac{1}{(n+\beta)^3}\Big(\frac{6n^3}{[4]_q[n]_q}(1+2[1+2\mu]_q)+\frac{9n^2\alpha}{[3]_q}\Big)x^2+\frac{4n^3}{(n+\beta)^3[4]_q}x^3$,

5. $T^*_{n,q}(t^4;x)\leq\frac{n}{(n+\beta)^4[n]_q}\Big(\frac{n^3}{[5]_q[n]_q^3}+\frac{4n^2\alpha}{[4]_q[n]_q^2}+\frac{6n\alpha^2}{[3]_q[n]_q}+\frac{4\alpha^3}{[2]_q}\Big)+\frac{\alpha^4}{(n+\beta)^4}+\Big(\frac{n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^3}+\frac{16n\alpha}{[4]_q[n]_q^2}+\frac{18\alpha^2}{[3]_q[n]_q}+\frac{8n\alpha}{[2]_q}\big)+\frac{2n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^2}+\frac{12n\alpha}{[4]_q[n]_q}+\frac{9\alpha^2}{[3]_q}\big)\frac{[1+2\mu]_q}{[n]_q}+\frac{2n^3}{(n+\beta)^4}\big(\frac{5n}{[5]_q[n]_q}+\frac{8\alpha}{[n]_q}\big)\frac{[1+2\mu]_q^2}{[n]_q^2}+\frac{5n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^3}{[n]_q^3}\Big)x+\Big(\frac{2n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^2}+\frac{12n\alpha}{[4]_q[n]_q}+\frac{9\alpha^2}{[3]_q}\big)+\frac{6n^3}{(n+\beta)^4}\big(\frac{5n}{[5]_q[n]_q}+\frac{8\alpha}{[n]_q}\big)\frac{[1+2\mu]_q}{[n]_q}+\frac{35n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^2}{[n]_q^2}\Big)x^2+\Big(\frac{2n^3}{(n+\beta)^4}\big(\frac{5n}{[5]_q[n]_q}+\frac{8\alpha}{[n]_q}\big)+\frac{5n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q}{[n]_q}\Big)x^3+\frac{5n^4}{(n+\beta)^4[5]_q}x^4$.

Proof. It is easily seen that

$$[k+1+2\mu\theta_k]_q = q[k+2\mu\theta_k]_q + 1. \tag{13}$$

So we get the followings

$$\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} 1 d_q t = \frac{1}{[n]_q}, \tag{14}$$

$$\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} t d_q t = \frac{1}{[2]_q[n]_q^2}\left(1+2q[k+2\mu\theta_k]_q\right), \tag{15}$$

$$\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} t^2 d_q t = \frac{1}{[3]_q[n]_q^3}\left(1+3q[k+2\mu\theta_k]_q+3q^2[k+2\mu\theta_k]_q^2\right), \tag{16}$$

$$\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} t^3 d_q t = \frac{1}{[4]_q[n]_q^4}\left(1+4q[k+2\mu\theta_k]_q+6q^2[k+2\mu\theta_k]_q^2+4q^3[k+2\mu\theta_k]_q^3\right), \tag{17}$$

and

$$\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} t^4 d_q t = \frac{1}{[5]_q[n]_q^5}\left(1+5q[k+2\mu\theta_k]_q+10q^2[k+2\mu\theta_k]_q^2+10q^3[k+2\mu\theta_k]_q^3+5q^4[k+2\mu\theta_k]_q^4\right) \tag{18}$$

From the Lemma 1.1 we have the following results:

$$\frac{1}{[n]_q}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q = x, \tag{19}$$

$$x^2+q^{2\mu}[1-2\mu]_q\frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q x)}\frac{x}{[n]_q} \leq \frac{1}{[n]_q^2}\frac{1}{e_{\mu,q}([n]_q x)}\sum_{k=0}^{\infty}\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^2 \leq x^2+[1+2\mu]_q\frac{x}{[n]_q}, \tag{20}$$

$$\frac{1}{[n]_q^3}\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^3 \leq x^3+3[1+2\mu]_q\frac{x^2}{[n]_q}+[1+2\mu]_q^2\frac{x}{[n]_q^2}, \tag{21}$$

$$\frac{1}{[n]_q^3}\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^3$$

$$\geq x^3+(2q+1)[1-2\mu]_q\frac{e_{\mu,q}(q[n]_qx)}{e_{\mu,q}([n]_qx)}\frac{x^2}{[n]_q}+q^{4\mu}[1-2\mu]_q^2\frac{e_{\mu,q}(q^2[n]_qx)}{e_{\mu,q}([n]_qx)}\frac{x}{[n]_q^2}, \tag{22}$$

and

$$\frac{1}{[n]_q^4}\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^4 \leq x^4+[1+2\mu]_q\frac{x^3}{[n]_q}+7[1+2\mu]_q^2\frac{x^2}{[n]_q^2}+[1+2\mu]_q^3\frac{x}{[n]_q^3}. \tag{23}$$

In view of (12) and (14), (1) is easily proved.

If $f(t)=t$, then (12), (15) and (19) imply that

$$\begin{aligned} T_{n,q}^*(t;x) &= \frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}}\left(\frac{nt+\alpha}{n+\beta}\right)d_qt \\ &= \frac{n}{n+\beta}\frac{1}{[2]_q[n_q]}\frac{1}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}(1+2q[k+2\mu\theta_k]_q)+\frac{\alpha}{n+\beta} \\ &= \frac{2qn}{(n+\beta)[2]_q}x+\frac{n}{(n+\beta)[2]_q[n]_q}+\frac{\alpha}{n+\beta}, \end{aligned}$$

which proves (2).

If $f(t) = t^2$, then (12), (16), (19) and (20) imply that

$$\begin{aligned}
T^*_{n,q}(t^2;x) &= \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \left(\frac{nt+\alpha}{n+\beta}\right)^2 d_q t \\
&= \frac{1}{(n+\beta)^2} \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \{ \frac{n^2}{[3]_q [n]_q^3} (1 + 3q[k+2\mu\theta_k]_q + 3q^2[k+2\mu\theta_k]_q^2) \\
&+ \frac{2n\alpha}{[2]_q [n]_q^2} (1 + 2q[k+2\mu\theta_k]_q) \} + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{n^2}{(n+\beta)^2} \frac{1}{[3]_q [n]_q^2 e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} (1 + 3q[k+2\mu\theta_k]_q + 3q^2[k+2\mu\theta_k]_q^2) \\
&+ \frac{2n\alpha}{(n+\beta)^2} \frac{1}{[3]_q [n]_q^2 e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} (1 + 2q[k+2\mu\theta_k]_q) + \frac{\alpha^2}{(n+\beta)^2} \\
&= \frac{n^2}{(n+\beta)^2} \frac{1}{[3]_q [n]_q^2} + \frac{2n\alpha}{(n+\beta)^2 [2]_q [n]_q} + \frac{\alpha^2}{(n+\beta)^2} + (\frac{n^2}{(n+\beta)^2} \frac{3q}{[3]_q [n]_q} \\
&+ \frac{2n\alpha}{(n+\beta)^2} \frac{2q}{[2]_q}) x + \frac{n^2}{(n+\beta)^2} \frac{3q^2}{[3]_q} \frac{1}{[n]_q^2 e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} [k+2\mu\theta_k]_q^2,
\end{aligned}$$

$$\begin{aligned}
T^*_{n,q}(t^2;x) &\geq \frac{n^2}{(n+\beta)^2} \frac{1}{[3]_q [n]_q^2} + \frac{2n\alpha}{(n+\beta)^2 [2]_q [n]_q} + \frac{\alpha^2}{(n+\beta)^2} + \Big(\frac{n^2}{(n+\beta)^2} \frac{3q}{[3]_q [n]_q} \\
&+ \frac{2n\alpha}{(n+\beta)^2} \frac{2q}{[2]_q} \Big) x + \frac{n^2}{(n+\beta)^2} \frac{3q^2}{[3]_q} \Big(x^2 + q^{2\mu} [1-2\mu]_q \frac{e_{\mu,q}(q[n]_q x)}{e_{\mu,q}([n]_q)} \frac{x}{[n]_q} \Big),
\end{aligned}$$

$$\begin{aligned}
T^*_{n,q}(t^2;x) &\leq \frac{n^2}{(n+\beta)^2} \frac{1}{[3]_q [n]_q^2} + \frac{2n\alpha}{(n+\beta)^2 [2]_q [n]_q} + \frac{\alpha^2}{(n+\beta)^2} + \Big(\frac{n^2}{(n+\beta)^2} \frac{3}{[3]_q [n]_q} \\
&+ \frac{2n\alpha}{(n+\beta)^2} \frac{2}{[2]_q} \Big) x + \frac{n^2}{(n+\beta)^2} \frac{3}{[3]_q} \Big(x^2 + [1+2\mu]_q \frac{x}{[n]_q} \Big),
\end{aligned}$$

which proves (3).

If $f(t)=t^3$, then (12), (17), (19), (20), (21) and (22) imply that

$$\begin{aligned}
T^*_{n,q}(t^3;x) &= \frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}}\left(\frac{nt+\alpha}{n+\beta}\right)^3 d_qt\\
&= \frac{n^3}{(n+\beta)^3}\frac{1}{[4]_q[n]_q^3e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}(1+4q[k+2\mu\theta_k]_q+6q^2[k+2\mu\theta_k]_q^2\\
&+ 4q^3[k+2\mu\theta_k]_q^3)+\frac{3n^2\alpha}{(n+\beta)^3}\frac{1}{[3]_q[n]_q^2e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}(1+3q[k+2\mu\theta_k]_q\\
&+ 3q^2[k+2\mu\theta_k]_q^2)+\frac{3n\alpha^2}{(n+\beta)^3}\frac{1}{[2]_q[n]_qe_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}(1+2q[k+2\mu\theta_k]_q)\\
&+ \frac{\alpha^3}{(n+\beta)^3},
\end{aligned}$$

$$\begin{aligned}
T^*_{n,q}(t^3;x) &\geq \frac{n}{(n+\beta)^3[n]_q}\Big(\frac{n^2}{[4]_q[n]_q^2}+\frac{3n\alpha}{[3]_q[n]_q}+\frac{3\alpha^2}{[2]_q}\Big)+\frac{\alpha^3}{(n+\beta)^3}+\frac{1}{(n+\beta)^3}\Big(n(\frac{4n^2q}{[4]_q[n]_q^2}\\
&+ \frac{9n\alpha}{[n]_q}\frac{q}{[3]_q}+6\alpha^2q\frac{q}{[2]_q})+(\frac{2n}{[4]_q[n]_q}+\frac{3\alpha}{[3]_q})\frac{3n^2q^{2(1+\mu)}}{[n]_q}[1-2\mu]_q)\frac{e_{\mu,q}(q[n]_qx)}{e_{\mu,q}([n]_qx)}\\
&+ \frac{4n^3q^{4\mu+3}}{[4]_q[n]_q^2}[1-2\mu]_q^2\frac{e_{\mu,q}(q^2[n]_qx)}{e_{\mu,q}([n]_qx)}\Big)x+\frac{n^2}{(n+\beta)^3}\Big(\frac{6nq^2}{[4]_q[n]_q}+\frac{9\alpha q^2}{[3]_q}+\frac{4nq^3}{[4]_q[n]_q}\\
&\times (2q+1)[1-2\mu]_q)\frac{e_{\mu,q}(q[n]_qx)}{e_{\mu,q}([n]_qx)}\Big)x^2+\frac{4n^3q^3}{(n+\beta)^3[4]_q}x^3,
\end{aligned}$$

$$\begin{aligned}
T^*_{n,q}(t^3;x) &\leq \frac{n}{(n+\beta)^3[n]_q}\Big(\frac{n^2}{[4]_q[n]_q^2}+\frac{3n\alpha}{[3]_q[n]_q}+\frac{3\alpha^2}{[2]_q}\Big)+\frac{\alpha^3}{(n+\beta)^3}+\frac{n}{(n+\beta)^3}\Big(\frac{4n^2}{[4]_q[n]_q^2}(1\\
&+ [1+2\mu]_q^2)+\frac{9n\alpha}{[3]_q[n]_q}(1+[1+2\mu]_q)+6(\frac{\alpha^2}{[2]_q}+\frac{n^2[1+2\mu]_q}{[4]_q[n]_q})\Big)x\\
&+ \frac{n^2}{(n+\beta)^3}\Big(\frac{6n}{[4]_q[n]_q}(1+2[1+2\mu]_q)+\frac{9\alpha}{[3]_q}\Big)x^2+\frac{4n^3}{(n+\beta)^3[4]_q}x^3,
\end{aligned}$$

which proves (4).

If $f(t)=t^4$, then (12), (18), (19), (20), (21) and (23) imply that

$$\begin{aligned}
T^*_{n,q}(t^4;x) &= \frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}}\left(\frac{nt+\alpha}{n+\beta}\right)^4 d_qt\\
&= \frac{1}{(n+\beta)^4e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}((\frac{n^4}{[5]_q[n]_q^4}+\frac{4n^3\alpha}{[4]_q[n]_q^3}+\frac{6n^3\alpha^2}{[3]_q[n]_q^2}+\frac{4n\alpha^3}{[2]_q[n]_q})\\
&+ (\frac{5n^4q}{[5]_q[n]_q^4}+\frac{16n^3\alpha}{[4]_q[n]_q^3}+\frac{18n^2\alpha^2q}{[3]_q[n]_q^2}+\frac{8n^3\alpha q}{[2]_q[n]_q})[k+2\mu\theta_k]_q+(\frac{10n^4q^2}{[5]_q[n]_q^4}+\frac{24n^3\alpha q^2}{[4]_q[n]_q^3}\\
&+ \frac{18n^2\alpha^2q^2}{[3]_q[n]_q^2})[k+2\mu\theta_k]_q^2+(\frac{10n^4q^3}{[5]_q[n]_q^4}+\frac{16n^3\alpha q^3}{[4]_q[n]_q^3})[k+2\mu\theta_k]_q^3+\frac{5n^4q^4}{[5]_q[n]_q^4}[k+2\mu\theta_k]_q^4)\\
&+ \frac{\alpha^4}{(n+\beta)^4}\\
&= \frac{n}{(n+\beta^4)[n]_q}(\frac{n^3}{[5]_q[n]_q^3}+\frac{4n^2\alpha}{[4]_q[n]_q^2}+\frac{6n\alpha^2}{[3]_q[n]_q}+\frac{4\alpha^3}{[2]_q})+\frac{n^2q}{(n+\beta)^4}(\frac{5n^2\alpha}{[5]_q[n]_q^3}+\frac{16n\alpha q}{[4]_q[n]_q^2}\\
&+ \frac{18\alpha^2q}{[3]_q[n]_q}+\frac{8n\alpha}{[2]_q})\frac{1}{[n]_qe_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q+\frac{2n^2q^2}{(n+\beta)^4}(\frac{5n^2}{[5]_q[n]_q^2}\\
&+ \frac{12n\alpha}{[4]_q[n]_q}+\frac{9\alpha^2q^2}{[3]_q})\frac{1}{[n]_q^2e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^2+\frac{2n^3q^3}{(n+\beta)^4}(\frac{5n\alpha}{[5]_q[n]_q}+\frac{8\alpha}{[4]_q})\\
&\times \frac{1}{[n]_q^3e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^3\\
&+ \frac{1}{[n]_q^4e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}[k+2\mu\theta_k]_q^4\frac{5n^4q^4}{(n+\beta)^4[5]_q}+\frac{\alpha^4}{(n+\beta)^4}\\
&\leq \frac{n}{(n+\beta)^4[n]_q}(\frac{n^3}{[5]_q[n]_q^3}+\frac{4n^2\alpha}{[4]_q[n]_q^2}+\frac{6n\alpha^2}{[3]_q[n]_q}+\frac{4\alpha^3}{[2]_q})+\frac{\alpha^4}{(n+\beta)^4}\\
&+ \frac{n^2}{(n+\beta)^4}(\frac{5n^2}{[5]_q[n]_q^3}+\frac{16n\alpha}{[4]_q[n]_q^2}+18\alpha^2[3]_q[n]_q+\frac{8\alpha^2}{[3]_q[n]_q}+\frac{8n\alpha}{[2]_q})x\\
&+ \frac{2n^2}{(n+\beta)^4}(\frac{5n^2}{[5]_q[n]_q^2}+\frac{12n\alpha}{[4]_q[n]_q}+\frac{9\alpha^2}{[3]_q})(x^2+[1+2\mu]_q\frac{x}{[n]_q})\\
&+ \frac{2n^3}{(n+\beta)^4}(\frac{5n}{[5]_q[n]_q}+\frac{8\alpha}{[n]_q})(x^3+3[1+2\mu]_q\frac{x^2}{[n]_q}+[1+2\mu]_q^2\frac{x}{[n]_q^2})\\
&+ \frac{5n^4}{(n+\beta)^4[5]_q}(x^4+[1+2\mu]_q\frac{x^3}{[n]_q}+7[1+2\mu]_q^2\frac{x^2}{[n]_q^2}+[1+2\mu]_q^3\frac{x}{[n]_q^3})\\
&= \frac{n}{(n+\beta)^4[n]_q}\Big(\frac{n^3}{[5]_q[n]_q^3}+\frac{4n^2\alpha}{[4]_q[n]_q^2}+\frac{6n\alpha^2}{[3]_q[n]_q}+\frac{4\alpha^3}{[2]_q}\Big)+\frac{\alpha^4}{(n+\beta)^4}\\
&+ \Big(\frac{n^2}{(n+\beta)^4}(\frac{5n^2}{[5]_q[n]_q^3}+\frac{16n\alpha}{[4]_q[n]_q^2}+\frac{18\alpha^2}{[3]_q[n]_q}+\frac{8n\alpha}{[2]_q})
\end{aligned}$$

$$+ \quad \frac{2n^2}{(n+\beta)^4}\Big(\frac{5n^2}{[5]_q[n]_q^2} + \frac{12n\alpha}{[4]_q[n]_q} + \frac{9\alpha^2}{[3]_q}\Big)\frac{[1+2\mu]_q}{[n]_q} + \frac{2n^3}{(n+\beta)^4}\Big(\frac{5n}{[5]_q[n]_q} + \frac{8\alpha}{[n]_q}\Big)\frac{[1+2\mu]_q^2}{[n]_q^2}$$
$$+ \quad \frac{5n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^3}{[n]_q^3}\Big)x + \Big(\frac{2n^2}{(n+\beta)^4}\Big(\frac{5n^2}{[5]_q[n]_q^2} + \frac{12n\alpha}{[4]_q[n]_q} + \frac{9\alpha^2}{[3]_q}\Big) + \frac{6n^3}{(n+\beta)^4}\Big(\frac{5n}{[5]_q[n]_q}$$
$$+ \quad \frac{8\alpha}{[n]_q}\Big)\frac{[1+2\mu]_q}{[n]_q} + \frac{35n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^2}{[n]_q^2}\Big)x^2 + \Big(\frac{2n^3}{(n+\beta)^4}\Big(\frac{5n}{[5]_q[n]_q} + \frac{8\alpha}{[n]_q}\Big) + \frac{5n^4}{(n+\beta)^4[5]_q}$$
$$\times \quad \frac{[1+2\mu]_q}{[n]_q}\Big)x^3 + \frac{5n^4}{(n+\beta)^4[5]_q}x^4,$$

which proves (5). $\square$

Lemma 2.2. *Let the operators $T^*_{n,q}(.\ ;\ .)$ be given by (12). Then*

1. $T^*_{n,q}(t-x;x) = \Big(\frac{2qn}{(n+\beta)[2]_q} - 1\Big)x + \frac{n}{(n+\beta)[2]_q[n]_q} + \frac{\alpha}{(n+\beta)}$,

2. $T^*_{n,q}((t-x)^2;x) \le \frac{n}{(n+\beta)^2[n]_q}\Big(\frac{n}{[3]_q[n]_q} + \frac{2\alpha}{[2]_q}\Big) + \frac{\alpha^2}{(n+\beta)^2} + \Big(\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q}(1+[1+2\mu]_q) + \frac{2n}{(n+\beta)[2]_q}\big(\frac{2\alpha}{(n+\beta)} - \frac{1}{[n]_q}\big) - \frac{2\alpha}{(n+\beta)}\Big)x + \Big(\frac{n}{(n+\beta)}\big(\frac{3n}{(n+\beta)[3]_q} - \frac{4n}{n+\beta)[2]_q}\big) + 1\Big)x^2$,

3. $T^*_{n,q}((t-x)^4;x) \le \frac{n}{(n+\beta)^4[n]_q}\Big(\frac{n^3}{[5]_q[n]_q^3} + \frac{4n^2\alpha}{[4]_q[n]_q^2} + \frac{6n\alpha^2}{[3]_q[n]_q} + \frac{4\alpha^3}{[2]_q}\Big) + \frac{\alpha^4}{(n+\beta)^4}$ $\Big(\frac{n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^3} + \frac{16n\alpha}{[4]_q[n]_q^2} + \frac{18\alpha^2}{[3]_q[n]_q} + \frac{8n\alpha}{[2]_q}\big) + \frac{2n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^2} + \frac{12n\alpha}{[4]_q[n]_q} + \frac{9\alpha^2}{[3]_q}\big)$ $\times\frac{[1+2\mu]_q}{[n]_q} + \frac{2n^3}{(n+\beta)^4}\big(\frac{5n}{[5]_q[n]_q} + \frac{8\alpha}{[n]_q}\big)\frac{[1+2\mu]_q^2}{[n]_q^2} + \frac{5n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^3}{[n]_q^3} - \frac{4n}{(n+\beta)^3[3]_q}\big(\frac{n^2}{[4]_q[n]_q^2} + \frac{3n\alpha}{[3]_q[n]_q} + \frac{3\alpha^2}{[2]_q}\big) - \frac{4\alpha^3}{(n+\beta)^3}\Big)x + \Big(\frac{2n^2}{(n+\beta)^4}\big(\frac{5n^2}{[5]_q[n]_q^2} + \frac{12n\alpha}{[4]_q[n]_q} + \frac{9\alpha^2}{[3]_q}\big) + \frac{6n^3}{(n+\beta)^4}$ $\big(\frac{5n}{[5]_q[n]_q} + \frac{8\alpha}{[n]_q}\big)\frac{[1+2\mu]_q}{[n]_q} + \frac{35n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q^2}{[n]_q^2} - \frac{4}{(n+\beta)^3}\big(\frac{4n^3}{[4]_q[n]_q^2}(1+[1+2\mu]_q^2)$ $+ \frac{9n^2\alpha}{[3]_q[n]_q}(1+[1+2\mu]_q) + 6n\big(\frac{\alpha^2}{[2]_q} + \frac{n^2}{[4]_q}\frac{[1+2\mu]_q^2}{[n]_q^2}\big)\big) + 6\big(\frac{n^2}{(n+\beta)^2}\frac{1}{[3]_q[n]_q^2}$ $+ \frac{2n\alpha}{(n+\beta)^2[2]_q[n]_q} + \frac{\alpha^2}{(n+\beta)^2}\big)\Big)x^2 + \Big(\frac{2n^3}{(n+\beta)^4}\big(\frac{5n}{[5]_q[n]_q} + \frac{8\alpha}{[n]_q}\big) + \frac{5n^4}{(n+\beta)^4[5]_q}\frac{[1+2\mu]_q}{[n]_q}$ $- \frac{4}{(n+\beta)^3}\big(\frac{6n^3}{[4]_q[n]_q}(1+2[1+2\mu]_q) + \frac{9n^2\alpha}{[3]_q}\big) + 6\big(\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q} + \frac{2n\alpha}{(n+\beta)^2}\frac{2}{[2]_q}\big)$ $+ \frac{6n^2}{(n+\beta)^2}\frac{3}{[3]_q}\frac{[1+2\mu]_q}{[n]_q} - 4\big(\frac{n}{(n+\beta)[2]_q[n]_q} + \frac{\alpha}{n+\beta}\big)\Big)x^3 + \Big(\frac{5n^4}{(n+\beta)^4[5]_q} - \frac{16n^3}{(n+\beta)^3[4]_q}$ $\frac{6n^2}{(n+\beta)^2}\frac{3}{[3]_q} - \frac{8n}{(n+\beta)[2]_q} + 1\Big)x^4.$

3 Main Results

We obtain the Korovkin's type approximation properties for our operators defined by (12).

Let $C_B(\mathbb{R}^+)$ be the space of all bounded and continuous functions on $\mathbb{R}^+ = [0, \infty)$, endowed with the norm

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let

$$H := \{f : x \in [0, \infty), \frac{f(x)}{1+x^2} \quad \text{is} \quad \text{convergent} \quad \text{as} \quad x \to \infty\}.$$

In order to obtain the convergence results for the operators $T^*_{n,q}(.\,;\,.)$, we take $q = q_n$ where $q_n \in (0, 1)$, such that

$$\lim_n q_n \to 1, \qquad \lim_n q_n^n \to a \tag{24}$$

Theorem 3.1. *Let $q = q_n$ satisfies (24), for $0 < q_n < 1$ and if $T^*_{n,q_n}(.\,;\,.)$ be the operators given by (12). Then for any function $f \in C[0, \infty) \cap H$,*

$$\lim_{n\to\infty} T^*_{n,q_n}(f; x) = f(x)$$

uniformly on each compact subset of $[0, \infty)$.

Proof. The proof is based on the well known Korovkin's theorem regarding the convergence of a sequence of linear and positive operators, so it is enough to prove the conditions

$$\lim_{n\to\infty} T^*_{n,q_n}((t^j; x) = x^j, \quad j = 0, 1, 2, \quad \{\text{as } n \to \infty\}$$

uniformly on $[0, 1]$.
Clearly from (24) and $\frac{1}{[n]_{q_n}} \to 0 \;\; (n \to \infty)$ we have

$$\lim_{n\to\infty} T^*_{n,q_n}(t; x) = x, \qquad \lim_{n\to\infty} T^*_{n,q_n}(t^2; x) = x^2,$$

which completes the proof. $\square$

We recall the weighted spaces of the functions on $\mathbb{R}^+$, which are defined as follows:

$$\begin{aligned} P_\rho(\mathbb{R}^+) &= \{f :\mid f(x) \mid\leq M_f\rho(x)\}, \\ Q_\rho(\mathbb{R}^+) &= \left\{f : f \in P_\rho(\mathbb{R}^+) \cap C[0,\infty)\right\}, \\ Q_\rho^k(\mathbb{R}^+) &= \left\{f : f \in Q_\rho(\mathbb{R}^+) \quad \text{and} \quad \lim_{x\to\infty} \frac{f(x)}{\rho(x)} = k(k \quad \text{is} \quad \text{a} \quad \text{constant})\right\}, \end{aligned}$$

where

$$\rho(x) = 1 + x^2$$

is a weight function and M_f is a constant depending only on f. Note that $Q_\rho(\mathbb{R}^+)$ is a normed space with the norm

$$\| f \|_\rho = \sup_{x\geq 0} \frac{\mid f(x) \mid}{\rho(x)}.$$

Theorem 3.2. *Let $q = q_n$ satisfying (24), for $0 < q_n < 1$ and if $T^*_{n,q_n}(.\ ;\ .)$ be the operators given by (12). Then for any function $f \in Q_\rho^k(\mathbb{R}^+)$ we have*

$$\lim_{n\to\infty} \| T^*_{n,q_n}(f;x) - f \|_\rho = 0.$$

Proof. From the Lemma 2.1, the first condition of (1) is fulfilled for $\tau = 0$. Now for $\tau = 1, 2$ it is easy to see that from (2), (3) of the Lemma 2.1 by using (24)

$$\| T^*_{n,q_n}(t)^\tau; x) - x^\tau \|_\rho = 0.$$

This completes the proof. $\square$

4 Rate of Convergence

In what follows we calculate the rate of convergence of the operators (12) by means of modulus of continuity and Lipschitz type maximal functions.

Let $f \in C[0,\infty]$. The modulus of continuity of f, denoted by $\omega(f,\delta)$, gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and is given by

$$\omega(f,\delta) = \sup_{|y-x|\leq\delta} \mid f(y) - f(x) \mid, \quad x, y \in [0,\infty). \tag{25}$$

It is known that $\lim_{\delta\to 0+}\omega(f,\delta)=0$ for $f\in C[0,\infty)$ and for any $\delta>0$ one has

$$| f(y)-f(x) |\leq \left(\frac{| y-x |}{\delta}+1\right)\omega(f,\delta). \tag{26}$$

Theorem 4.1. *Let $T^*_{n,q}(.\ ;\ .)$ be the operators defined by (12). Then for $f\in\tilde{C}[0,\infty),\quad x\geq 0,\quad 0<q<1$ we have*

$$|T^*_{n,q}(f;x)-f(x)|\leq \Big(1+\sqrt{\phi_n(x)}\Big)\omega\Big(f;\frac{1}{\sqrt{[n]_q}}\Big),$$

where

$$\begin{aligned}\phi_n(x) &= \frac{n}{(n+\beta)^2[n]_q}\Big(\frac{n}{[3]_q[n]_q}+\frac{2\alpha}{[2]_q}\Big)+\frac{\alpha^2}{(n+\beta)^2}+\Big(\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q}(1+[1+2\mu]_q)\\ &+ \frac{2n}{(n+\beta)[2]_q}(\frac{2\alpha}{(n+\beta)}-\frac{1}{[n]_q})-\frac{2\alpha}{(n+\beta)}\Big)x+\Big(\frac{n}{(n+\beta)}(\frac{3n}{(n+\beta)[3]_q}-\frac{4n}{(n+\beta)[2]_q})+1\Big)x^2,\end{aligned}$$

and $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $\mathbb{R}^+$ and $\omega(f,\delta)$ is the modulus of continuity of the function $f\in\tilde{C}[0,\infty)$ defined in (25).

Proof. Making use of (25), (26) and the Cauchy-Schwarz inequality,we obtain

$| T^*_{n,q}(f;x)-f(x) |$

$$\begin{aligned}
&\leq \frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} | f(t)-f(x) | d_q(t)\\
&\leq \frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \left(1+\frac{1}{\delta}| t-x |\right)d_q(t)\omega(f;\delta)\\
&= \left\{1+\frac{1}{\delta}\left(\frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} | t-x | d_q(t)\right)\right\}\omega(f;\delta)\\
&\leq \left\{1+\frac{1}{\delta}\left(\frac{[n]_q}{e_{\mu,q}([n]_qx)}\sum_{k=0}^{\infty}\frac{([n]_qx)^k}{\gamma_{\mu,q}(k)}\int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} (t-x)^2d_q(t)\right)^{\frac{1}{2}}\left(T^*_{n,q}(1;x)\right)^{\frac{1}{2}}\right\}\omega(f;\delta)\\
&= \left\{1+\frac{1}{\delta}\left(T^*_{n,q}(t-x)^2;x\right)^{\frac{1}{2}}\right\}\omega(f;\delta),
\end{aligned}$$

if we choose $\delta = \delta_n = \sqrt{\frac{1}{[n]_q}}$, then we get the result. $\square$

Now we give the rate of convergence of the operators $T^*_{n,q}(f;x)$ defined in (12) in terms of the elements of the usual Lipschitz class $Lip_M(\nu)$. Let $f \in C[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f :\mid f(\zeta_1) - f(\zeta_2) \mid \leq M \mid \zeta_1 - \zeta_2 \mid^{\nu} \quad (\zeta_1, \zeta_2 \in [0,\infty))\} \tag{27}$$

Theorem 4.2. *Let $T^*_{n,q}(.\ ;\ .)$ be the operators defined by (12). Then for each $f \in Lip_M(\nu)$, satisfying (27), we have*

$$\mid T^*_{n,q}(f;x) - f(x) \mid \leq M\left(\lambda_n(x)\right)^{\frac{\nu}{2}},$$

where

$$\lambda_n(x) = T^*_{n,q}\left((t-x)^2;x\right).$$

Proof. We prove it by using (27) and the Hölder's inequality.

$$\begin{aligned} \mid T^*_{n,q}(f;x) - f(x) \mid &\leq \mid T^*_{n,q}(f(t) - f(x);x) \mid \\ &\leq T^*_{n,q}\left(\mid f(t) - f(x) \mid; x\right) \\ &\leq MT^*_{n,q}\left(\mid t - x \mid^{\nu}; x\right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\mid T^*_{n,q}(f;x) - f(x) \mid \\
&\leq M \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid t-x \mid^{\nu} d_q(t) \\
&\leq M \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \left(\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \right)^{\frac{\nu}{2}} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid t-x \mid^{\nu} d_q(t) \\
&\leq M \left(\frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} d_q(t) \right)^{\frac{2-\nu}{2}} \\
&\times \left(\frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid t-x \mid^{2} d_q(t) \right)^{\frac{\nu}{2}} \\
&= M \left(T^*_{n,q}(t-x)^2; x \right)^{\frac{\nu}{2}},
\end{aligned}
$$

which completes the proof. □

Let $C_B[0,\infty)$ denote the space of all bounded and continuous functions on $\mathbb{R}^+ = [0,\infty)$ and

$$C_B^2(\mathbb{R}^+) = \{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\} \tag{28}$$

with the norm

$$\parallel g \parallel_{C_B^2(\mathbb{R}^+)} = \parallel g \parallel_{C_B(\mathbb{R}^+)} + \parallel g' \parallel_{C_B(\mathbb{R}^+)} + \parallel g'' \parallel_{C_B(\mathbb{R}^+)} . \tag{29}$$

Also,

$$\parallel g \parallel_{C_B(\mathbb{R}^+)} = \sup_{x \in \mathbb{R}^+} \mid g(x) \mid . \tag{30}$$

Theorem 4.3. *Let $T^*_{n,q}(. \,; \,.)$ be the operator defined by (12). Then for any $g \in C_B^2(\mathbb{R}^+)$, we have*

$$\mid T^*_{n,q}(f;x) - f(x) \mid \leq \left(\left(\frac{2qn}{(n+\beta)[2]_q} - 1 \right) x + \frac{n}{(n+\beta)[2]_q[n]_q} + \frac{\alpha}{(n+\beta)} + \frac{\lambda_n(x)}{2} \right) \parallel g \parallel_{C_B^2(\mathbb{R}^+)},$$

where $\lambda_n(x)$ is given in Theorem 4.2.

Proof. Let $g \in C_B^2(\mathbb{R}^+)$, then by using the generalized mean value theorem in the Taylor series expansion we have

$$g(t) = g(x) + g'(x)(t-x) + g''(\psi)\frac{(t-x)^2}{2}, \quad \psi \in (x,t).$$

Operating by $T_{n,q}^*$ on both sides, we have

$$T_{n,q}^*(g,x) - g(x) = g'(x)T_{n,q}^*\left((t-x);x\right) + \frac{g''(\psi)}{2}T_{n,q}^*\left((t-x)^2;x\right),$$

which implies that
$\mid T_{n,q}^*(g;x) - g(x) \mid$

$$\begin{aligned}
&\leq \left(\left(\frac{2qn}{(n+\beta)[2]_q} - 1\right)x + \frac{n}{(n+\beta)[2]_q[n]_q} + \frac{\alpha}{(n+\beta)}\right) \parallel g' \parallel_{C_B(\mathbb{R}^+)} \\
&+ \left(\frac{n}{(n+\beta)^2[n]_q}\left(\frac{n}{[3]_q[n]_q} + \frac{2\alpha}{[2]_q}\right) + \frac{\alpha^2}{(n+\beta)^2} + \left(\frac{n^2}{(n+\beta)^2}\frac{3}{[3]_q[n]_q} \times (1+[1+2\mu]_q)\right.\right. \\
&+ \left.\left.\frac{2n}{(n+\beta)[2]_q}\left(\frac{2\alpha}{(n+\beta)} - \frac{1}{[n]_q}\right) - \frac{2\alpha}{(n+\beta)}\right)x + \left(\frac{n}{n+\beta}\left(\frac{3n}{(n+\beta)[3]_q} - \frac{4n}{(n+\beta)[2]_q}\right) + 1\right)x^2\right) \\
&\times \frac{\parallel g'' \parallel_{C_B(\mathbb{R}^+)}}{2}.
\end{aligned}$$

On using (29), $\parallel g' \parallel_{C_B[0,\infty)} \leq \parallel g \parallel_{C_B^2[0,\infty)}$ completes the proof by (2) of the Lemma 2.2. □

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{C_B^2(\mathbb{R}^+)}\left\{\left(\parallel f-g \parallel_{C_B(\mathbb{R}^+)} + \delta \parallel g'' \parallel_{C_B^2(\mathbb{R}^+)}\right) : g \in \mathcal{W}^2\right\}, \tag{31}$$

where

$$\mathcal{W}^2 = \left\{g \in C_B(\mathbb{R}^+) : g', g'' \in C_B(\mathbb{R}^+)\right\}. \tag{32}$$

There exits a positive constant $C > 0$ such that $K_2(f,\delta) \leq C\omega_2(f,\delta^{\frac{1}{2}})$, $\delta > 0$, where the second order modulus of continuity is defined by

$$\omega_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}}\sup_{x\in\mathbb{R}^+} \mid f(x+2h) - 2f(x+h) + f(x) \mid. \tag{33}$$

Theorem 4.4. *Let $T^*_{n,q}(.\,;\,.)$ be the operator defined by (12) and $C_B[0,\infty)$ be the space of all bounded and continuous functions on $\mathbb{R}^+$. Then for $x\in\mathbb{R}^+$, $f\in C_B(\mathbb{R}^+)$, we have*

$$| T^*_{n,q}(f;x)-f(x) | \leq 2M\Bigg\{\omega_2\left(f;\sqrt{\frac{\left(\frac{4qn}{(n+\beta)[2]_q}-2\right)x+\left(\frac{2n}{(n+\beta)[2]_q[n]_q}+\frac{2\alpha}{n+\beta}\right)+\lambda_n(x)}{4}}\right) + \min\left(1,\frac{\left(\frac{4qn}{(n+\beta)[2]_q}-2\right)x+\left(\frac{2n}{(n+\beta)[2]_q[n]_q}+\frac{2\alpha}{n+\beta}\right)+\lambda_n(x)}{4}\right)\| f \|_{C_B(\mathbb{R}^+)}\Bigg\},$$

where M is a positive constant, $\lambda_n(x)$ is as in the Theorem 4.2 and $\omega_2(f;\delta)$ is the second order modulus of continuity of the function f defined in (33).

Proof. We prove this by using the Theorem (4.3)

$$\begin{aligned} | T^*_{n,q}(f;x)-f(x) | &\leq | T^*_{n,q}(f-g;x) | + | T^*_{n,q}(g;x)-g(x) | + | f(x)-g(x) | \\ &\leq 2\| f-g \|_{C_B(\mathbb{R}^+)} + \frac{\lambda_n(x)}{2}\| g \|_{C^2_B(\mathbb{R}^+)} \\ &+ \left(\left(\frac{2qn}{(n+\beta)[2]_q}-1\right)x+\frac{n}{(n+\beta)[2]_q[n]_q}+\frac{\alpha}{(n+\beta)}\right)\| g \|_{C_B(\mathbb{R}^+)}. \end{aligned}$$

From (29) clearly we have $\| g \|_{C_B[0,\infty)} \leq \| g \|_{C^2_B[0,\infty)}$.
Therefore,

$$| T^*_{n,q}(f;x)-f(x) | \leq 2\left(\| f-g \|_{C_B(\mathbb{R}^+)} + \frac{\left(\frac{4qn}{(n+\beta)[2]_q}-2\right)x+\frac{2n}{(n+\beta)[2]_q[n]_q}+\frac{2\alpha}{(n+\beta)}+\lambda_n(x)}{4}\| g \|_{C^2_B(\mathbb{R}^+)}\right)$$

where $\lambda_n(x)$ is given in Theorem 4.2.

By taking infimum over all $g\in C^2_B(\mathbb{R}^+)$ and by using (31), we get

$$| T^*_{n,q}(f;x)-f(x) | \leq 2K_2\left(f;\frac{\left(\frac{4qn}{(n+\beta)[2]_q}-2\right)x+\frac{2n}{(n+\beta)[2]_q[n]_q}+\frac{2\alpha}{n+\beta}+\lambda_n(x)}{4}\right).$$

For an absolute constant $D>0$ in [5] we use the relation

$$K_2(f;\delta)\leq D\{\omega_2(f;\sqrt{\delta})+\min(1,\delta)\| f \|\}.$$

This completes the proof. $\square$

Atakut and Ispir [1] introduced the weighted modulus of continuity of $f \in Q_\rho^k(\mathbb{R}^+)$ defined as

$$\Omega(f,\delta) = \sup_{x\in[0,\infty),|h|\leq\delta} \frac{| f(x+h) - f(x) |}{(1+h^2)(1+x^2)}. \tag{34}$$

The two main properties of this modulus of continuity are

$$\lim_{\delta\to 0} \Omega(f,\delta) \to 0$$

and

$$| f(t) - f(x) | \leq 2\left(1 + \frac{| t-x |}{\delta}\right)(1+\delta^2)(1+x^2)(1+(t-x)^2)\Omega(f,\delta), \tag{35}$$

where $f \in Q_\rho^k(\mathbb{R}^+)$ and $t, x \in [0,\infty)$.

Theorem 4.5. *Let $T_{n,q}^*(.\,;\,.)$ be the operators defined by (12). Then for $f \in Q_\rho^k(\mathbb{R}^+)$, $0<q<1$ and $x \geq 0$ we have*

$$\sup_{x\in[0,\infty)} \frac{| T_{n,q}^*(f;x) - f(x) |}{(1+x^2)} \leq C_\mu\Big(1 + \frac{1}{[n]_q}\Big)\Omega\Big(f, \frac{1}{\sqrt{[n]_q}}\Big),$$

where C_μ is constant independent of n.

Proof. We prove it by using (34), (35) and the Cauchy-Schwarz inequality.

$$
\begin{aligned}
&\mid T^*_{n,q}(f;x) - f(x) \mid \\
&\leq \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid f(t) - f(x) \mid d_q(t) \\
&\leq 2(1+\delta^2)(1+x^2)\Omega(f;\delta) \frac{[n]_q}{e_{\mu,q}([n]_q x)} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \left(1 + \frac{1}{\delta} \mid t-x \mid \right) \left(1+(t-x)^2\right) d_q(t) \\
&= 2(1+\delta^2)(1+x^2)\Omega(f;\delta) \frac{[n]_q}{e_{\mu,q}([n]_q x)} \\
&\times \left\{ \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} + \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} (t-x)^2 d_q(t) \right. \\
&+ \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid t-x \mid d_q(t) \\
&+ \left. \frac{1}{\delta} \sum_{k=0}^{\infty} \frac{([n]_q x)^k}{\gamma_{\mu,q}(k)} \int_{\frac{q[k+2\mu\theta_k]_q}{[n]_q}}^{\frac{[k+1+2\mu\theta_k]_q}{[n]_q}} \mid t-x \mid (t-x)^2 d_q(t) \right\} \\
&\leq 2(1+\delta^2)(1+x^2)\Omega(f;\delta) \\
&\times \left(1 + T^*_{n,q}((t-x)^2;x) + \frac{1}{\delta}\sqrt{T^*_{n,q}((t-x)^2;x)} + \frac{1}{\delta}\sqrt{T^*_{n,q}((t-x)^2;x) T^*_{n,q}((t-x)^4;x)}\right)
\end{aligned}
$$

where $T^*_{n,q}((t-x)^2;x)$ and $T^*_{n,q}((t-x)^4;x)$ are defined in (2) and (3) of the Lemma 2.2. If we choose $\delta = \delta_n = \sqrt{\frac{1}{[n]_q}}$, then we get the result. □

References

[1] Ç. Atakut and N. Ispir, Approximation by modified Szász-Mirakjan operators on weighted spaces, *Proc. Indian Acad. Sci. Math. Sci.* 112 (2002), 571-578.

[2] A. Aral, V. Gupta and R.P. Agarwal, *Applications of q-Calculus in Operator Theory*, Springer, New York (2013).

[3] S.N. Bernstein, Démonstration du théoréme de Weierstrass fondée sur le calcul des probabilités, *Commun. Soc. Math. Kharkow*, 2(13)(2012), 1-2.

[4] B. Cheikh, Y. Gaied and M. Zaghouani, A q-Dunkl-classical q-Hermite type polynomials, *Georgian Math. J.*, 21(2)(2014), 125-137.

[5] A. Ciupa, A class of integral Favard-Szász type operators, *Stud. Univ. Babeş-Bolyai, Math.*, 40(1) (1995), 39-47.

[6] S. Ersan, O. Doğru, Statistical approximation properties of q-Bleimann, Butzer and Hahn operators, *Math. Comput. Modell.*, 49 (1995), 1595-1606.

[7] V. Gupta, C. Radu, Statistical approximation properties of q-Baskokov-Kantorovich operators, *Cent. Eur. J. Math.*, 7(4) (2009), 809-818.

[8] G. İçōz, B. Çekim, Stancu type generalization of Dunkl analogue of Szász-Kantorovich operators, *Math. Meth. Appl. Sci.*, 39(7) (2016), 1803-1810.

[9] G. İçōz, B. Çekim, Dunkl generalization of Szász operators via q-calculus, *J. Inequal. Appl.*, 2015 (2015): 284.

[10] A. Lupaş, A q-analogue of the Bernstein operator, *In Seminar on Numerical and Statistical Calculus*, University of Cluj-Napoca, Cluj-Napoca, 9 (1987), 85-92.

[11] N.I. Mahmudov, Statistical approximation of Baskakov and Baskakov-Kantorovich operators based on the q-integers, *Cent. Eur. Jour. Math.*, 8(4) (2010), 816–826.

[12] N.I. Mahmudov and V. Gupta, On certain q-analogue of Szász Kantorovich operators, *Jour. Appl. Math. Comput.*, 37 (2011), 407-419.

[13] M. Mursaleen, A. Khan, Statistical approximation properties of modified q- Stancu-Beta operators, *Bull. Malays. Math. Sci. Soc.* (2)36(3) (2013), 683-690.

[14] M. Mursaleen, A. Khan, Generalized q-Bernstein-Schurer operators and some approximation theorems, *Jour. Function Spaces Appl.*, 2013 (2013), Article ID 719834, 1-7. Volume, 7 pages (2013).

[15] M. Mursaleen, K. J. Ansari and A. Khan, On (p, q)-analogue of Bernstein operators, *Appl. Math. Comput.*, 266 (2015, 874-882).

[16] M. Mursaleen, Md. Nasiruzzaman and Ashirbayev Nurgali, Some approximation resluts on Bernstein-Schurer operators defined by (p, q)-integers, *J. Inequal. Appl.* 2015 (2015):249.

[17] M. Mursaleen, K. J. Ansari and A. Khan, Some approximation resluts by (p, q)-analogue of Bernstein-Stancu operators, *Appl. Math. Comput.*, 264 (2015), 392-402.

[18] M. Örkcü and O. Doğru, Weighted statistical approximation by Kantorovich type q-Szász Mirakjan operators, *Appl. Math. Comput.*, 217 (2011), 7913-7919.

[19] G.M. Phillips, Bernstein polynomials based on the q- integers, *Ann. Numer. Math.*, 4 (1997), 511-518.

[20] C. Radu, Statistical approximation properties of Kantorovich operators based on q-integers, *Creat. Math. Inform.*, 17(2) (2008), 75-84.

[21] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, *Oper. Theory, Adv. Appl.*, 73 (1994), 369-396.

[22] H. M. Srivastava, M. Mursaleen, A. Alotaibi, Md. Nasiruzzaman and A.H. Al-Abied, Some approximation results involving the q-Szsz-Mirakjan-Kantorovich type operators via Dunkl's generalization, *Math. Meth. Appl. Sci.*, 40(15) (2017), 5437-5452.

[23] S. Sucu, Dunkl analogue of Szász operators, *Appl. Math. Comput.*, 244 (2014), 42-48.

[24] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Natl. Bur. Stand.*, 45 (1950), 239-245.

Mohammad Mursaleen
Professor of Mathematics
Department of Mathematics
Aligarh Muslim University
Aligarh, India
E-mail: mursaleenm@gmail.com

Taqseer Khan
Assistant Professor of Mathematics
Department of Mathematics
Jamia Millia Islamia
New Delhi, India
E-mail: taqi.khan91@gmail.com

Md Nasiruzzaman
Assistant Professor of Mathematics
Department of Computer Science (SEST)
Jamia Hamdard
New Delhi-110062, India
E-mail: nasir3489@gmail.com