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The Odd Log-Logistic Poisson-G Family of Distributions

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Abstract. We propose a new class of continuous distributions with two extra shape parameters named the Odd Log-Logistic Poisson-G family. Some of its mathematical properties including moments, quantile, generating functions and order statistics are obtained. We estimate the model parameters by the maximum likelihood method and present a Monte Carlo simulation study. The importance of the proposed family is demonstrated by means of three real data applications. Empirical results indicate that proposed family provides better fits than other well-known classes of distributions in real applications.

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1 Introduction

Recently, many odd log-logistic-G families were developed such as the Zografos-Balakrishnan odd log-logistic family of distributions (by Cordeiro et al. 2016a), the generalized odd log-logistic family (by Cordeiro et al. 2016b), the beta odd log-logistic generalized family of distributions (by Cordeiro et al. 2016c), the Kumaraswamy odd log-logistic family of distributions (by Alizadeh et al. 2016) and a new generalized odd log-logistic family of distributions (by Haghbin et al. 2017).

In this study, we provide and study a new extesion of the odd log-logistic family of continuous distributions named the odd log-logistic Poisson-G (OLLP-G) family based on the odd log-logistic (OLL) family of distribution, originally developed by Gleaton and Lynch (2004) and (2006). They called this family as generalized log-logistic (GLL) family. The cumulative distribution function (cdf) of this family is given by

$$F_{\text{OLL-G}}(x,\alpha,\psi)=\frac{G(x,\psi)^{\alpha}}{G(x,\psi)^{\alpha}+\overline{G}(x,\psi)^{\alpha}},\alpha>0. \tag{1}$$

The probability density function (pdf) corresponding to (1) is given by

$$f_{\text{OLL-G}}(x,\alpha,\psi)=\frac{\alpha g(x,\psi)\left[G(x,\psi)\overline{G}(x,\psi)\right]^{\alpha-1}}{\left[G(x,\psi)^{\alpha}+\overline{G}(x,\psi)^{\alpha}\right]^{2}}.$$

Mixing lifetime distribution with Poisson distribution was studied by several authors. For example, Kus (2007) and Cancho et al. (2011) studied Poisson-Exponential distribution. Suppose $Z_1, ..., Z_N$ be independent identically random variable (iid) with common cdf $G(x)$ and N be random variable with

$$P(N=n)=\frac{1}{e^{\beta}-1}\frac{\beta^{n}}{n!}\quad n=1,2,...,\beta>0$$

and define $M_N=\max(Z_1,...,Z_N)$, then the cdf and pdf of M_N is given by

$$\begin{aligned} G_{Poisson-G}(x;\beta,\psi) &= \sum_{n=1}^{\infty} P(M_N \le x|N=n)P(N=n) \\ &= \sum_{n=1}^{\infty} [G(x;\beta,\psi)]^n \frac{1}{e^{\beta}-1}\frac{\beta^n}{n!} \\ &= \frac{e^{\beta\, G(x;\beta,\psi)}-1}{e^{\beta}-1} \end{aligned} \tag{2}$$

and

$$g_{Poisson-G}(x;\beta,\psi) = \frac{\beta\, g(x;\beta,\psi)\, e^{\beta\, G(x;\beta,\psi)}}{e^{\beta}-1},$$

respectively. Using (1) and (2), the cdf of the OLLP-G can be written as

$$F(x;\alpha,\beta,\boldsymbol{\psi}) = \frac{\left[\frac{e^{\beta\, G(x,\boldsymbol{\psi})}-1}{e^{\beta}-1}\right]^{\alpha}}{\left[\frac{e^{\beta\, G(x,\boldsymbol{\psi})}-1}{e^{\beta}-1}\right]^{\alpha} + \left[1-\frac{e^{\beta\, G(x,\boldsymbol{\psi})}-1}{e^{\beta}-1}\right]^{\alpha}}, \tag{3}$$

where $\alpha, \beta > 0$ and ψ is the vector of parameter for baseline G. The corresponding pdf is given by

$$f(x;\alpha,\beta,\boldsymbol{\psi}) = \frac{\alpha\,\beta\, g(x,\boldsymbol{\psi})\, e^{\beta\, G(x,\boldsymbol{\psi})}\left[e^{\beta\, G(x,\boldsymbol{\psi})}-1\right]^{\alpha-1}\left[e^{\beta}-e^{\beta\, G(x,\boldsymbol{\psi})}\right]^{\alpha-1}}{(e^{\beta}-1)\left\{\left[e^{\beta\, G(x,\boldsymbol{\psi})}-1\right]^{\alpha}+\left[e^{\beta}-e^{\beta\, G(x,\boldsymbol{\psi})}\right]^{\alpha}\right\}^{2}}. \tag{4}$$

The hazard rate functions (hrf), difened as $f(x;\alpha,\beta,\boldsymbol{\psi})/\left[1-F(x;\alpha,\beta,\boldsymbol{\psi})\right]$, can be written as

$$\tau(x;\alpha,\beta,\boldsymbol{\psi}) = \frac{\alpha\,\beta\, g(x,\boldsymbol{\psi})\, e^{\beta\, G(x,\boldsymbol{\psi})}\left[e^{\beta\, G(x,\boldsymbol{\psi})}-1\right]^{\alpha-1}}{(e^{\beta}-1)\left[e^{\beta}-e^{\beta\, G(x,\boldsymbol{\psi})}\right]\left\{\left[e^{\beta\, G(x,\boldsymbol{\psi})}-1\right]^{\alpha}+\left[e^{\beta}-e^{\beta\, G(x,\boldsymbol{\psi})}\right]^{\alpha}\right\}}. \tag{5}$$

An interpretation of the OLLP-G family can be given as follows. Let T be a random variable describing a stochastic system by the cdf $G_{\text{Poisson-G}}(x;\beta,\psi)$. If the random variable X represents the odds ratio, the risk that the system following the lifetime T will be not working at time x is given by

$G_{\text{Poisson-G}}(x;\beta,\psi)/[1-G_{\text{Poisson-G}}(x;\beta,\psi)]$. If we are interested in modeling the randomness of the odds ratio by the exponentiated half-logistic cdf $R(t)=\frac{t^{\alpha}}{1+t^{\alpha}}$ (for $t>0$), the cdf of X is given by

$$Pr(X\leq x)=R\left[\frac{G_{\text{Poisson-G}}(x;\beta,\psi)}{1-G_{\text{Poisson-G}}(x;\beta,\psi)}\right]=\frac{\left[e^{\beta\,G(x,\boldsymbol{\psi})}-1\right]^{\alpha}}{\left[e^{\beta\,G(x,\boldsymbol{\psi})}-1\right]^{\alpha}+\left[e^{\beta}-e^{\beta\,G(x,\boldsymbol{\psi})}\right]^{\alpha}}.$$

Furthemore, the basic motivations for using the OLLP-G family in practice are the following: to make the kurtosis more flexible compared to the baseline model; to produce a skewness for symmetrical distributions; to construct heavy-tailed distributions that are not longer-tailed for modeling real data; to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped; to define special models with flexible types of the hrf; to provide consistently better fits than other generated models under the same baseline distribution.

Remark 1: Although, we have stated that $\beta\in(0,\infty)$, Eq. (3) is still a cdf if $\beta<0$. Hence, we can consider the OLLP-G family defined for any $\beta\in\mathcal{R}-\{0\}$.

Theorem 1. provides a relation of the OLLP-G family with log-logistic distribution.

Theorem 1. Let $X\sim OLLP-G\,(\alpha,\beta,\boldsymbol{\psi})$, then

$$Y=e^{\left(\frac{G(x;\boldsymbol{\psi})}{\bar{G}(x;\boldsymbol{\psi})}\right)^{\alpha}}-1,$$

has log-logistic cdf

$$F_Y(y)=y^{\alpha}/\left(1+y^{\alpha}\right)^{-1},\ y>0.$$

For $\alpha=1$, we obtain Poisson-G, for $\beta\to0^{+}$ we obtain OLL-G, for $\alpha=1$ and $\beta\to0^{+}$ we obtain baseline G. The OLLP-G density function (4) allows for greater flexibility of its tails and mode of distribution and can be widely applied in many areas of engineering and science.

Theorem 2: If distribution $G(x)$ has a moment generating function, then distribution function $F(x)$ has a moment generating function.

Let $m = \inf\{x|G(x) \geq 0.5\}$, then

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}}\,\mathrm{f(x)dx} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{g(x)} \left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha-1} \left[1-\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha-1}}{(\mathrm{e}^{\beta}-1)\left\{\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}+\left[1-\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}\right\}^2}\,\mathrm{dx} \\ &\leq \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{e}\,\mathrm{g(x)}}{(\mathrm{e}^{\beta}-1)\left\{\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}+\left[1-\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}\right\}^2}\,\mathrm{dx} \\ &= \int_{-\infty}^{m} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{e}\,\mathrm{g(x)}}{(\mathrm{e}^{\beta}-1)\left\{\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}+\left[1-\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}\right\}^2}\,\mathrm{dx} \\ &+ \int_{m}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{e}\,\mathrm{g(x)}}{(\mathrm{e}^{\beta}-1)\left\{\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}+\left[1-\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{\alpha}\right\}^2}\,\mathrm{dx} \end{aligned}$$

The first integral in last line is finite, the second integral is no greater than

$$\int_{m}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{e}\,\mathrm{g(x)}}{(\mathrm{e}^{\beta}-1)\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{2\alpha}}\,\mathrm{dx}$$

For $x > m$, we have $G(x) \geq 0.5$, so that

$$\int_{m}^{\infty} \mathrm{e}^{\mathrm{t\,x}} \frac{\alpha\beta\,\mathrm{e}\,\mathrm{g(x)}}{(\mathrm{e}^{\beta}-1)\left[\frac{\mathrm{e}^{\beta\,\mathrm{G(x)}}-1}{\mathrm{e}^{\beta}-1}\right]^{2\alpha}}\,\mathrm{dx} < \left(\frac{\mathrm{e}^{\beta/2}-1}{\mathrm{e}^{\beta}-1}\right)^{-2\alpha} \int_{\mathrm{m}}^{\infty} \mathrm{e}^{\mathrm{t\,x}}\,\mathrm{g(x)dx} < \infty.$$

Then $M_X(t) < \infty$.

Corollary: Every distribution in OLLP-G class has exactly the same number of moments of $G(x)$.

Here, we introduce two special models obtained from this family because it extends several widely-known distributions in the literature.

2 Special OLLP models

2.1 OLLP-Weibull distribution

Taking $G(x)$ to be the Weibull cdf with scale parameter $b > 0$ and shape parameter $a > 0$, the OLLP-W density function (for $x > 0$) is given by

$$
\begin{aligned}
f(x;\alpha,\beta,a,b) \; &= \; \frac{a\,\alpha\beta\, x^{a-1}\,\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\,\mathrm{e}^{\beta\left[1-\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\right]}}{b^{a}\left(\mathrm{e}^{\beta}-1\right)\left\{\left[\mathrm{e}^{\beta\left[1-\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\right]}-1\right]^{\alpha}+\left[\mathrm{e}^{\beta}-\mathrm{e}^{\beta\left[1-\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\right]}\right]^{\alpha}\right\}^{2}} \\
&\times \; \left[\mathrm{e}^{\beta\left[1-\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\right]}-1\right]^{\alpha-1}\left[\mathrm{e}^{\beta}-\mathrm{e}^{\beta\left[1-\mathrm{e}^{-(\mathrm{x}/\mathrm{b})^{\mathrm{a}}}\right]}\right]^{\alpha-1}.
\end{aligned}
$$

In Figure 1, the OLLP-W distribution has flexible shape of pdf and hrf. The pdf includes right and left skew unimodal- bimodal shapes and on the other hand, hrf can model datasets with increasing, decreasing, unimodal and bathtube hrf.

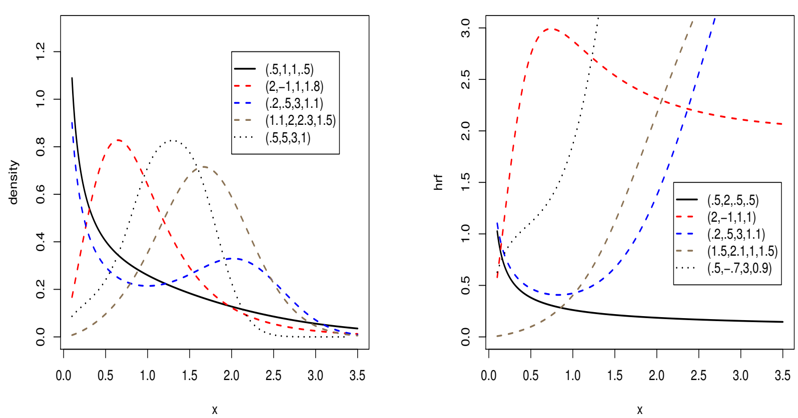

Figure 1: Odd log-logistic Poisson-Weibull distribution: pdf (left), hrf (right).

2.2 OLLP-Lindley distribution

The pdf and cdf of the Lindley (L) distribution are (for $x, \theta > 0$)

$$g(x) = \frac{\theta^2\,(1+x)\mathrm{e}^{-\theta\,\mathrm{x}}}{1+\theta},$$

$$G(x) = 1 - (1 + \frac{\theta\,x}{1+\theta})\mathrm{e}^{-\theta\,\mathrm{x}}.$$

Inserting these expressions into (4), we obtain the OLLP-L density function (for $x > 0$)

$$\begin{aligned} f(x;\alpha,\beta,\theta) \quad &= \quad \frac{\theta^2\,\alpha\beta\,(1+x)\,\mathrm{e}^{-\theta\,\mathrm{x}}\,\mathrm{e}^{\beta\left[1-(1+\frac{\theta\,\mathrm{x}}{1+\theta})\,\mathrm{e}^{-\theta\,\mathrm{x}}\right]}}{\left\{\left[\mathrm{e}^{\beta\left[1-(1+\frac{\theta\,\mathrm{x}}{1+\theta})\,\mathrm{e}^{-\theta\,\mathrm{x}}\right]}-1\right]^{\alpha}+\left[\mathrm{e}^{\beta}-\mathrm{e}^{\beta\left[1-(1+\frac{\theta\,\mathrm{x}}{1+\theta})\,\mathrm{e}^{-\theta\,\mathrm{x}}\right]}\right]^{\alpha}\right\}^{2}} \\ &\times \quad \left[\mathrm{e}^{\beta\left[1-(1+\frac{\theta\,\mathrm{x}}{1+\theta})\,\mathrm{e}^{-\theta\,\mathrm{x}}\right]}-1\right]^{\alpha-1}\,\left[\mathrm{e}^{\beta}-\mathrm{e}^{\beta\left[1-(1+\frac{\theta\,\mathrm{x}}{1+\theta})\,\mathrm{e}^{-\theta\,\mathrm{x}}\right]}\right]^{\alpha-1}. \end{aligned}$$

Figure 2 shows that the OLLP-L distribution has more flexibility than Lindley distribution in term of shape of pdf and hrf.

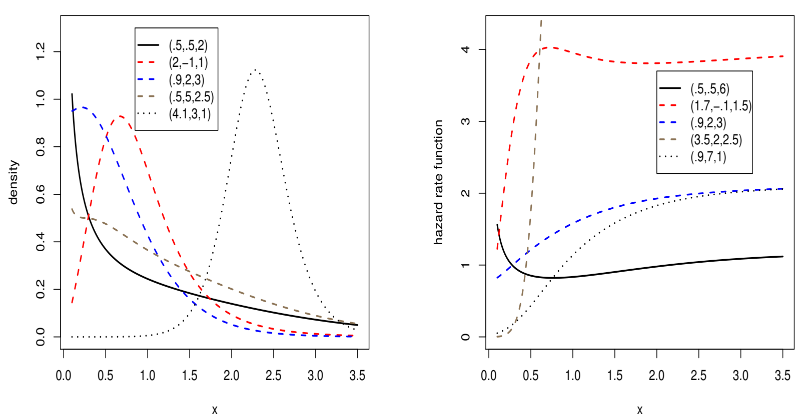

Figure 2: Odd log-logistic Poisson-Lindley distribution: pdf (left), hrf (right).

3 Linear representations

In this section, mixture representations for Equations (3) and (4) are obtained, firstly we have

$$\left[\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right]^{\alpha}=\sum_{j=0}^{\infty}a_j\,\left[\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right]^{j},$$

where $a_j=\sum_{i=j}^{\infty}(-1)^{i+j}\binom{\alpha}{i}\binom{i}{j}$ and

$$\left[\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right]^{\alpha}+\left[1-\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right]^{\alpha}=\sum_{j=0}^{\infty}b_j\,\left[\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right]^{j},$$

where $b_j=a_j+(-1)^j\binom{\alpha}{j}$, then the OLLP-G cdf in (3) can be written as

$$F(x)=\frac{\sum_{j=0}^{\infty}a_j\,\left(\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right)^{j}}{\sum_{j=0}^{\infty}b_j\,\left(\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right)^{j}}=\sum_{j=0}^{\infty}c_j\,\left(\frac{e^{\beta\,G(x)}-1}{e^{\beta}-1}\right)^{j},$$

where $c_0=\frac{a_0}{b_0}$ and for $j\geq 1$, we have

$$c_j=\frac{1}{b_0}\left(a_j-\frac{1}{b_0}\sum_{r=1}^{j}b_r\,c_{j-r}\right),$$

then

$$F(x)=\sum_{k=0}^{\infty}d_k\,H_{k+1}(x), \tag{6}$$

where

$$d_k=\sum_{j=0}^{\infty}\sum_{l=0}^{j}\frac{(1+j)c_{j+1}\,(-1)^{j+l}\,(1+l)^k}{(e^{\beta}-1)^{j+1}\,(k+1)!}\binom{j}{l}$$

and $H_{\delta}(x)=G(x)^{\delta}$ is cdf of exponentiated-G (Exp-G) distribution with power parameter (δ). The corresponding OLLP-G density function is obtained by differentiating (6) as

$$f(x) = \sum_{k=0}^{\infty} d_k \, h_{k+1}(x), \tag{7}$$

where $h_\delta(x) = \delta g(x)\, G(x)^\delta$ is the pdf of the Exp-G distribution with power parameter (δ). Equation (6) and (7) reveal that pdf of OLLP-G is a linear combination of Exp-G densities. Thereby, some properties of the proposed family such as moments and generating function can be determined by means of Exp-G distribution. The properties of Exp-G distributions have been studied by many authors in recent years, see Mudholkar and Srivastava (1993) and Mudholkar et al. (1995) for exponentiated Weibull, Gupta et al. (1998) for exponentiated Pareto, Gupta and Kundu (1999) for exponentiated exponential, Nadarajah (2005) for the exponentiated-type distributions, Nadarajah and Kotz (2006) for exponentiated Gumbel, Shirke and Kakade (2006) for exponentiated log-normal and Nadarajah and Gupta (2007) for exponentiated gamma distributions , among others.

4 Mathematical properties

4.1 Asymptotics

Let $d = \inf\{x|G(x) > 0\}$, the asymptotics of equations (3), (4) and (5) as $x \to d$ are given by

$$F(x) \sim \left[\frac{\beta\, G(x;\boldsymbol{\psi})}{e^\beta - 1}\right]^\alpha \quad as \quad x \to d,$$
$$f(x) \sim \frac{\alpha\beta^\alpha}{(e^\beta - 1)^\alpha} g(x;\boldsymbol{\psi})G(x;\boldsymbol{\psi}) \quad as \quad x \to d,$$

and

$$h(x) \sim \frac{\alpha\beta^\alpha}{(e^\beta - 1)^\alpha} g(x;\boldsymbol{\psi})G(x;\boldsymbol{\psi})^{\alpha-1} \quad as \quad x \to d.$$

The asymptotics of equations (3), (4) and (5) as $x \to \infty$ are given by

$$1-F(x) \sim \left\{ \frac{\beta\,[1-G(x;\boldsymbol{\psi})]}{e^{\beta}-1} \right\}^{\alpha} \quad as \quad x \to \infty,$$

$$f(x) \sim \frac{\alpha\beta^{\alpha}}{(e^{\beta}-1)^{\alpha}} g(x;\boldsymbol{\psi})\,[1-G(x;\boldsymbol{\psi})] \quad as \quad x \to \infty,$$

and

$$h(x) \sim \alpha g(x;\boldsymbol{\psi})/\,[1-G(x;\boldsymbol{\psi})]^{-1} \quad as \quad x \to \infty.$$

These equations show the effect of parameters on tails of OLLP-G distributions.

4.2 Moments, incomplete moments and generating function

The rthordinary moment of X is given by $\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r\, f\,(x)\,dx$. Then we obtain

$$\mu_r' = \sum_{k=0}^{\infty} d_k E(Y_{k+1}^r). \tag{8}$$

Henceforth, Y_δ denotes the Exp-G model with power parameter (δ). For $\delta > 0$, we have $E\left(Y_\delta^r\right) = \delta\, \int_{-\infty}^{\infty} x^r\ g\,(x;\psi)\ \ G\,(x;\psi)^{\delta-1}\ dx$, which can be computed numerically in terms of the baseline quantile function (qf) $Q_G\,(u;\psi) = G^{-1}\,(u;\psi)$ as $E\left(Y_\delta^n\right) = \delta\, \int_0^1\ Q_G\,(u;\psi)^n\ \ u^{\delta-1}du$. Setting $r = 1$ in (8), we have the mean of X. The last integration can be computed numerically for most parent distributions. The rthincomplete moment, say $I_r\,(t)$, of X can be expressed from (7) as

$$I_r\,(t) = \int_{-\infty}^{t} x^r f\,(x)\,dx = \sum_{k=0}^{\infty} d_k\, \int_{-\infty}^{t} x^r\, h_{k+1}(x)dx. \tag{9}$$

The first incomplete moment $I_1\,(t)$ given by (9) with $r = 1$. A general equation for $I_1\,(t)$ can be derived from (9) as $I_1\,(t) = \sum_{k=0}^{\infty} d_k\, J_{k+1}\,(x)\,,$ where $J_\delta\,(x) = \int_{-\infty}^{t} x\, h_\delta\,(x)\,dx$ is the first incomplete moment of the Exp-G model. The moment generating function (mgf) $M_X\,(t) = E\left(e^{t\,X}\right)$ of X can be derived from equation (8) as $M_X\,(t) = \sum_{k=0}^{\infty} d_k M_{k+1}\,(t)$,where $M_\delta\,(t)$ is the mgf of Y_δ. Hence, $M_X\,(t)$ can be determined from the

Exp-G generating function. The cf of X, $\phi(t) = E\left(e^{it\,X}\right)$ is given by $\phi(t) = \sum_{k=0}^{\infty} d_k\ \phi_{k+1}(t)$ where $\phi_\delta(t)$ is the cf of Y_δ and $i = \sqrt{-1}$. For the OLLP-W model we have

$$\mu_r' = \Gamma\left(1 + \frac{r}{a}\right) \sum_{k,w=0}^{\infty} \upsilon_{w,k}^{(r,k+1)},\ \forall\ r > -a$$

and

$$I_r(t) = \gamma\left(1 + \frac{r}{a}, \left(\frac{1}{bt}\right)^a\right) \sum_{k,w=0}^{\infty} \upsilon_{w,k}^{(r,k+1)}\ ,\ \forall\ r > -a,$$

where

$$\upsilon_{w,k}^{(r,k+1)} = d_k \frac{(k+1)(-1)^w}{(1/b)^r (w+1)^{(r+a)/a}} \binom{k}{w}.$$

Figure 3 displays the mean and variance plots of OLLP-W distribution for parameters $a = 2$ and $b = 1$. Based on these plots, we conclude that: if β increases, mean increases; if α increases, mean decreases and variance increases. The parameter β has not significant effect on variance.

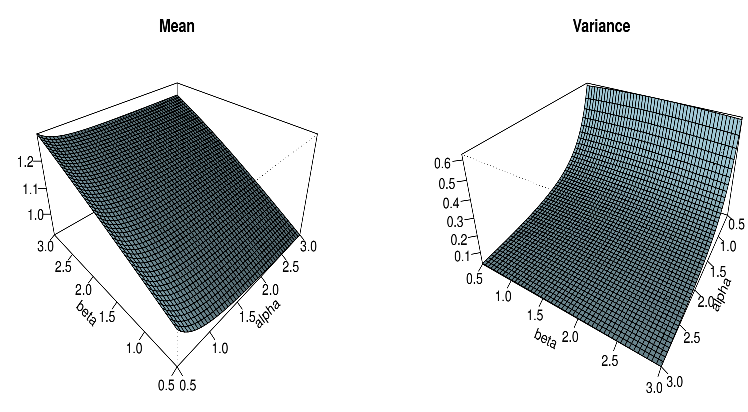

Figure 3: Mean and variance plot for OLLP-W distribution with shape=2 and scale=1 parameters

4.3 Quantile function

The OLLP-G family can easily be simulated by inverting (3) as follows: if $U \sim U(0,1)$, then the random variable X_U can be obtained from the baseline qf, say $Q_G(u) = G^{-1}(u)$. In fact, the random variable

$$X_U = Q_G\left\{\frac{1}{\beta}\log\left[1+\frac{(e^{\beta}-1)U^{\frac{1}{\alpha}}}{U^{\frac{1}{\alpha}}+(1-U)^{\frac{1}{\alpha}}}\right]\right\},$$

has cdf (5). The effects of the shape parameters on the skewness and kurtosis can be based on quantile measures. We obtain skewness and kurtosis measures using the quantile function of OLLP-G family. The Bowley's skewness measure is given by

$$Skewness = \frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)},$$

and the Moors's kurtosis measure is

$$Kurtosis = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)},$$

these measures enjoy the advantage of having less sensitivity to outliers. Moreover, they do exist for distribution without moments. Both measures equal zero for the normal distribution. Plots of skewness and kurtosis of the OLLP-W distribution is presented in Figure 4. This plot indicates that parameter α controls the skewness and kurtosis measures.

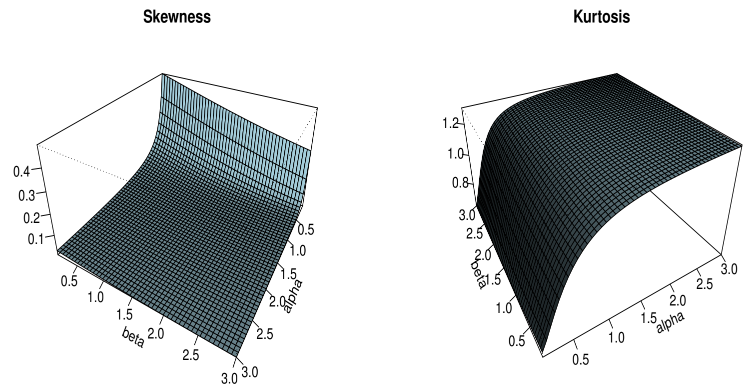

Figure 4: Skewness and Kurtosis plot for OLLP-W distribution with shape=2 and scale=1 parameters

4.4 Moments of residual and reversed residual life

The nthmoment of the residual life say, $z_n(t) = E[(X-t)^n \mid X > t]$, $n = 1, 2, \ldots$, uniquely determines $F(x)$. The nthmoment of the residual life of X is given by $z_n(t) = \frac{1}{1-F(t)} \int_t^\infty (x-t)^n dF(x)$. Therefore

$$z_n(t) = \frac{1}{1-F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} d_k \,(1-t)^n \int_t^\infty x^r h_{k+1}(x) dx.$$

The nthmoment of the reversed residual life say, $Z_n(t) = E\left[(t-X)^n \mid X \le t\right]$ for $t > 0$ and $n = 1, 2, \ldots$ uniquely determines $F(x)$. We obtain $Z_n(t) = \frac{1}{F(t)} \int_0^t (t-x)^n dF(x)$. Then, the nthmoment of the reversed residual life (RRL) of X becomes

$$Z_n(t) = \frac{1}{F(t)} \sum_{k=0}^{\infty} \sum_{r=0}^{n} d_k \,(-1)^r \binom{n}{r} t^{n-r} \int_0^t x^r h_{k+1}(x) dx.$$

The mean residual life (MRL) function or the life expectation at age t defined by $z_1(t) = E\left[(X-t) \mid X > t\right]$, which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained when $n=1$ in $z_n(t)$ equation. The mean inactivity time (MIT) or mean waiting time also called the mean reversed residual life function (MRRL) is given by $Z_1(t) = E[(t-X) \mid X \leq t]$, and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the OLLP-G family of distributions can be obtained easily when $n=1$ in $Z_n(t)$ equation. For the OLLP-W model we have

$$z_n(t) = \frac{\gamma\left(1+\frac{n}{a}, \left(\frac{1}{bt}\right)^a\right)}{1-F(t)} \sum_{k,w=0}^{\infty} \sum_{r=0}^{n} \upsilon_{w,k,r}^{(n,k+1)}, \ \forall\ n > -a$$

and

$$Z_n(t) = \frac{\gamma\left(1+\frac{n}{a}, \left(\frac{1}{bt}\right)^a\right)}{F(t)} \sum_{k,w=0}^{\infty} \sum_{r=0}^{n} \vartheta_{w,k,r}^{(n,k+1)}, \ \forall\ n > -a,$$

where

$$\upsilon_{w,k,r}^{(n,k+1)} = d_k \frac{(k+1)\,(-1)^w\,(1-t)^n}{(1/b)^n\,(w+1)^{(n+a)/a}} \binom{k}{w}$$

and

$$\vartheta_{w,k,r}^{(n,k+1)} = d_k \frac{(k+1)\,(-1)^{w+r}\,t^{n-r}}{(1/b)^n\,(w+1)^{(n+a)/a}} \binom{n}{r} \binom{k}{w}.$$

4.5 Order statistics

Suppose $X_1, \ldots, X_n$ is a random sample from any OLLP-G model, let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as $f_{i:n}(x) = \frac{f(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}$, following similar algebraic developments of Nadarajah et al. (2015), we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{l,k=0}^{\infty} d_{l,k}\, h_{l+k+1}(x), \tag{10}$$

where, $d_{l,k} = \frac{n!\,(l+1)\,(i-1)!\,d_{l+1}}{(l+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j\,\zeta_{j+i-1,k}}{(n-i-j)!\,j!}$ and the quantities $\zeta_{j+i-1,k}$ can be determined with $\zeta_{j+i-1,0} = d_0^{j+i-1}$ and recursively for $k \geq 1$,

$\zeta_{j+i-1,k} = (k\, d_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\;\; d_m\; \zeta_{j+i-1,k-m}$. Equation (10) is the main result of this section. It reveals that the pdf of the OLLP-G order statistics is a linear combination of Exp-G density functions. So, several mathematical quantities of the OLLP-G order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the Exp-G distribution. For the OLLP-W model we have

$$E\left(X_{i:n}^{q}\right) = \Gamma\left(1 + \frac{q}{a}\right) \sum_{l,k,w=0}^{\infty} \upsilon_{l,k,w}^{(q,l+k+1)}\;, \forall\; q > -a,$$

where

$$\upsilon_{l,k,w}^{(q,l+k+1)} = d_{l,k} \frac{(l+k+1)\,(-1)^{w}}{(1/b)^{q}\,(w+1)^{(q+a)/a}} \binom{l+k}{w}.$$

5 Maximum likelihood estimation

Here, we consider estimation of the unknown parameters of the OLLP-G distribution by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample from the OLLP-G distribution with a $(q+2) \times 1$ parameter vector $\mathbf{\Psi} = (\alpha, \beta, \psi)^{\intercal}$, where ψ is a $q \times 1$ baseline parameter vector. The log-likelihood function for $\mathbf{\Psi}$ is given by

$$\begin{aligned} \ell(\mathbf{\Psi}) \;\; = \;\; & n \log \alpha + n \log \beta - n \log\left(e^{\beta} - 1\right) + \sum_{i=1}^{n} \log g\left(x_i; \psi\right) + \beta \sum_{i=1}^{n} G(x_i; \psi) \\ & + (\alpha - 1) \sum_{i=1}^{n} \log\left(\eta_i - 1\right) + (\alpha - 1) \sum_{i=1}^{n} \log\left(e^{\beta} - \eta_i\right) \\ & - 2 \sum_{i=1}^{n} \log\left[(\eta_i - 1)^{\alpha} + \left(e^{\beta} - \eta_i\right)^{\alpha}\right], \end{aligned}$$

where $\eta_i = e^{\beta\, G(x_i;\psi)}$. The components of the score vector, $U(\mathbf{\Psi}) = \frac{\partial \ell}{\partial \mathbf{\Psi}} = \left(\frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \psi}\right)^{\intercal}$, are given as

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log(\eta_i - 1) + \sum_{i=1}^{n} \log\left(e^{\beta} - \eta_i\right) \\ &\quad -2\sum_{i=1}^{n} \frac{(\eta_i - 1)^{\alpha} \log(\eta_i - 1) + \left(e^{\beta} - \eta_i\right)^{\alpha} \log\left(e^{\beta} - \eta_i\right)}{(\eta_i - 1)^{\alpha} + \left(e^{\beta} - \eta_i\right)^{\alpha}},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} - \frac{ne^{\beta}}{e^{\beta} - 1} + \sum_{i=1}^{n} G(x_i;\psi) + (\alpha - 1)\sum_{i=1}^{n} \frac{e^{\beta} - b_i}{e^{\beta} - \eta_i} \\ &\quad -2\sum_{i=1}^{n} \frac{\alpha\left[b_i(\eta_i - 1)^{\alpha-1} + \left(e^{\beta} - b_i\right)\left(e^{\beta} - \eta_i\right)^{\alpha-1}\right]}{(\eta_i - 1)^{\alpha} + \left(e^{\beta} - \eta_i\right)^{\alpha}},\end{aligned}$$

and (for $r = 1, ..., q$)

$$\begin{aligned}\frac{\partial \ell}{\partial \psi_r} &= +\sum_{i=1}^{n} \frac{\partial g(x_i;\psi)/\partial \psi_r}{g(x_i;\psi)} + \beta\sum_{i=1}^{n}\left[\partial G(x_i;\psi)/\partial \psi_r\right] \\ &\quad + (\alpha - 1)\sum_{i=1}^{n} \frac{a_{i,r}}{\eta_i - 1} + (\alpha - 1)\sum_{i=1}^{n} \frac{-a_{i,r}}{e^{\beta} - \eta_i} \\ &\quad -2\sum_{i=1}^{n} \frac{\alpha a_{i,r}\left[(\eta_i - 1)^{\alpha-1} - \left(e^{\beta} - \eta_i\right)^{\alpha-1}\right]}{(\eta_i - 1)^{\alpha} + \left(e^{\beta} - \eta_i\right)^{\alpha}},\end{aligned}$$

where, $b_i = G(x_i;\psi)e^{\beta\, G(x_i;\psi)}$ and $a_{i,r} = \beta\left[\partial G(x_i;\psi)/\partial \psi_r\right] e^{\beta\, G(x_i;\psi)}$. Setting the nonlinear system of equations $U_{\alpha} = U_{\beta} = U_{\psi_r} = 0$ (for $r = 1, \ldots, q$) and solving them simultaneously yields the MLEs $\widehat{\mathbf{\Psi}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\psi}^{\intercal})^{\intercal}$. To solve these equations, it is more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell(\mathbf{\Psi})$. For interval estimation of the parameters, we can evaluate numerically the elements of the $(q+2) \times (q+2)$ observed information matrix $J(\mathbf{\Psi}) = \{-\frac{\partial^2 \ell}{\partial \theta_r\, \theta_s}\}$. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\mathbf{\Psi}}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\mathbf{\Psi}})^{-1})$ distribution to construct approximate confidence intervals for the parameters, where p is the number of parameters. Here, $J(\widehat{\mathbf{\Psi}})$ is the total observed information matrix evaluated at $\widehat{\mathbf{\Psi}}$.

6 Simulation study

In this section, we assess the performance of the MLEs of the OLLP-W parameters. The precision of the MLEs is based on the following measures: bias, mean square error (MSE), estimated average length (AL) and coverage probability (CP). We generate $N = 1,000$ samples of sizes $n = 50, 55, \dots, 1000$ from the OLLP-W distribution with $\alpha = 3, \beta = 3, a = 2, b = 2$ by using the inverse transform method. The MLEs of the model parameters are obtained for each generated sample, say $(\hat{\alpha}_i, \hat{\beta}_i, \hat{a}_i, \hat{b}_i)$, for $i = 1, 2, \cdots, N$. The standard errors of the MLEs are evaluated by inverting the observed information matrix, namely $(s_{\hat{\alpha}_i}, s_{\hat{\beta}_i}, s_{\hat{a}_i}, s_{\hat{b}_i})$ for $i = 1, 2, \cdots, N$. The estimated biases and MSEs are given by

$$\begin{array}{l} \widehat{Bias}_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon) \\ \widehat{MSE}_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2 \\ CP_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\epsilon}_i - 1.95996 s_{\hat{\epsilon}_i}, \hat{\epsilon}_i + 1.95996 s_{\hat{\epsilon}_i}) \\ AL_\epsilon(n) = \frac{3.919928}{N}\sum_{i=1}^{N} s_{\hat{\epsilon}_i} \end{array} \tag{11}$$

where $\epsilon = \alpha, \beta, a, b$.

The numerical results for the above measures are displayed in the plots of Figure 5. We can note that the estimated biases decrease as the sample size n increases. Further, the estimated MSEs decay toward zero when n increases. This fact reveals the consistency property of the MLEs. The CP is near to 0.95 and approaches to the nominal value when the sample size increases. Moreover, if the sample size increases, the AL decreases in each case. The reported results are obtained for a selected parameter vector (α, β, a, b). However, similar results hold for several parameter values.

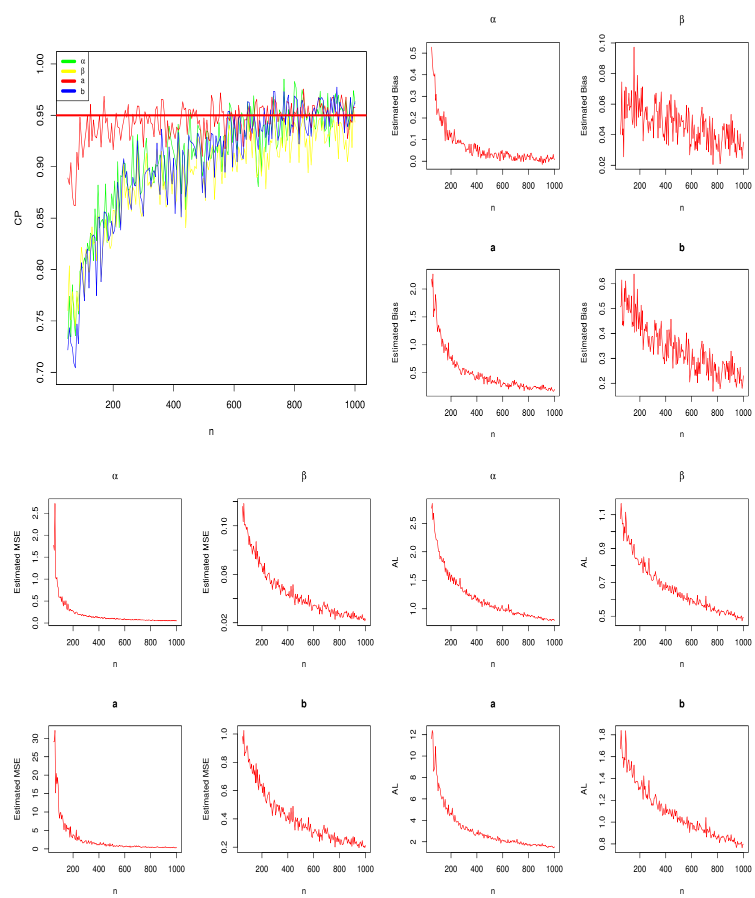

Figure 5: Estimated CPs, biases, MSEs and ALs for the selected parameters.

7 Application

We illustrate the importance of the OLLP-G family in an application to real data set. In the two last decades, several extensions of the Weibull distributions have been introduced in the literature. Some of 4 parameters generalized Weibull distribution that is used in this section are Kumaraswamy Weibull (Cordeiro et al., 2010), Beta-Weibull (Lee et al., 2007), Generalized Modified Weibull (Carrasco et al., 2008), Exponentiated modified Weibull extension (Sarhan and Apaloo, 2013) and (P-A-L) Extended Weibull Distribution (Al-Zahrani et al., 2015). Also the sub model of OLLP-W are Poisson-Weibull (Lu and Shi, 2012), OLL-Weibull (Cooray, 2006), Weibull, OLLP-exponential, OLL-exponential (Gleaton and Lynch, 2006), Poisson-exponential and exponential distributions are used in this section. The measures of goodness-of-fit including the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Consistent Akaike information criterion (CAIC) are calculated to compare the fitted rival models. In general, the smaller the values of these statistics, the better the fit to the data. Also the likelihood ratio test is used for comparing proposed model with submodels. The required computations are carried out using the R software.

The data set represent the lifetime of a certain device reported in Sylwia (2007). These data have a bathtub shaped hazard function and are given by 0.0094, 0.05, 0.4064, 4.6307, 5.1741, 5.8808, 6.3348, 7.1645, 7.2316, 8.2604, 9.2662, 9.3812, 9.5223, 9.8783, 9.9346, 10.0192, 10.4077, 10.4791, 11.076, 11.325, 11.5284, 11.9226, 12.0294, 12.074, 12.1835, 12.3549, 12.5381, 12.8049, 13.4615, 13.853.

The maximum likelihood estimates of the parameters and the goodness of fit statistics are reported in Tables 1 and 2. Table 3 includes LR test for comparing OLLP-W with submodels. Figure 6 shows fitted distributions on histogram of data set. In the applications, the information about the hazard shape can help in selecting a particular model. For this aim, a device called the total time on test (TTT) plot (Aarset, 1987) is useful. The TTT plot is obtained by plotting

$$G(r/n) = \left[\left(\sum_{i=1}^{r} y_{(i)}\right) + (n-r)y_{(r)}\right] / \sum_{i=1}^{n} y_{(i)},$$

where $r = 1, ..., n$ and $y_{(i)}$ $(i = 1, ..., n)$ are the order statistics of the sample, against r/n. If the shape is a straight diagonal the hazard is constant. It is convex shape for decreasing hazards and concave shape for increasing hazards. The bathtub-shaped hazard is obtained when the first convex and then concave and for bimodal shape hazard, the TTT plot is first concave and then convex. In Figure 7, The TTT plot of real data set and hrf function of fitted OLLP-W based on MLE are shown. This figure illustrates that the hazard function of OLLP-W is similar to data sets. In this real data set, the results show that the proposed distribution yields a better fit than other distributions.

Table 1:Parameters estimates and log likelihood function.

Model	Estimates (Standard Error)	−Log Likelihood
OLLP-W	0.103, 11.347, 5.801, 7.475	71.912
(α, β, a, b)	(0.016), (0.024), (0.043), (0.042)	
Kw-W	0.381, 15.234, 4.214, 0.018	91.829
(a, b, λ, β)	(0.074), (9.629), (0.118), (0.002)	
B-W	0.011, 76.025, 0.068, 96.938	79.140
(a, b, λ, β)	(0.005), (3.162), (0.081), (14.125)	
GM-W	0.003, 0.249, 0.438, 0.699	72.352
$(\alpha, \gamma, \lambda, \beta)$	(0.001), (0.170), (0.048), (0.159)	
PALEW	1.903, 0.943, 83.822, 1.248	81.872
(α, β, ν, p)	(1.101), (0.219), (16.501), (4.389)	
EMWE	22.832, 168.036, 3.239, 0.346	86.282
$(\lambda, \alpha, \beta, \gamma)$	(1.828), (0.322), (0.021), (0.064)	
OLL-W(Submodel)	0.113, 6.543, 7.999	75.949
(α, a, b)	(0.017), (0.002), (0.002)	
P-W(Submodel)	3.782, 1.319, 5.823	87.467
(β, a, b)	(1.212), (0.242), (1.056)	
W(Submodel)	1.619, 9.582	92.729
(a, b)	(0.277), (1.095)	
OLLP-E(Submodel)	0.970, 4.516, 4.485	88.486
(α, β, b)	(0.224), (1.550), (0.884)	
OLL-E(Submodel)	1.439, 11.240	94.596
(α, b)	(0.275), (1.769)	
P-E(Submodel)	4.397, 4.567	88.495
(β, b)	(1.218), (0.638)	
E(Submodel)	9.040	96.047
(b)	(1.650)	

Table 2:Goodness of fit statistics.

Model	AIC	BIC	HQIC	CAIC
OLLP-W	151.824	157.429	153.617	153.424
Kw-W	191.658	197.263	193.451	193.258
B-W	166.280	171.884	168.073	167.880
GM-W	152.706	158.310	154.499	154.306
PALEW	171.744	177.348	173.537	173.344
EMWE	180.565	186.170	182.358	182.165

Table 3: Likelihood Ratio test for Submodel

	Hypothesis	LR	P-Value
1	$H_0 : OLL - PE\, versus\, H_1 : OLLP - W$	6.33×10^{-8}	8.53×10^{-9}
2	$H_0 : OLL - W\, versus\, H_1 : OLLP - W$	0.017	0.004
3	$H_0 : P - W\, versus\, H_1 : OLLP - W$	1.75×10^{-7}	2.43×10^{-8}
4	$H_0 : W\, versus\, H_1 : OLLP - W$	9.10×10^{-10}	9.14×10^{-10}
5	$H_0 : OLL - E\, versus\, H_1 : OLLP - W$	1.40×10^{-10}	1.43×10^{-10}
6	$H_0 : P - E\, versus\, H_1 : OLLP - W$	6.28×10^{-8}	6.08×10^{-8}
7	$H_0 : E\, versus\, H_1 : OLLP - W$	3.29×10^{-11}	1.86×10^{-10}

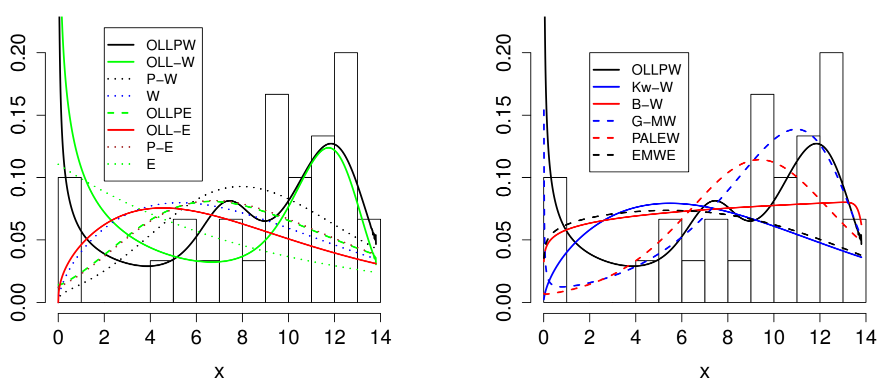

Figure 6: Fitted pdfs on histogram: (left) submodels, (right) rival models

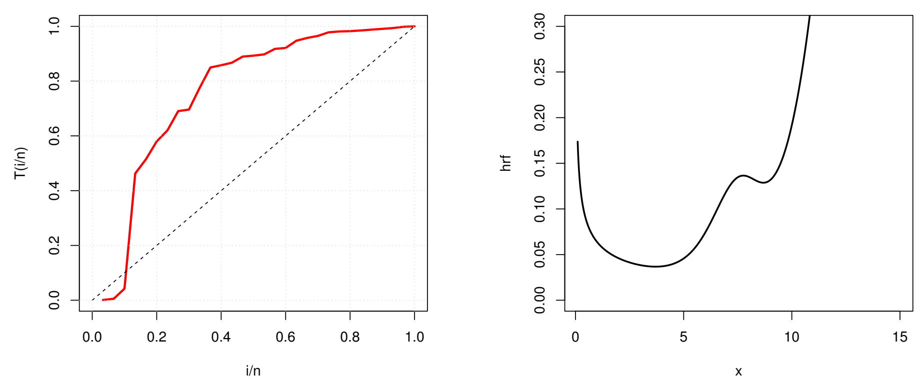

Figure 7: (left) TTT plot of data set, (right) hrf function of fitted OLLP-W for data set

8 Conclusions

We introduce a new class of continuous distributions named the *Odd Log-Logistic Poisson-G* (OLLP-G) family. Some of its mathematical properties are obtained such as the moments, quantile and generating functions. The maximum likelihood method is used to estimate the model parameters and the performance of the maximum likelihood estimators are discussed in terms of biases, mean squared errors, coverage probability and estimated average length by means of Monte-Carlo simulations. Three applications of the proposed family prove empirically its flexibility to model the real data sets. Empirical findings show that proposed family provides an opportunity to model data sets with different characteristics.

Appendix A: Regularity conditions of MLE

1. The random variables $X_i, i = 1, 2, ..., n$ are independent and identically distributed with density $f(x; \Psi)$ where Ψ is the unknown parameter vector.

2. The parameter space Θ is compact.
3. The unknown parameter value Ψ_0 is defined as $\Psi_0 = \arg\max_{\Psi\in\Theta} E_{\Psi_0} \log f(x_i; \Psi)$.
4. The log-likelihood function, $\ell(\Psi) = \sum_{i=1}^{n} \log f(x_i; \Psi)$, is continuous for all θ and $E_{\Psi_0} \log f(x_i; \Psi)$ exists.
5. The log-likelihood function $\ell(\Psi)$ is twice continuously differentiable in a neighbourhood of Ψ_0.
6. The information matrix, $J(\mathbf{\Psi}) = \{-\frac{\partial^2 \ell}{\partial\theta_r \theta_s}\}$, exists and non-singular.

More detail information about regularity conditions can be found in Hoadley (1971).

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