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Convergence of Euler-Maruyama Method for Stochastic Differential Equations Driven by $\alpha-$stable Lévy Motion

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Abstract. In the literature, the Euler-Maruyama (EM) method for approximation purposes of stochastic differential Equations (SDE) driven by α-stable Lévy motions is reported. Convergence in probability of that method was proven but it is surrounded by some ambiguities. To accomplish the but without ambiguities, this article has derived convergence in probability of numerical EM method based on diffusion given by semimartingales for SDEs driven by α-stable processes. Some examples are provided, their numerical solution are obtained and theoretical results are reconfirmed. The adopted method could be applied to other subclasses of semimartingales.

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1 Introduction

According to Ito's stochastic integral theory, it is usually supposed that continuous diffusion process $\{X(t) : t \geq 0\}$ with drifts μ and dispersion coefficients σ can be considered as a solution of stochastic differential equation

$$\begin{cases} dX(t) = \mu(t, X(t))\ dt + \sigma(t, X(t))\ dB(t), \\ X(0) = X_0,\ t \geq 0. \end{cases} \tag{1.1}$$

where $\{B(t) : t \geq 0\}$ stands for the Brownian motion.The theory of these types of stochastic differential equations has been studied in a lot of papers [4, 9, 22] and books [1, 2, 10, 15, 17].
Some researchers tried to extend this theory to some other classes such as stable Lévy processes [1, 23, 24]. Zanzotto [23], in 1997, considered the SDE

$$dX(t) = \sigma(X(t))\ dS_\alpha(t),\ t \geq 0. \tag{1.2}$$

where σ is a Borel measurable function and S_α denotes an α stable motion, $0 < \alpha < 1$ or $1 < \alpha \leq 2$. He studied the problem of the existence of nontrivial weak solutions. This result was extended later by him [24] to the time-independent case, for strictly stable processes with $\alpha \in (0, 2]$, and for $X(0)$ with an arbitrary distribution on $\mathbb{R}$

$$X(t) = X(0) + \int_{(0,t]} \sigma(X(s^-))\ dS_\alpha(s),\ t \geq 0. \tag{1.3}$$

Weak solutions were investigated and specially when $\alpha = 1$, i.e. Cauchy process, a sufficient existence condition was established. Kurenok [9] considered the driftless and time dependent SDE

$$dX(t) = \sigma(t, X(t))\ dS_\alpha(t),\ t \geq 0. \tag{1.4}$$

for $X(0) = x_0 \in \mathbb{R}$ as well as S_α which is a symmetric α stable process, $1 < \alpha \leq 2$. He studied the existence of non exploding solution of (1.4) through the existence of solution of the equation:

$$dA(t) = |\sigma(t, \overline{S}_\alpha(A(t)))|^\alpha\ dt,\ t \geq 0, \tag{1.5}$$

in the class of time change process. In (1.5), $\overline{S}_\alpha$ is distributed as S_α.

He also proved that for any arbitrary initial value $X(0) = x_0 \in R$, there exists a solution of stochastic differential equation (1.4) if and only if $\mathcal{M}_\alpha \subseteq \mathcal{N}$, $1 < \alpha \leq 2$, where

$$M_\alpha =: \left\{ y : \int_{V(y)} |\sigma(x)|^\alpha \ L(dx) = \infty \right\}$$

where $V(y)$ is any open neighborhood of y, and

$$\mathcal{N} =: \{y : \sigma(y) = 0\}$$

and $L(dx)$ is the Lebesgue measure on $\mathbb{R}^+ \times \mathbb{R}$.

Now let $(\Omega, \mathcal{F}, \mathcal{L}, \{\mathcal{F}_t\})$ be a stochastic basis. The purpose of this research is to approximate a real valued and $\{\mathcal{F}_t\}$ adapted diffusion process $\{X(t) : t \geq 0\}$ obtained by $\alpha-$stable stochastic measures with given drift $\mu(.,.)$ and dispersion coefficient $\sigma(.,.)$, which is the solution of SDE:

$$\begin{cases} dX(t) = \mu(t, X(t)) \ dt \ + \sigma(t, X(t)) \ dS_\alpha(t), \\ X(0) = X_0, \ t \geq 0. \end{cases} \tag{1.6}$$

It should be noted that since $S_2(t) = \sqrt{2} \ B(t)$, the SDE (1.1) is a special case of SDE (1.6). The stochastic equation described by (1.6) is regarded as a special case of general SDE given by semimartingales, i.e. equations

$$\begin{cases} X(t) = X(0) \ + \int_0^t f(X(s^-)) \ dZ(s), \\ X(0) = X_0, \ t \geq 0. \end{cases} \tag{1.7}$$

where $\{Z(t)\}$ stands for a given semimartingale process. The aforementioned processes have been discussed in detail in [17, 19] and the references therein.

In Section 2, we first provide some basic convergence and two theorems concerning the existence and uniqueness of solution of (1.7), along with convergence in probability of its numerical solution. Then, a brief discussion on Lévy processes is brought and, as shown in the literature, we point out that Lévy processes are semimartingale. Our next focus is on the family of $\alpha-$stable processes as a member of Lévy processes family.

Section 3 studies stochastic differential equations driven by $\alpha-$stable Lévy motion. Based on the discussion of section 2, we conclude that the Euler-Maruyama numerical method converging in probability to the exact solution of (1.6). The innovation in the present study is to make this heuristic argument rigorous. This is accomplished by proving Theorem 3.2.

Although the convergence in probability of Euler-Maruyama method has already been claimed, it is fraught with ambiguities. Our method proves it clearly and could be used for all subfamilies of semimartingales.

In Section 4 a simulation study and some classical examples are provided to support the conclusion.

2 Semimartingales

In this section, some definitions and theorems concerning SDEs of semimartingales are presented.

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq\infty}$ be a filtration of sub σ fields of $\mathcal{F}$. The random variable T : $\Omega \to [0, \infty]$ is a stopping time if $\{\omega : T(\omega) \leq t \ \forall t \geq 0\} \in \mathcal{F}_t$. A stochastic process $X(t)$ is called adapted if $X(t) \in \mathcal{F}_t$ for all $t \geq 0$. It is càdlàg if its paths are right continuous and have left limit. The class of all adapted and càdlàg process is denoted by $\mathbb{D}$. The stochastic processes $H(t)$ are said to be simple predictable if they admit the following form:

$$H(t) = H(0)\mathbf{1}_{\{0\}}(t) + \sum_{i=1}^{n-1} H(i)\mathbf{1}_{(T_i,T_{i+1}]}(t). \tag{2.1}$$

where $\mathbf{1}_A$ is indicator function and $0 \leq T_0 \leq T_1 \leq \cdots \leq T_n < \infty$ are stopping times and $H(i) \in L^\infty(\mathcal{F}_{T_i})$ where $i = 0, 1, \cdots, n$. The class of all simple predictable processes is denoted by $\mathcal{S}$. A norm on $\mathcal{S}$ is

$$\|H\|_u = \sup_{s\geq 0} \|H_s\|_\infty.$$

When $\mathcal{S}$ is equipped with the topology, it induced by $\|.\|_u$ and is denoted by $\mathcal{S}_u$. The class of almost surely finite valued stochastic processes

$X \in \mathcal{F}$ with the topology induced by convergence in probability under $\mathbb{P}$ is shown by L^0. In other words:

$$L^0 = L^0(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in \mathcal{F} : X \textit{ is finite valued a.s.}\}.$$

Now, for $H \in \mathcal{S}$ with representation (2.1) and for stochastic process X, we define

$$I_X(H) = H(0)X(0) + \sum_{i=1}^{n-1} H(i)(X(T_{i+1}) - X(T_i)), \tag{2.2}$$

Which is the stochastic integral of H with respect to X.

A càdlàg stochastic process X is called a total semimartingale if the map $I_X : \mathcal{S}_u \to L^0$ is continuous. The processes X is a semimartingale if X^t is total semimartingale for all $t \geq 0$, where for a process X and a stopping time T, X^t stands for the process $(X(t \wedge T))_{t\geq 0}$. càdlàg locally square integrable local martingales, local martingales with continuous path and the Brownian motion are examples of semimartingales. An adapted càdlàg process X is said to be decomposable if it can be decomposed as

$$X(t) = X(0) + \mathcal{M}(t) + \mathcal{A}(t) \tag{2.3}$$

where, $\mathcal{M}(0) = \mathcal{A}(0) = 0$, $\mathcal{M}$ is locally square integrable martingale, i.e., $E(\mathcal{M}^2(T_n)\mathbf{1}_{(T_n \geq 0)}) < \infty$, and $\mathcal{A}$ is càdlàg adapted with path of finite variation on compact sets [17].

Theorem 2.1. *A decomposable process is semimartingale.*

Proof. See [17] for a proof. □

A sequence of stochastic process $\{X^n\}_{n=1}^{\infty}$ converges uniformly on compacts in probability, abbreviated as ucp, if $(X^n - X)_t^*$ tend to zero in probability for all t, where

$$X_t^* = \sup_{0 \leq s \leq t} |X_s|.$$

We say that the operator $F : \mathbb{D} \to \mathbb{D}$ is functional Lipschitz if for any X, Y in $\mathbb{D}$ and any stopping time T, $X^{T-} = Y^{T-}$ implies $F(X)^{T-} = F(Y)^{T-}$ and $| F(X)_t - F(Y)_t | \leq k(t) \parallel X - Y \parallel_t^*$ a.s, for some finitely increasing processes $(k(t))_{t\geq 0}$. The following two theorems are specially significant in this section. The proofs are given in [17, 19].

Theorem 2.2. *Let Z be a $\mathbb{R}$ valued vector of semimartingales with $Z(0) = 0$, F be a matrix valued operator which is functional Lipschitz and J is a $\mathbb{R}$ vector processes in $\mathbb{D}$. Then the following SDE has a unique solution in $\mathbb{D}$.*

$$X(t) = J(t) + \int_0^t F(X)_{s_-} dZ(s). \tag{2.4}$$

Moreover, if $J = (J(t))_{t\geq 0}$ is a semimartingale, then so is $X = (X(t))_{t\geq 0}$.

It should be noted that a functional Lipschitz operator F will typically be of the following form

$$F(X) = f(t, w, X_s, s \leq t).$$

Theorem 2.3. *Suppose that Z, F and J satisfy the conditions of Theorem 3.1. Define inductively the following SDE*

$$X^{(n+1)}(t) = J(t) + \int_0^t F(X^{(n)})_s \; dZ(s). \tag{2.5}$$

with $X^{(1)}$ in $\mathbb{D}$. Then $X^{(n)}$ converges in ucp to X, the solution of (2.4).

2.1 Lévy process

An adapted process $X = \{X(t)\}_{t\geq 0}$ with $X(0) = 0$ a.s. is called a Lévy process if:

i) it is independent increment of the past, i.e. $X(t) - X(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s \leq t < \infty$.

ii) $X(t) - X(s)$ is distributed as $X(t - s)$ for all $0 \leq s \leq t < \infty$.

iii) $X(t)$ is continuous in probability, i.e. $X(s) \to X(t)$ in probability as $s \to t$.

It should be mentioned that X has a unique modification which is càdlàg and also Lévy process [5, 17]. Poisson and Wiener processes are examples of Lévy process. The following theorem shows how a Lévy process can be decomposed. In comparison with Theorem 2.2, it is concluded that the Lévy processes are semimartingale.

Theorem 2.4 ([17]). *Let $X = (X(t))_{t\geq 0}$ be a Lévy processes. Then X can be decomposed $X(t) = Y(t)+Z(t)$ where Y and Z are Lévy processes, Y is martingale with bounded jumps, $E \mid Y(t) \mid^p < \infty$ for all $p \geq 1$ and Z has paths of finite variation on compacts.*

It is possible to characterize the Lévy process by looking to its characteristic function, known as Lévy-Khinchin formula

$$\begin{aligned}\Phi_{X(t)} =& E(e^{i\theta X(t)})\\ =& \exp\left\{ ait\theta - \frac{1}{2}\sigma^2\theta^2 + t\int_{\mathbb{R}-\{0\}} (e^{i\theta x} - 1 - i\theta x\mathbf{1}_{\{|x|<1\}})\nu(dx)\right\}\end{aligned} \tag{2.6}$$

where, $a \in \mathbb{R}$, $\sigma \geq 0$ and $\mathbf{1}_A$ is indicator function of the Set A. The Lévy measure ν must be in such a way that

$$\int_{\mathbb{R}-\{0\}} \min\{x^2, 1\}\nu(dx) < \infty.$$

A Lévy process has three components: drift, diffusion, and jump. The Lévy-Khinchin formula with triple (a, σ^2, ν) fully determines the processes; therefore, one can see that a purely continuous Lévy process is Brownian motion with drift.

2.2 $\alpha-$stable process

The present study focuses on $\alpha-$stable stochastic processes. A real random variable X is said to have stable distribution whenever X_1 and X_2 are independent random variables, having the same distribution as X, then $c_1X_1 + c_2X_2$ has the same distribution as $cX + d$ for some c, d depending on c_1 and c_2. Moreover, there exists a positive real constant α, called the index of stability, such that $c^\alpha = c_1^\alpha + c_2^\alpha$. X is strictly stable if $d = 0$ for all c_1 and c_2. The characteristic function of the stable random variable X is

$$\begin{aligned}\varphi(\theta) =& E(e^{i\theta X})\\ =& \begin{cases} \exp\{-\sigma^\alpha|\theta|^\alpha\{1 - i\beta(\text{sign } \theta)\ \tan(\frac{\pi\alpha}{2})\} + i\mu\theta\}, & \alpha \neq 1, \\ \exp\{-\sigma|\theta|\{1 + i\beta\frac{2}{\pi}(\text{sign } \theta)\ln|\theta|\} + i\mu\theta\}, & \alpha = 1. \end{cases}\end{aligned} \tag{2.7}$$

where, $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\sigma \in [0, \infty)$ and $\mu \in \mathbb{R}$ are called indices, skewness, scale and shift parameter, respectively, then $X \sim S_\alpha(\sigma, \beta, \mu)$. X is called a symmetric $\alpha-$stable random variable if $\beta = \mu = 0$. In other words, if X has the form $S_\alpha(\sigma, 0, 0)$, in that case we also write $X \sim S\alpha S$. The probability densities of α stable random variables are continuous but do not have closed forms except for $\alpha = \frac{1}{2},\ \alpha = 1,\ \alpha = 2$. For more details refer to [14, 18].

A stochastic process $\{X_t :\ t \in T\}$, where T is an arbitrary set, is stable if all its finite dimensional distributions

$$(X(t_1), X(t_2), \ldots, X(t_n)),\ t_1, t_2, \ldots, t_n \in T,\ n \geq 1.$$

are stable for the same index. It is symmetric stable if all its finite dimensional distributions are symmetric stable for the same index as well [14, 18].

Definition 2.5. A stochastic process $\{X(t) : t \geq 0\}$ is called (standard) $\alpha-$stable lèvy motion if

1. $X(0) = 0\ \ a.s.$;
2. $\{X(t) : t \geq 0\}$ has independent increment;
3. $X(t) - X(s) \sim S_\alpha((t-s)^{\frac{1}{\alpha}}, \beta, 0)$.

Regarding processes having independent increments without Brownian component, that is, processes with the Laplace transform [7]:

$$E(e^{-uX(t)}) = \exp\{btu + t\int_{\mathbb{R}\setminus\{0\}}\{1 - ux\mathbf{1}_{\{|x|<1\}} - e^{-ux}\}\nu(dx)\} \tag{2.8}$$

where ν and b present Lévy measure and the drift for $\alpha-$stable processes, respectively, and $\nu(dx) = \frac{c\ dx}{|x|^{\alpha+1}}$ for some constant c. Taking $u = -i\theta$ in (2.8), we get to (2.6). It reconfirms that the $\alpha-$stable motions are Lévy processes.

3 SDE's driven by $\alpha-$stable Lévy motion

In this section, an SDE involving stochastic integrals with $\alpha-$stable integrators are defined by α stable Lévy motion,

$$dX(t) = \mu(t, X(t))\ dt\ + \sigma(t, X(t))\ dS_\alpha(t), \qquad X(0) = X_0, \tag{3.1}$$

which the integral form

$$X(t) = X_0 + \int_0^t \mu(s, X(s_-))\, ds + \int_0^t \sigma(s, X(s_-))\, dS_\alpha(s), \tag{3.2}$$

is taken into consideration, where $\{S_\alpha(t)\}_{t\geq 0}$ is a $S_\alpha(t^{\frac{1}{\alpha}}, \beta, 0)$ Lévy motion. It includes SDEs with Brownian processes integrator: $S_2(t) = \sqrt{2}\, B(t),\ \beta = 0$. According to the discussions of Sections 2.1 and 2.2 the integral equation (3.2) is a special case of the equation (2.4). To verify this claim [19], let

$$Z_t = [t, S_\alpha(t)]', \quad J(t) = X_0, \quad F(X)_{t_-} = [\mu(t, X(t_-)), \sigma(t, X(t_-))].$$

Therefore, we can state Theorem 3.1 as follows:

Theorem 3.1. *The integral equation (3.2) has a unique solution, provided the functions μ and σ satisfy the Lipchitz condition (note that a Lipschitz function is also functional Lipschitz).*

Having the above suitable choices of $J(t), F(t), Z(t)$ in (2.5) and (3.2), the following equation is obtained

$$X^{(n+1)}(t) = X_0 + \int_0^t \left[\mu(s, X^{(n)}(s)), \sigma(s, X^{(n)}(s))\right] d\begin{bmatrix} s \\ S_\alpha(s)\end{bmatrix} \tag{3.3}$$

which is equivalence to

$$X^{(n+1)}(t) = X_0 + \int_0^t \mu(s, X^{(n)}(s_-))\, ds + \int_0^t \sigma(s, X^{(n)}(s_-))\, dS_\alpha(s). \tag{3.4}$$

In which, ucp to $X(t)$, the solution of (3.2). On the other hand, for the simple functions,

$$\hat{\mu}(s, X(s_-)) = \sum_{i=0}^{k-1} \mu(t_i, X(t_i))\mathbf{1}_{[t_i, t_{i+1})}(s) \tag{3.5}$$

$$\hat{\sigma}(s, X(s_-)) = \sum_{i=0}^{k-1} \sigma(t_i, X(t_i))\mathbf{1}_{[t_i, t_{i+1})}(s) \tag{3.6}$$

approximate $\mu(s, X(s))$ and $\sigma(s, X(s))$ respectively, for each $s \in [0, T]$. Now, by helping Theorem 16 and its corollary and argument given on page 276 of [17], we obtain that the following integral equation

$$\hat{X}(t) = X_0 + \int_0^t \hat{\mu}(s, X(s_-))\ ds + \int_0^t \hat{\sigma}(s, X(s_-))\ dS_\alpha(s) \tag{3.7}$$

$$\begin{aligned} \widehat{X}(t) = X_0 + &\int_0^t \sum_{i=0}^{k-1} \mu(t_i, X(t_i)) \mathbf{1}_{[t_i, t_{i+1}]}(s) ds \\ &+ \int_0^t \sum_{i=0}^{k-1} \sigma(t_i, X(t_i)) \mathbf{1}_{[t_i, t_{i+1}]}(s) dS_\alpha(s) \end{aligned} \tag{3.8}$$

i.e. converges in probability to $X(t)$, the exact solution of (3.2).

In (3.7), the time intervals $[0, T]$ are fixed by schemes based on equidistant time discretization points $t_n = nh,\ n = 0, 1, \ldots, N$ with step size $h = \dfrac{T}{N},\ N = 1, 2, \ldots$ Then

$$\begin{aligned} X(t_1) = X(t_0) + &\mu(t_0, X(t_0))(t_1 - t_0) \\ &+ \sigma(t_0, X(t_0))\ (S_\alpha(t_1) - S_\alpha(t_0)) \end{aligned} \tag{3.9}$$

Inductively for $t = t_{n+1}, i = 0, 1, \ldots, N-1$ in (3.9) we get to

$$\begin{aligned} X(t_{n+1}) = X(t_n) + &\mu(t_n, X(t_n))(t_{n+1} - t_n) \\ &+ \sigma(t_n, X(t_n))\ (S_\alpha(t_{n+1}) - S_\alpha(t_n)) \end{aligned} \tag{3.10}$$

or equivalently

$$\begin{aligned} X(t_{n+1}) = X(t_n) + &\mu(t_n, X(t_n))\Delta t_n \\ &+ \sigma(t_n, X(t_n))\Delta S_\alpha(t_n),\ n = 0, 1, ..., N-1 \end{aligned} \tag{3.11}$$

The above argument proves the following theorem.

Theorem 3.2. *The family* $\Big\{ X(t_n) :\ \ n = 0, 1, ..., N \Big\}$ *given by* (3.11) *with drift* $\mu(t, x)$ *and cofficient function* $\sigma(t, x)$ *define on* $[0, T] \times \mathbb{R}$ *under Lipschitz condition converges in probability to the exact solution of* (3.2) *on* $[0, T]$ *as* h *tends to zero.*

4 Simulation of $\alpha-$stable stochastic processes and diffusions driven by $\alpha-$stable processes

One of the main goals in this study is to derive some computational methods of simulating α stable processes and numerical solution of SDEs of α stable processes. To obtain an SαS random variable with unit dispersion, $\alpha \in (0, 2]$, the method given in [7, 14] is followed. Let U be a random variable distributed uniformly on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and independent of E, which is distributed exponentially with mean 1, then S is distributed as $S_\alpha(1, 0, 0)$.

$$S = \begin{cases} \dfrac{\sin(\alpha U)}{(\cos U)^{\frac{1}{\alpha}}} \left[\dfrac{\cos((1-\alpha)U)}{E}\right]^{\dfrac{1-\alpha}{\alpha}}, & \alpha \neq 1, \\ \tan(U), & \alpha = 1. \end{cases} \tag{4.1}$$

Also, for skew parameter $\beta \in [-1, 1]$, define $\theta = \dfrac{\arctan(\beta \tan \frac{\pi\alpha}{2})}{\alpha}$, when $\alpha \neq 1$ then S has $S_\alpha(1, \beta, 0)$ distribution. It is noted that $S' = \sigma^{\frac{1}{\alpha}} S + \mu$ is distributed as $S_\alpha(\sigma, \beta, \mu)$.

$$S = \begin{cases} \dfrac{\sin(\alpha(\theta + U))}{(\cos(\alpha\theta)\cos(U))^{\frac{1}{\alpha}}} \left[\frac{\cos(\alpha\theta + (1-\alpha)U)}{E}\right]^{\frac{1-\alpha}{\alpha}}, & \alpha \neq 1, \\ \dfrac{2}{\pi}\left[(\dfrac{\pi}{2} + \beta U)\tan(U) - \beta \log(\dfrac{\frac{\pi}{2} E\cos(U)}{\dfrac{\pi}{2} + \beta U})\right], & \alpha = 1. \end{cases} \tag{4.2}$$

In order to approximate the process $X(t)$, $t \in [0, T]$, the solution of Equation (3.1), we partition $[0, T]$ as $h = \dfrac{T}{N}, t_n = nh, n = 0, 1, \cdots, N$. The stochastic stable measure $S_\alpha([t_{n-1}, t_n)) \overset{\mathrm{d}}{=} S_\alpha(t_n) - S_\alpha(t_{n-1}) \sim S_\alpha(h, \beta, 0)$ and $X(0) = X_0 \sim S_\alpha(0.001, \beta, 0)$ are constructed by (4.1) and (4.2), respectively, and Equation (3.11) is used to evaluate $X(t_{n+1}), n = 0, 1, \cdots, N-1$. The order of convergence of this method which is known as Euler-Maruyama method is not obtained yet in the α stable case.

4.1 Result and conclusion

Concerning convergence in probability of semimartingle, Theorems 2.2 and 2.3 have been stated. By using Theorem 2.4, it has been poven that $\alpha-$stable Lévy motion are semimartingle which is discussed in subsection 2.2. Moreover, it has been proven that Euler-Maruyama equation is special case of equation (2.5) and it convergence to $X(t)$ in probability. This means that SDE driven by $\alpha-$stable Lévy motions have removed the ambiguities.

4.2 Examples

Denoting $\widetilde{X}^{(k)}(t_n)$, the EM approximation of $X^{(k)}(t_n)$, the exact solution at step point t_n in the kth simulation of all 1000 simulations, we define the difference mean of exact and estimated solution at $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ and the difference mean at T by Equations (4.3) and (4.4), respectively.

$$E_1 = \frac{1}{1000} \sum_{k=1}^{1000} \sum_{n=0}^{N} |X^{(k)}(t_n) - \widetilde{X}^{(k)}(t_n)| \tag{4.3}$$

$$E_2 = \frac{1}{1000} \sum_{k=1}^{1000} |X^{(k)}(T) - \widetilde{X}^{(k)}(T)| \tag{4.4}$$

In Examples 1,2 and 3, the exact solutions are known so we obtain E_1 and E_2. We consider "Logistic Model of Population Growth" in Example 4. The exact solution is not yet known and the averages of 1000 EM simulations at $t_0, t_1, \cdots, t_N$ are plotted for different values of α and β.

Example 4.1. [3, page 46][Geometric $\alpha-$stable process] Consider a stochastic differential equation driven by an α stable process

$$\begin{cases} dX(t) = \lambda X(t)\ dt + \mu X(t)\ dS_\alpha(t), \\ X(0) \in \mathbb{R}, t \in [0, 2]. \end{cases} \tag{4.5}$$

The following equation is the exact solution of (4.5)

$$X(t) = X(0)\ \exp\{(\lambda - \frac{1}{2}\mu^2)t + \mu S_\alpha(t)\}. \tag{4.6}$$

Table 4.1 shows the mean of absolute errors, i.e. E_1 and E_2, for equation (4.5) when $\beta = -0.7$, $\lambda = 0.25$, and $\mu = 0.1$ and for different values of α at two different distances of $h = 10^{-2}$ and $h = 10^{-3}$.

Table 4.1: Mean of absolute error for Equation (4.5), $\beta = -0.7,\ \lambda = 0.25,\ \mu = 0.01$

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-2}	$1.6\ 10^{-6}$	$3.9\ 10^{-6}$	$1.7\ 10^{-5}$	$4.0\ 10^{-5}$	$1.3\ 10^{-4}$	$3.1\ 10^{-4}$	$3.4\ 10^{-6}$	$8.1\ 10^{-6}$
10^{-3}	$3.2\ 10^{-7}$	$7.4\ 10^{-7}$	$1.5\ 10^{-6}$	$3.5\ 10^{-6}$	$2.2\ 10^{-5}$	$5.0\ 10^{-5}$	$2.6\ 10^{-7}$	$6.1\ 10^{-7}$

Table 4.2 shows the mean of absolute error, i.e. E_1 and E_2 for equation (4.5) when $\beta = 0$, $\lambda = 0.25$, and $\mu = 0.1$ and for different values of α at two different distances of of $h = 10^{-2}$ and $h = 10^{-3}$.

Table 4.2: Mean of absolute error for Equation (4.5), $\beta = 0,\ \lambda = 0.25,\ \mu = 0.1$

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-2}	$2.1\ 10^{-6}$	$4.9\ 10^{-6}$	$1.1\ 10^{-5}$	$2.5\ 10^{-5}$	$7.4\ 10^{-4}$	$1.3\ 10^{-3}$	$4.3\ 10^{-5}$	$1.0\ 10^{-4}$
10^{-3}	$1.5\ 10^{-7}$	$8.6\ 10^{-7}$	$2.9\ 10^{-6}$	$6.8\ 10^{-6}$	$2.0\ 10^{-4}$	$4.6\ 10^{-4}$	$4.8\ 10^{-5}$	$1.1\ 10^{-4}$

Figure 4.1 presents the mean of 1000 discredited $\alpha-$stable paths for equations (4.6) and Euler-Maruyama method when $\lambda = 0.25$, $\mu = 0.1$ and $h = 10^{-3}$.

Example 4.2. [7, page 109][Ornstein-Uhlenbeck process]
Let's study the following SDE

$$\begin{cases} dX(t) = -\lambda X(t)\ dt + dS_\alpha(t), \\ X(0) \in \mathbb{R}, \lambda > 0, t \in [0, 2]. \end{cases} \tag{4.7}$$

The following $\alpha-$stable stochastic process is explicit solution of (4.7) [7, page 109]

$$X(t) = X(0)\ e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}\ dS_\alpha(s). \tag{4.8}$$

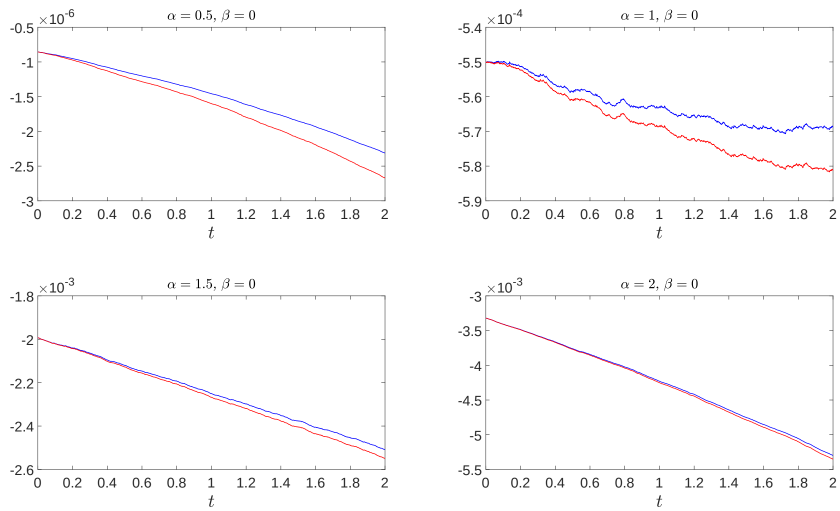

Figure 4.1: Mean of 1000 discredited $\alpha-$stable paths for Equation (4.6) denoted with red line and Euler-Maruyama method denoted by blue line for $\lambda = 0.25,\ \mu = 0.01$ and $h = 10^{-3}$.

The stochastic integral in (4.8) is approximated by

$$\int_0^{t_i} e^{-\lambda(t-s)}\ dS_\alpha(s) \approx \sum_j e^{-\lambda(t_i - s_j)}\ \Delta S_\alpha(s_j).$$

Similar to Example 4.1, Figure 4.2 compares the exact solution with the Euler-Maruyama estimated solution for 1000 $\alpha-$stable paths.

Table 4.3 shows the mean of absolute errors, i.e. E_1 and E_2 for equation (4.7) when $\beta = 0$, $\lambda = 2$, and for different values of α at three different distances of $h = 10^{-1}$, $h = 10^{-2}$, and $h = 10^{-3}$.

Table 4.3: Mean of absolute error for Equation (4.7), $\beta = 0,\ \lambda = 2$.

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-1}	$3.3\ 10^{-2}$	$4.7\ 10^{-2}$	$1.4\ 10^{-1}$	$2.3\ 10^{-1}$	$2.2\ 10^{-2}$	$4.3\ 10^{-2}$	$5.2\ 10^{-2}$	$8.2\ 10^{-2}$
10^{-2}	$1.0\ 10^{-3}$	$1.4\ 10^{-3}$	$1.8\ 10^{-2}$	$2.5\ 10^{-2}$	$2.0\ 10^{-3}$	$5.0\ 10^{-3}$	$1.5\ 10^{-3}$	$8.0\ 10^{-3}$
10^{-3}	$2.0\ 10^{-4}$	$2.8\ 10^{-4}$	$3.5\ 10^{-3}$	$4.5\ 10^{-3}$	$5.5\ 10^{-4}$	$7.2\ 10^{-4}$	$4.9\ 10^{-4}$	$8.8\ 10^{-4}$

Table 4.4 shows the mean of absolute errors, i.e. E_1 and E_2 for equation (4.7) when $\beta = 0.7,\ \lambda = 2$, and for different values of α at three different distances of $h = 10^{-1}$, $h = 10^{-2}$, and $h = 10^{-3}$.

Table 4.4: Mean of absolute error for Equation (4.7), $\beta = 0.7,\ \lambda = 2$.

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-1}	$4.4\ 10^{-2}$	$4.9\ 10^{-2}$	$4.9\ 10^{-3}$	$8.5\ 10^{-2}$	$3.2\ 10^{-2}$	$6.1\ 10^{-2}$	$1.4\ 10^{-2}$	$6.8\ 10^{-2}$
10^{-2}	$6.0\ 10^{-3}$	$6.6\ 10^{-3}$	$1.4\ 10^{-2}$	$1.5\ 10^{-2}$	$5.8\ 10^{-3}$	$8.5\ 10^{-3}$	$5.5\ 10^{-3}$	$9.2\ 10^{-3}$
10^{-3}	$3.4\ 10^{-4}$	$5.1\ 10^{-4}$	$3.1\ 10^{-3}$	$3.3\ 10^{-3}$	$2.7\ 10^{-3}$	$2.9\ 10^{-3}$	$5.7\ 10^{-4}$	$9.4\ 10^{-4}$

Figure 4.2 presents the mean of 1000 discredited $\alpha-$stable paths for equation (4.7) and Euler-Maruyama method when $\lambda = 2$, and $h = 10^{-3}$.

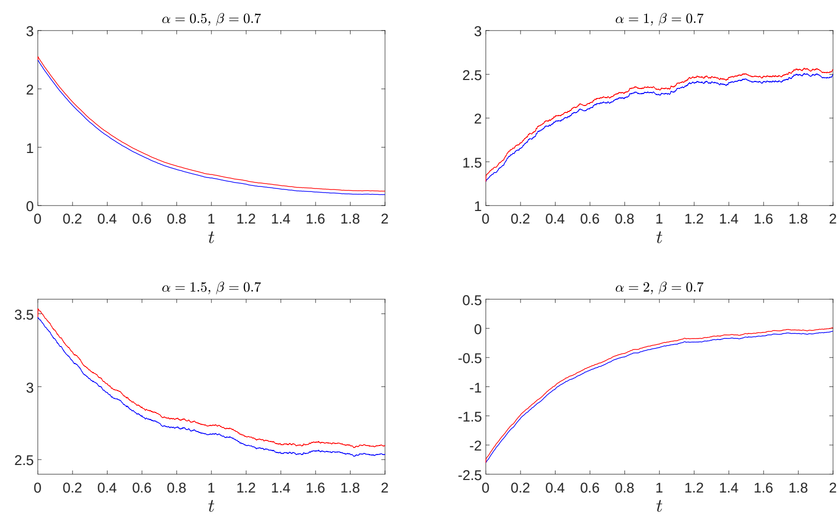

Figure 4.2: Mean of 1000 discredited $\alpha-$stable paths for Equation (4.8) denoted by red line and Euler-Maruyama method denoted by blue line for $\lambda = 2$ and $h = 10^{-3}$.

Example 4.3 (Resisitive - inductive electrical circuit)**.** Consider the following $\alpha-$stable SDE

$$\begin{cases} dX(t) = (4\sin(t) - X(t))\ dt + \frac{1}{L}\ dS_\alpha(t), \\ X(0) = X_0 \in \mathbb{R}, L > 0, t \in [0, 2]. \end{cases} \tag{4.9}$$

First, we prove that Equation (4.9) has exact solution, then similar to previous examples, the exact and estimated solutions are compared. Explicit solution can be obtained, as ordinary differential equations case, by the variational of parameters technique. First, the solution following homogeneous linear $\alpha-$stable equation is obtained:

$$\begin{cases} dX_0(t) = -X_0(t)\ dt, \\ X_0(0) = 1, t \geq 0. \end{cases} \tag{4.10}$$

According to Example 4.1, the following equation is explicit form for Equation (4.10)

$$X_0(t) = e^{-t},\ t \geq 0. \tag{4.11}$$

Now, writing the solution of non-homogeneous Equation (4.9) as

$$X(t) = Y(t)X_0(t) = e^{-t}Y(t),$$

the problem is determined by $Y(t) = X_0^{-1}(t)X(t) = e^t X(t)$, which gives

$$\begin{aligned} dY(t) = d(e^t X(t)) &= e^t X(t)\ dt + e^t\ dX(t) \\ &= e^t\ X(t)\ dt + e^t\ ((4\sin(t) - X(t))\ dt + \frac{1}{L}\ dS_\alpha(t)) \\ &= 4\ e^t \sin(t)\ dt + \frac{1}{L}\ e^t dS_\alpha(t) \end{aligned}$$

therefore,

$$\begin{aligned} Y(t) &= Y(0) + 4\int_0^t e^s \sin(s)\ ds + \frac{1}{L}\int_0^t e^s\ dS_\alpha(s) \\ &= Y(0) + 2(1 + e^t \sin(t) - e^t \cos(t)) + \frac{1}{L}\int_0^t e^s\ dS_\alpha(s) \end{aligned}$$

hence, $X(t)$, the solution of Equation (4.9), can be written by

$$X(t) = 2\ (e^{-t} + \sin(t) - \cos(t)) + e^{-t}\ X(0) + \int_0^t \frac{1}{L}\ e^{s-t}\ dS_\alpha(s). \tag{4.12}$$

We approximat the last part of (4.12) by

$$\int_0^{t_i} \frac{1}{L}\, e^{s-t}\, dS_\alpha(s) \approx \frac{1}{L} \sum_j\, e^{s_j - t_i}\, \Delta S_\alpha(s_j).$$

Table 4.5 shows the mean of absolute errors, i.e. E_1 and E_2 for equation (4.9) when $\beta = -0.7$, $L = 2$, and for different values of α at three different distances of $h = 10^{-1}$, $h = 10^{-2}$, and $h = 10^{-3}$.

Table 4.5: Mean of absolute error for Equation (4.9), $\beta = -0.7 and L = 2$.

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-1}	$3.0\ 10^{-2}$	$8.3\ 10^{-2}$	$3.1\ 10^{-2}$	$4.6\ 10^{-2}$	$3.3\ 10^{-2}$	$3.7\ 10^{-2}$	$2.8\ 10^{-2}$	$6.0\ 10^{-2}$
10^{-2}	$4.3\ 10^{-2}$	$5.3\ 10^{-2}$	$5.8\ 10^{-3}$	$9.6\ 10^{-3}$	$2.6\ 10^{-3}$	$4.2\ 10^{-3}$	$3.1\ 10^{-3}$	$4.3\ 10^{-3}$
10^{-3}	$3.0\ 10^{-3}$	$3.3\ 10^{-3}$	$2.6\ 10^{-3}$	$2.7\ 10^{-3}$	$2.2\ 10^{-3}$	$2.2\ 10^{-3}$	$2.7\ 10^{-4}$	$5.9\ 10^{-4}$

Table 4.6 shows the mean of absolute errors, i.e. E_1 and E_2 for equation (4.9) when $\beta = 0.7$, $L = 2$, and for different values of α at three different distances of $h = 10^{-1}$, $h = 10^{-2}$, and $h = 10^{-3}$.

Table 4.6: Mean of absolute error for Equation (4.9), $\beta = 0.7 and L = 2$.

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	E_1	E_2	E_1	E_2	E_1	E_2	E_1	E_2
10^{-1}	$3.1\ 10^{-2}$	$4.4\ 10^{-2}$	$3.3\ 10^{-2}$	$1.0\ 10^{-1}$	$2.9\ 10^{-2}$	$3.8\ 10^{-2}$	$3.4\ 10^{-2}$	$4.1\ 10^{-2}$
10^{-2}	$4.3\ 10^{-2}$	$7.3\ 10^{-2}$	$4.3\ 10^{-3}$	$1.2\ 10^{-2}$	$9.5\ 10^{-3}$	$1.9\ 10^{-2}$	$2.8\ 10^{-3}$	$5.2\ 10^{-3}$
10^{-3}	$2.7\ 10^{-4}$	$5.6\ 10^{-4}$	$2.5\ 10^{-4}$	$7.3\ 10^{-4}$	$9.8\ 10^{-4}$	$2{,}0\ 10^{-3}$	$2.9\ 10^{-4}$	$4.6\ 10^{-4}$

Figure 4.3 presents the mean of 1000 discredited $\alpha-$stable paths for equation (4.12) and Euler-Maruyama method when $L = 2$, and $h = 10^{-3}$.

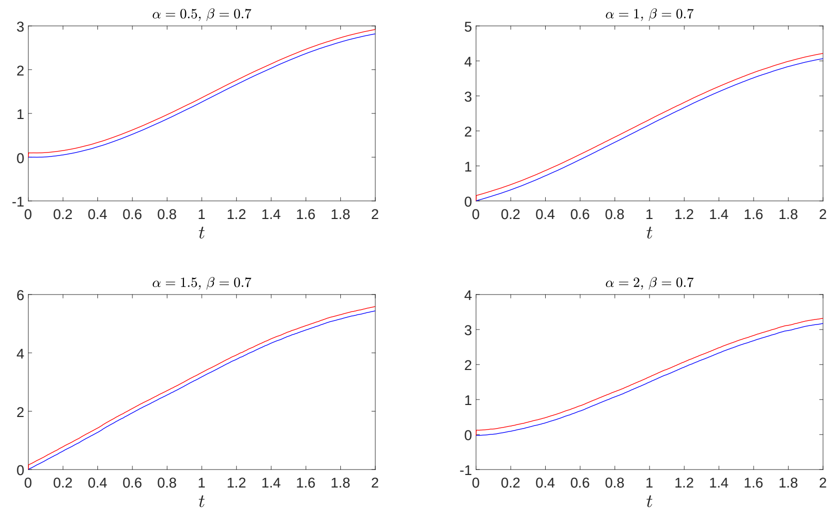

Figure 4.3: Mean of 1000 discredited $\alpha-$stable paths for Equation (4.12) denoted by red line and Euler-Maruyama method denoted by blue line for $L = 2$ and $h = 10^{-3}$.

Example 4.4. (Logistic Model of Population Growth)
The logistic model to describe the growth of a population subject to a fixed food supply with $\alpha-$stable random effect is

$$\begin{cases} dX(t) = X(t)\ (K - X(t))\ dt + a_k\ X(t)\ dS_\alpha(t)., \\ X(0) \in \mathbb{R}, t \in [0, 2]. \end{cases} \tag{4.13}$$

where K is the food supply in the population unit. To the best of our knowledge, the exact solution has not been obtained yet. Therefore, just the solutions are simulated in $[0, 2]$.

Figure 4.4 presents the mean of 1000 discredited $\alpha-$stable paths for equation (4.13) and Euler-Maruyama method when $L = 2$, and $h = 10^{-3}$.

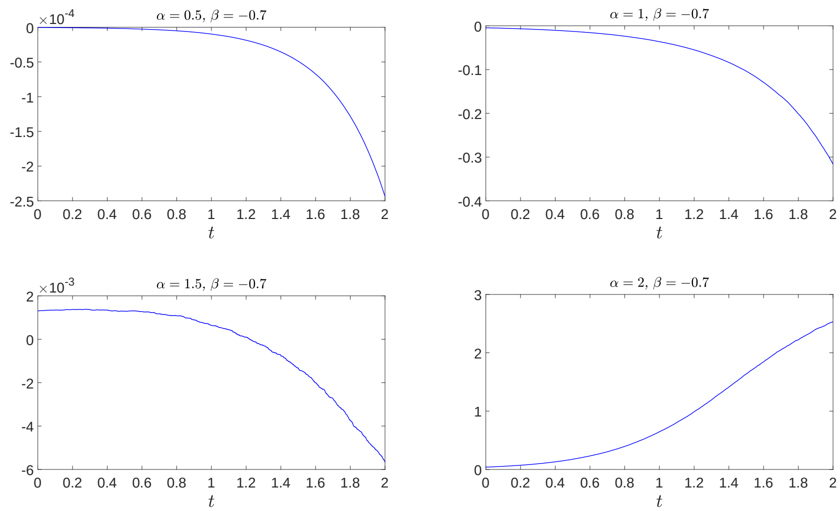

Figure 4.4: Mean of 1000 paths of Euler-Maruyama method for Equation (4.13) for $K = 3, a_k = 0.3$ and $h = 10^{-3}$.

5 Compare with other methods

Regarding comparison of the presented method in our article with other methods, it should be mentioned that we have considered $\alpha \in (0, 2]$ while some authors [11] have used $\alpha = 2$ (Wiener process). Moreover, Example 4.1 for our article was applied to the method presented by Janicki et al [6]. In this section we denot the NEu and Eu approximation, the approximation of equation (4.5) by method (5.1) and the approximation of equation (4.5) by method (3.11), respectively. The result indicated that the E_1 and E_2 values obtained by their method where higher than our presented method in this article.

$$\begin{aligned} X_n\left(\frac{k+1}{n}\right) =& X_n\left(\frac{k}{n}\right) + \frac{1}{n} a\left(\frac{k}{n}, X_n\left(\frac{k}{n}\right)\right) \\ & + b\left(\frac{k}{n}, X_n\left(\frac{k}{n}\right)\right) L_{\alpha,\beta}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right)\right) \end{aligned} \tag{5.1}$$

Table 5.1: Comparison of E_2 error in methods (5.1) and (3.11).

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	NEu	Eu	NEu	Eu	NEu	Eu	NEu	Eu
10^{-2}	$1.2\ 10^{-6}$	$5.2\ 10^{-9}$	$6.1\ 10^{-5}$	$3.4\ 10^{-6}$	$3.3\ 10^{-3}$	$8.7\ 10^{-5}$	$2.1\ 10^{-3}$	$6.7\ 10^{-6}$
10^{-3}	$8.0\ 10^{-8}$	$5.7\ 10^{-11}$	$4.9\ 10^{-5}$	$7.5\ 10^{-9}$	$2.3\ 10^{-3}$	$3.1\ 10^{-6}$	$1.7\ 10^{-3}$	$1.2\ 10^{-6}$

Table 5.2: Comparison of E_2 error in methods (5.1) and (3.11).

h	$\alpha = 0.5$		$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$	
	NEu	Eu	NEu	Eu	NEu	Eu	NEu	Eu
10^{-2}	$3.4\ 10^{-8}$	$2.2\ 10^{-10}$	$1.2\ 10^{-3}$	$9.0\ 10^{-6}$	$4.2\ 10^{-3}$	$2.4\ 10^{-5}$	$4.1\ 10^{-4}$	$1.3\ 10^{-6}$
10^{-3}	$3.3\ 10^{-7}$	$1.6\ 10^{-11}$	$3.1\ 10^{-4}$	$1.8\ 10^{-7}$	$4.8\ 10^{-3}$	$2.2\ 10^{-6}$	$3.8\ 10^{-3}$	$2.9\ 10^{-6}$

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