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New Results on Monotonicity

Z. S. Mirsaney
University of Isfahan

M. Rezaie*
University of Isfahan

Abstract. The purpose of this paper is to show that under reasonable assumptions Debrunner and Flor Theorem can be extended to arbitrary θ-monotone operators. This generalization provides some tools for further analysis of the θ-monotone operators, which allows us for establishing some key facts related to domains and ranges of θ-maximal monotone operators.

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1. Introduction

In the recent years the concept of monotonicity for multivalued operators defined on a Banach space and taking values in its dual have turned out to be very useful in the study of various branches of mathematics, such as differential equations, economics, engineering, management science, probability theory, bifunctions etc. (See [1, 3, 4, 5, 7, 14, 15]). In 2012 Szilárd László in [12] introduced a new monotonicity concept for an operator, namely θ-monotonicity, and presented some fundamental properties of the operators having this monotonicity property.

Since the θ-monotonicity concept is more general than most of the monotonicity notions known in literature, it would be interesting to study these operators.

A very important fact concerning monotone sets is the classical extension theorem of Debrunner and Flor [8]. This theorem is used for establishing some key facts related to domains and ranges of maximal monotone operators. Inspired by this theorem and its proof in [6], we establish some result in a rather more general situation in which we work with θ-monotone operators. The paper is

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*Corresponding author

organized as follows. After preliminaries, in main results, under reasonable assumptions, we generalize Debrunner and Flor Theorem. This generalization is further analyzed and illustrated by an example. Finally we present an application of our result in which we will show that a maximal θ-monotone operator with bounded range must be defined everywhere.

2. Notations and Preliminaries

Through this paper we will assume that X is a real Banach space and X^* is its dual. The norm in X and X^* will be denoted by $\|.\|$. Also, by w and w^*, we denote weak and weak star topology in X and X^*, respectively. The symbol $< .,. >$ will be used for the associated duality pairing, i.e., $< x^*, x >= x^*(x)$, for all $x \in X$ and $x^* \in X^*$.
Given any set $D \subseteq X$, the convex hull of D, denoted co(D), is the smallest convex set containing D.
Let $T : X \to 2^{X^*}$ be a multivalued operator. We denote its domain by $D(T) = \{x \in X : T(x) \neq \emptyset\}$ and its range by $R(T) = \bigcup\limits_{x \in D(T)} T(x)$. The graph of the operator T is the set $G(T) = \{(x, u) \in X \times X^* : u \in T(x)\}$. Let $\theta : X \times X \longrightarrow \mathbb{R}$ be a given function with the property that $\theta(x, y) = \theta(y, x)$ for all $x, y \in X$.

From now on, θ is positive (negative), this means that $\theta(x, y) > 0$ (< 0) for all $x,\ y \in X$ such that $x \neq y$.

Definition 2.1. *Given a subset M of $X \times X^*$, M is said to be θ-monotone if*

$$\langle u - v, x - y\rangle \geqslant \theta(x, y)\|x - y\|, \quad \forall(x, u), (y, v) \in M.$$

Also, $T : X \to 2^{X^*}$ is θ-monotone, if its graph is θ-monotone, which means that

$$\langle u - v, x - y\rangle \geqslant \theta(x, y)\|x - y\|, \quad \forall(x, u), (y, v) \in G(T). \tag{1}$$

Moreover a θ-monotone operator T is maximal θ-monotone if for every operator $T' : X \to 2^{X^*}$ which is θ-monotone with $G(T) \subseteq G(T')$, one has $T = T'$.
If in Definition 2.1, $\theta(x, y) = 0$ for all $x, y \in D(T)$ we obtain the concept of Minty-Browder monotonicity (See [1,5, 14, 15]), i.e.

$$\langle u - v, x - y\rangle \geqslant 0, \quad \forall(x, u), (y, v) \in G(T). \tag{2}$$

Definition 2.2. *We say that the pair (x, x^*) is θ-monotonically related to a subset M of $X \times X^*$ if*

$$< x^* - y^*, x - y >\geqslant \theta(x, y)\|x - y\|, \quad \forall(y, y^*) \in M.$$

We will make use of the following result from [12].

Proposition 2.3. *A θ-monotone operator $T : X \to 2^{X^*}$ is maximal θ-monotone if and only if whenever a pair $(x, u) \in X \times X^*$ is θ-monotonically related to $G(T)$, it holds that $u \in T(x)$.*

One of the most basic theorems concerning monotone sets the so-called Debrunner and Flor Theorem.

Theorem 2.4. [Debrunner-Flor] *Let $C \subseteq X^*$ be a w^*-compact and convex set, $\varphi : C \longrightarrow X$ be a continuous from the w^* topology to the strong topology, and $M \subseteq X \times X^*$ be a monotone set. Then there exists $v \in C$ such that $\{(\varphi(v), v)\} \cup N$ is monotone.*

Proof. The proof can be founded in [6, Theorem 4.3.1]. $\square$

3. Main Results

A very important fact concerning θ-monotone sets is the classical extension theorem of Debrunner and Flor [8] by weaker monotonicity condition than Minty-Browder monotonicity. The idea and proof of the following theorem is in essence contained in [6], where only monotonicity were considered. We need some topological preliminaries to prove the main result.

Definition 3.1. *Let K be a subset of X, and $V = \{V_1, ..., V_p\}$ be a finite covering of K of open sets. A partition of unity associated with V is a set $\{\alpha_1, ..., \alpha_p\}$ of continuous functions $\alpha_i : X \longrightarrow \mathbb{R}(1 \leqslant i \leqslant p)$ satisfying*

a) $\sum_{i=1}^{p} \alpha_i(x) = 1,$ *for all* $x \in K$.

b) $\alpha_i(x) \geqslant 0,$ *for all* $x \in K$*and for all* $i \in \{1, ..., p\}$.

c) $\{x \in X : \alpha_i(x) > 0\} \subseteq V_i,$ *for all* $i \in \{1, ..., p\}$.

Proposition 3.2. *If K is compact, then for every finite covering V there exists a partition of unity associated with V.*

Proof. See Theorem VIII.4.2 in [9], where the result is proved under a condition on K weaker than compactness, namely paracompactness. $\square$

Theorem 3.3. *Let X be a finite-dimensional Euclidean space, K be a convex and compact subset of X, and $h : K \longrightarrow K$ be a continuous function. Then there exists $\bar{x} \in K$ such that $h(\bar{x}) = \bar{x}$.*

Proof. See the proof for the case in which K is the unit ball in [Theorem XVI.2.1 and Corollary XVI.2.2]. The extension to a general convex and compact K is elementary. $\square$

Before giving the main result, we recall the fact (See [2]):

Fact 3.4. By bdw^* we denote w^*-convergence for bounded nets and hence include all w^*-convergent sequences. It is known that $< .,. >$ is $\|.\| \times bdw^*$-continuous.

Theorem 3.5. *Let $C \subseteq X^*$ be a w^*-compact convex set, $N = M \times M^* \subseteq X \times C$ be θ-monotone set and $\varphi : C \longrightarrow X$ be a continuous function from the w^*-topology to the strong topology. Assume that all of the conditions below hold:*

- **(i)** *θ is negative and $\theta(x,x) = 0$ for all $x \in X$.*
- **(ii)** *$\theta(x,y) \geqslant \theta(x^{'},y^{'})$ for all $(x,y) \in M \times M$ and $(x^{'},y^{'}) \in (M \times M)^c$.*
- **(iii)** *$\varphi(C) \subseteq M^c$.*
- **(iv)** *θ is continuous on $\{(x,y) \in X \times X : x \neq y$ and $y \in M^c\}$.*

Then there exists $u \in C$ such that $\{(\varphi(u), u)\} \cup N$ is θ-monotone.

Proof. For each element $(y,v) \in N$ define the set

$$U(y,v) := \{u \in C :< u - v, \varphi(u) - y >< \theta(\varphi(u), y)\|\varphi(u) - y\|\}. \tag{3}$$

Clearly, if $u \in U(y,v)$ then $\varphi(u) \neq y$. Define the function $p : C \longrightarrow \mathbb{R}$ as $p(u) =< u - v, \varphi(u) - y > -\theta(\varphi(u), y)\|\varphi(u) - y\|$. The Fact 3.4, together with continuity of φ and w^*-compactness of C (as a consequence, every net in C is bounded) yields p is w^*- continuous and then the sets $U(y,v)$ are w^*-open.
Assume that the conclusion of the theorem is not true. This means that for every $u \in C$, there exists $(y,v) \in N$ such that $u \in U(y,v)$. Thus the family of open sets $\{U(y,v)\}_{(y,v)\in N}$ is an open covering of C. By compactness of C, there exists a finite subcovering $\{U(y_1,v_1),...,U(y_n,v_n)\}$ of C.
By Proposition 3.2, associated with this finite subcovering there exists a partition of unity as in Definition 3.1; that is, functions $\alpha_i : X^* \longrightarrow \mathbb{R}(1 \leqslant i \leqslant n)$ that are w^*- continuous and satisfy

(a) $\sum_{i=1}^{n} \alpha_i(u) = 1$ for all $u \in C$.

(b) $\alpha_i(u) \geqslant 0$ for all $u \in C$ and all $i \in \{1,...,n\}$.

(c) $\{u \in C : \alpha_i(u) > 0\} \subseteq U_i := U(y_i, v_i)$ for all $i \in \{1,...,n\}$.

Let $K := \mathrm{co}\{v_1, ..., v_n\} \subseteq C$. Then K can be identified with a finite-dimensional convex and compact set. Consider the function $\rho : K \longrightarrow K$ defined as $\rho(u) := \sum_{i=1}^{n} \alpha_i(u)v_i$. Clearly, ρ is w^*- continuous. Since w^*- continuous coincides with strong continuous in the finite-dimensional vector space spanned by K, thus we can invoke Theorem 3.3 in order to conclude that there exists $w \in K$ such that $\rho(w) = w$.
Therefore, we have

$$\begin{aligned} 0 &=< \rho(w) - w, \sum_j \alpha_j(w)y_j - \varphi(w) > \\ &=< \sum_i \alpha_i(w)(v_i - w), \sum_j \alpha_j(w)y_j - \varphi(w) > \\ &= \sum_{i,j} \alpha_i(w)\alpha_j(w) \underbrace{< v_i - w, y_j - \varphi(w) >}_{a_{ij}}. \end{aligned} \tag{4}$$

Note that

$$a_{ij} + a_{ji} = a_{ii} + a_{jj} + < v_i - v_j, y_j - y_i > \leqslant a_{ii} + a_{jj} - \theta(y_i, y_j)\|y_i - y_j\|.$$

Combining these facts with (4) gives

$$\begin{aligned} 0 &= \sum_{i,j} \alpha_i(w)\alpha_j(w)a_{ij} \\ &= \sum_i \alpha_i(w)^2 a_{ii} + \sum_{i<j} \alpha_i(w)\alpha_j(w)(a_{ij} + a_{ji}) \\ &\leqslant \sum_i \alpha_i(w)^2 a_{ii} + \sum_{i<j} \alpha_i(w)\alpha_j(w)(a_{ii} + a_{jj}) \\ &- \sum_{i<j} \alpha_i(w)\alpha_j(w)\theta(y_i, y_j)\|y_i - y_j\|. \end{aligned} \tag{5}$$

Call $I(w) := \{i \in \{1, ..., n\} : w \in U_i\}$. By using property (c), triangle inequality and condition (ii), the inequality in (5) can be rewritten as

$$\begin{aligned} 0 \leqslant & \sum_{i\in I(w)} \alpha_i(w)^2 a_{ii} + \sum_{\substack{i < j \\ i, j \in I(w)}} \alpha_i(w)\alpha_j(w)(a_{ii} + a_{jj}) \\ & - \sum_{\substack{i < j \\ i, j \in I(w)}} \alpha_i(w)\alpha_j(w)\theta(y_i, y_j)\|y_i - y_j\| \\ \leqslant & \sum_{i\in I(w)} \alpha_i(w)^2 a_{ii} + \sum_{\substack{i < j \\ i, j \in I(w)}} \alpha_i(w)\alpha_j(w)\theta(y_i, \varphi(w))\|y_i - \varphi(w)\| \end{aligned}$$

$$+ \sum_{\substack{i<j \\ i,j\in I(w)}} \alpha_i(w)\alpha_j(w)\theta(y_j,\varphi(w))\|y_j-\varphi(w)\|$$

$$- \sum_{\substack{i<j \\ i,j\in I(w)}} \alpha_i(w)\alpha_j(w)\theta(y_i,y_j)\|y_i-y_j\|$$

$$\leqslant \sum_{i\in I(w)} \alpha_i(w)^2 a_{ii}$$

$$+ \sum_{\substack{i<j \\ i,j\in I(w)}} \alpha_i(w)\alpha_j(w)\|y_i-\varphi(w)\|(\theta(y_i,\varphi(w))-\theta(y_i,y_j))$$

$$+ \sum_{\substack{i<j \\ i,j\in I(w)}} \alpha_i(w)\alpha_j(w)\|y_j-\varphi(w)\|(\theta(y_j,\varphi(w))-\theta(y_i,y_j))$$

$$\text{by (ii), (iii)} \quad < 0.$$

The above inequalities yield $\alpha_i(w) = 0$ for all $i = 1, ..., n$, but this contradicts the fact that $\sum_{i=1}^{n} \alpha_i(u) = 1$ for all $u \in C$, completing the proof. □

The next corollary is an easy consequence of the Theorem 3.5.

Corollary 3.6. *Let $N = M \times M^* \subset X \times X^*$ be a θ-monotone set. Suppose that there exist a set $C \subseteq X^*$ and a function $\varphi : C \longrightarrow X$ such that C, φ and θ satisfying the conditions in Theorem 3.5, then N is not a maximal θ-monotone set.*

Example 3.7. Let $\varphi : [1, 2] \longrightarrow \mathbb{R}$ be such that $\varphi(x) = -x^2$. Suppose that θ is defined by

$$\theta(x,y) = \begin{cases} -\dfrac{\varepsilon}{2}, & \text{if } x, y \in M, x \neq y \\ 0, & \text{if } x = y \\ -\varepsilon, & \text{o. w} \end{cases}$$

It can be verified that $T : [2, 4] \longrightarrow [1, 2]$, $T(x) = \dfrac{x}{2}$ is a θ-monotone operator. It is easy to check that $M = [2, 4]$, $N = G(T)$, $C = [1, 2]$ and θ satisfy all conditions in Theorem 3.5.

Remark 3.8. *Obviously we obtain the same result if in Theorem 3.5 we replace the conditions* (i) -(iv) *by the condition in which $\theta(x, y) = -\varepsilon$. This means* *that N is ε-monotone (the concept of ε-monotonicity was considered in [11, 13]). The argument is similar to the one in the proof of Theorem 3.5, but we use the definition of θ in the last inequalities in the proof and conclude that*

$$0 \leqslant \sum_{i \in I(w)} \alpha_i(w)^2 a_{ii} < 0.$$

This will lead to a contradiction.

One of the more important applications of Theorem 3.5 is discussed in the following result, which states that a maximal θ-monotone operator with bounded range must be defined everywhere.

Theorem 3.9. *Let $T : X \to 2^{X^*}$ be maximal θ-monotone such that $R(T)$ is bounded and θ satisfies conditions in Theorem 3.5, with $N = G(T)$. Then $D(T) = X$.*

Proof. By boundedness of $R(T) \subseteq X^*$ and Banach–Alaoglu theorem we can find a w^*-compact convex set C containing $R(T)$. Suppose $x \in X \setminus D(T)$. Define $\varphi : C \longrightarrow X$ as $\varphi(v) = x$. Our assumption implies $\varphi(C) \subseteq D(T)^c$. By Theorem 3.5 and w^*–strong continuity of φ, there exists $v \in C$ such that the set $\{(\varphi(v), v)\} \cup G(T) = \{(x, v)\} \cup G(T)$ is θ-monotone. Since T is maximal, by Proposition 2.3, we have $v \in T(x)$, yielding $x \in D(T)$, which contradicts the assumption on x. Therefore all $x \in X$ are in $D(T)$. $\square$

From here comes a new idea for future investigations, namely to show celebrated results of maximal monotone operator with full domain remain true to maximal θ-monotone operator with full domain.

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Zahra Sadat Mirsaney
Assistant Professor of Mathematics
Department of Mathematics
University of Isfahan
Isfahan, Iran
E-mail: z.mirsaney@sci.ui.ac.ir

Mahboubeh Rezaie
Associate Professor of Mathematics
Department of Mathematics
Associate Professor of Mathematics
University of Isfahan
Isfahan, Iran
E-mail: mrezaie@sci.ui.ac.ir