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Numerical Solution of Lane-Emden Type Equation by Using Hybrid Third Kind Chebyshev Polynomials and Block-Pulse Functions Operational Matrix of Differentiation

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Abstract. In this paper, first, a numerical method is presented for solving generalized linear and nonlinear Lane-Emden type equations. The operational matrix of derivative is obtained by introducing hybrid third kind Chebyshev polynomials and Block-pulse functions. This matrix with the tau method is then utilized to transform the differential equation into a system of algebraic equations. Finally, the convergence analysis is investigated and the efficiency of the proposed method is indicated by some numerical examples.

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1. Introduction

In recent years, the studies of singular initial value problems for second order ordinary differential equations have attracted the attention of many mathematicians and physicists. One of the equations describing this type is the Lane-Emden type equation which is formulated as:

$$y'' + \frac{\alpha}{x} y'(x) + f(x, y) = g(x), \quad 0 < x \leqslant 1, \quad \alpha \geqslant 0, \tag{1}$$

Subject to the initial conditions

$$y(0) = a, \quad y'(0) = 0, \tag{2}$$

where x, a are constants, $f(x, y)$ is a nonlinear function of x and y, and $g(x)$ is an analytical function. Equation (1) was named after the astrophysicists Jonathan H. Lane and Robert Emden (1870), as it was first studied by them. Equation (1) was used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas sphere, and theory of thermionic currents [1, 2]. Due to the multiple applicability of Lane-Emden equation, many researchers focused their attention to give good approximate solution to these equations and many methods were proposed. Horedt [3] presented a method based on Runge-Kutta type. Solutions based on Adomian decomposition method presented in [4,5] and the difficulty of singular point overcame by an alternative Adomian decomposition method designed by Wazwaz. Dehghan and Shakeri [6] presented the variational iteration method to solve Lane-Emden type equation. In [7,8] the authors used Legendre and Bernstein operational matrix of differentiation for solving Lane-Emden type equation. Balaji in [9] a new Bernoulli wavelet operational matrix of derivative method for the solution of nonlinear singular Lane-Emden type equations arising in astrophysics. Maleknejad and Hashemizadeh's [10] used numerical method for solving Lane-Emden type equation arising in astrophysics.

In this paper, a new numerical method for solving homogeneous and nonhomogeneous Lane-Emden type equation is presented. The method is based on combination of third kind Chebyshev polynomials and Block-pulse functions called the hybrid of third kind Chebyshev polynomials and Block-pulse functions (HTKCPBPF).

The paper is organized as follows: In section 2, we review briefly about Block-pulse functions and third kind Chebyshev polynomials and hybrid of them. Section 3 is devoted to function approximation. In Sections 4, we construct the operational matrices of derivative based on the (HTKCPBPF). Convergence analysis of the proposed method is done in Section 5. In Section 6 and 7, we show the validity and efficiency of the proposed method, we present some numerical examples. Finally, Section 8 concludes the paper.

2. Function and Hybrid Function

2.1 Block-pulse functions

A set of Block-pulse functions $b_i(x), i = 1, 2, ..., N$, on the interval $[0, 1)$ are defined as [12]:

$$b_i(x) = \begin{cases} 1, & \frac{i-1}{N} \leqslant x < \frac{i}{N} \\ 0, & otherwise. \end{cases}$$

These functions satisfy in the following properties:
i- Disjointness

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & for \ \ i = j, \\ 0, & for \ \ i \neq j. \end{cases}$$

ii- Orthogonality

$$\int_0^1 b_i(x)b_j(x)dt = \frac{1}{N}\delta_{ij}$$

where $i, j = 1, 2, ..., N$, and δ_{ij} is the Kronecker delta,
iii- Completeness
for every $f \in L^2[0, 1)$ when m approach to the infinity, parsevals identity hold:

$$\int_0^1 f^2(x)dx = \sum_0^\infty (f_i^2 \|b_i(x)\|^2),$$

where $f_i = N \int_0^1 f(x)b_i(x)dx$.

2.2 Third kind of Chebyshev polynomials

The third kind of Chebyshev polynomial $V_n(x)$ is a polynomial of degree n in x defined by [13] :

$$V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos(\frac{1}{2})\theta},$$

where $x = \cos\theta$. clearly from (2.2), fundamental recurrence relation as follows:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \quad n = 2, 3, ...,$$

where

$$V_0(x) = 1, \quad V_1(x) = 2x - 1,$$

These polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $\omega(x) = \sqrt{\frac{1+x}{1-x}}$, that is

$$\int_{-1}^{1} V_i(x)V_j(x)w(x)dx = \pi\delta_{ij}.$$

2.3 Hybrid functions

For $n = 1, ..., N$ and $m = 0, ..., M - 1$, the HTKCPBPF are defined as follows [11]:

$$\varphi_{nm}(x) = \begin{cases} \sqrt{\frac{2}{N}} V_m(2Nx - 2n + 1), & \frac{n-1}{N} \leqslant x < \frac{n}{N}, \\ 0, & otherwise, \end{cases}$$

with the following weight function

$$\omega_n(x) = \omega(2Nx - 2n + 1).$$

3. Function Approximation

A function $f(x) \in L^2[0, 1)$ may be expanded as:

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(x), \tag{3}$$

where

$$c_{nm} = \frac{\langle f(x), \varphi_{nm}(x)\rangle}{\langle \varphi_{nm}(x), \varphi_{nm}(x)\rangle} = \frac{N^2}{\pi} \int_0^1 \omega_n(x)\varphi_{nm}(x)f(t)dx. \tag{4}$$

In (4), $\langle ., .\rangle_{L^2_\omega[0,1)}$ denotes the inner product in $L^2_\omega[0, 1)$, with weight function $w_n(x)$. If the infinite series in (3) is truncated, then equation (3) can be written as:

$$f(x) \simeq \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x) = C^T \varphi(x),$$

where C and $\varphi(x)$ are $NM \times 1$ matrices given by:

$$C = [c_{10}, c_{11}, c_{12}, ..., c_{1,M-1}, c_{20}, ..., c_{N,M-1}]^T,$$

$$\varphi(x) = [\varphi_{10}(x), \varphi_{11}(x), ..., \varphi_{1,M-1}(x), \varphi_{20}(x)..., \varphi_{N,M-1}(x)]^T. \tag{5}$$

The differentiation of vector $\varphi(x)$ can be obtained by:

$$\frac{d\varphi(x)}{dx} = D\varphi(x).$$

We derive the matrix D in the following section for some particular values of N and M.

4. Operational Matrix of Derivative

In this section, we figure out the precise derivative of the HTKCPBPF with $N = 2$ and $M = 3$. In this case, the six basis functions are given by:

$$\begin{aligned}
\varphi_1 &= \varphi_{10}(x) = 1, \\
\varphi_2 &= \varphi_{11}(x) = 8x - 3, \\
\varphi_3 &= \varphi_{12}(x) = 64x^2 - 40x + 5,
\end{aligned} \tag{6}$$

for $t \in [0, \frac{1}{2})$, and

$$\begin{aligned}
\varphi_4 &= \varphi_{20}(x) = 1, \\
\varphi_5 &= \varphi_{21}(x) = 8x - 7, \\
\varphi_6 &= \varphi_{22}(x) = 64x^2 - 104x + 41,
\end{aligned} \tag{7}$$

for $t \in [\frac{1}{2}, 1)$. Let $\varphi_6(t) = (\varphi_{10}(t) \quad \varphi_{11}(t) \quad \varphi_{12}(t) \quad \varphi_{20}(t) \quad \varphi_{21}(t) \quad \varphi_{22}(t))$. By differentiation $(6), (7)$ from 0 to t, and representing them in the matrix form, we obtain

$$\begin{aligned}
\frac{d\varphi_1}{dx} &= 0, \\
\frac{d\varphi_2}{dx} &= 8 = 8\varphi_{10}, \\
\frac{d\varphi_3}{dx} &= 128x - 40 = 16\varphi_{11} + 8\varphi_{10}, \\
\frac{d\varphi_4}{dx} &= 0, \\
\frac{d\varphi_5}{dx} &= 8 = 8\varphi_{20}, \\
\frac{d\varphi_6}{dx} &= 128x - 104 = 16\varphi_{21} + 8\varphi_{20}.
\end{aligned}$$

Thus, we have

$$\frac{d\varphi(x)}{dx} = D_{6\times 6}\varphi(x).$$

Where

$$D_{6\times 6} = 2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 8 & 0 \end{bmatrix}.$$

The matrix $D_{6\times 6}$ can be written as

$$D_{6\times 6} = 2\begin{bmatrix} F_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & F_{3\times 3} \end{bmatrix},$$

where

$$F_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 8 & 0 \end{bmatrix}.$$

In general,for $M \geqslant 4$, we have

$$\frac{d\varphi(x)}{dx} = D\varphi(x),$$

where $\varphi(x)$ is given in (5) and D is a $NM \times NM$ matrix given by

$$D = N\begin{bmatrix} F & 0 & 0 & \cdots & 0 \\ 0 & F & 0 & \cdots & 0 \\ 0 & 0 & F & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & F \end{bmatrix},$$

where $F = a_{(ij)}$ is $M \times M$ matrices, whose the elements are given explicitly by:

$$a_{ij} = \begin{cases} 2(i+j-1), & i > j, (i+j) odd, \\ 2(i-j), & i > j, (i+j) even, \\ 0, & otherwise. \end{cases}$$

For example if $M = 7$ as follows:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 12 & 0 & 0 & 0 & 0 \\ 8 & 12 & 4 & 16 & 0 & 0 & 0 \\ 12 & 8 & 16 & 4 & 20 & 0 & 0 \\ 12 & 16 & 8 & 20 & 4 & 24 & 0 \end{bmatrix}_{7\times 7},$$

Using the above procedure, the operational matrix of nth derivative can be derived as:

$$\frac{d^n\varphi(x)}{dx^n} = D^n\varphi(x). \tag{8}$$

5. Convergence Analysis

The following theorem gives the convergence and accuracy estimation of HTKCPBPF.

Theorem 5.1. Let $f(x)$ be a second-order derivative square-integrable function defined on $[0,1)$ with bounded second-order derivative, say $|f''(x)| \leqslant A$ for some constant A , then

(i) $f(x)$ can be expanded as an infinite sum of the HTKCPBPF and the series converges to $f(x)$ uniformly, that is

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(t),$$

where $c_{nm} = \langle f(x), \varphi_{nm}(x)\rangle_{L^2_\omega[0,1)}$.

(ii)

$$\beta_{f,n,M} \leqslant \frac{\pi A^2}{8}\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(m-1)^4},$$

where $\beta_{f,n,M} = [\int_0^1 |f(x) - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx]^{\frac{1}{2}}$.

Proof. To prove (i), we have:

$$\begin{aligned}(i)c_{nm} = \langle f(x), \varphi_{nm}(x)\rangle_{L^2_\omega[0,1)} &= \frac{N^2}{\pi}\int_0^1 \omega_n(x)\varphi_{nm}(x)f(x)dx \\ &= \frac{N^2}{\pi}\int_{\frac{n-1}{N}}^{\frac{n}{N}} f(x)\sqrt{\frac{2}{N}}V_m(2Nx-2n+1)\omega(2Nx-2n+1)dx.\end{aligned}$$

Let $t = (2Nx - 2n + 1)$ then $dt = 2Ndx$. Clearly, we have

$$c_{nm} = \frac{N}{2\pi}\sqrt{\frac{2}{N}}\int_{-1}^{1} f(\frac{t+2n-1}{2N})V_m(t)\sqrt{\frac{1+t}{1-t}}dt.$$

By letting $t = cos\theta$ and the definition of the HTKCPBPF , it follows that

$$\begin{aligned}c_{nm} &= \frac{N}{2\pi}\sqrt{\frac{2}{N}}\int_0^{\pi} f(\frac{cos\theta+2n-1}{2N})(cos\ m\theta + cos(m+1)\theta)d\theta \\ &= \frac{N}{2\pi}\sqrt{\frac{2}{N}}[\int_0^{\pi} f(\frac{cos\theta+2n-1}{2N})cos\ m\theta + \int_0^{\pi} f(\frac{cos\theta+2n-1}{2N})cos(m+1)\theta d\theta].\end{aligned}$$

Using the integration by parts, we have

$$c_{nm} = \sqrt{\frac{2}{N}}\frac{1}{4\pi}[\frac{1}{m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})(sin\ m\theta sin\theta)d\theta +$$

$$\frac{1}{m+1}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})(sin(m+1)\theta sin\theta)d\theta] = \sqrt{\frac{2}{N}}\frac{1}{4\pi}[I_1 + I_2], \quad (9)$$

where

$$I_1 = [\frac{1}{m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})(sin\ m\theta sin\theta)d\theta,$$

and

$$I_2 = \frac{1}{m+1}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})(sin(m+1)\theta sin\theta)d\theta.$$

Now, we estimate I_1 and I_2, respectively. A simple computation shows that

$$\begin{aligned} I_1 &= \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})[cos(m-1)\theta - cos(m+1)\theta]d\theta \\ &= \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})[cos(m-1)\theta d\theta - \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})cos(m+1)\theta]d\theta \\ &= I_{11} - I_{12}, \end{aligned}$$

where

$$I_{11} = [\frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})cos(m-1)\theta d\theta,$$

and

$$I_{12} = \frac{1}{2m}\int_0^{\pi} f'(\frac{cos\theta + 2n - 1}{2N})cos(m+1)\theta d\theta.$$

By using the integration by parts, and for $m > 1$, we get

$$\begin{aligned} I_{11} &= \frac{1}{4mN(m-1)}\int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[sin(m-1)\theta sin\theta]d\theta \\ &= \frac{1}{8mN(m-1)}\int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[cos(m-2)\theta d\theta - cos\ m\theta]d\theta, \end{aligned}$$

$$I_{12} = \frac{1}{4mN(m+1)} \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[sin(m+1)\theta sin\theta]d\theta$$
$$= \frac{1}{8mN(m+1)} \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[cos\ m\theta - cos(m+2)\theta]d\theta.$$

Thus, for $m > 1$, we conclude that

$$I_1 = \frac{1}{8mN} \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta,$$

and hence

$$|I_1|^2 = |\frac{1}{8mN} \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta|^2$$
$$= \frac{1}{64m^2N^2}| \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})[\frac{cos(m-2)\theta - cos\ m\theta}{(m-1)} - \frac{cos\ m\theta - cos(m+2)\theta}{(m+1)}]d\theta|^2.$$

By the fact that $|f''(x)| \leqslant A$ and Schwartz inequality , it follows that

$$|I_1|^2 \leqslant \frac{1}{64m^2N^2}| \int_0^{\pi} f''(\frac{cos\theta + 2n - 1}{2N})|^2 d\theta \times$$
$$\int_0^{\pi} |\frac{(m+1)cos(m-2)\theta - 2mcos\ m\theta + (m-1)cos(m+2)\theta}{(m+1)(m-1)}|^2 d\theta$$
$$\leqslant \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2} \int_0^{\pi} |(m+1)cos(m-2)\theta + 2mcos\ m\theta + (m-1)cos(m+2)\theta|^2 d\theta$$
$$= \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2} \times$$
$$[\int_0^{\pi} (m+1)^2 cos^2(m-2)\theta d\theta + \int_0^{\pi} 4m^2 cos^2 m\theta d\theta + \int_0^{\pi} (m-1)^2 cos^2(m+2)\theta d\theta]$$
$$= \frac{\pi A^2}{64m^2N^2(m-1)^2(m+1)^2}[\frac{\pi}{2}(m+1)^2 + \frac{\pi}{2}4m^2 + \frac{\pi}{2}(m-1)^2]$$
$$= \frac{\pi^2 A^2(3m^2+1)}{64m^2N^2(m-1)^2(m+1)^2} \leqslant \frac{\pi^2 A^2}{4N^2(m-1)^4}.$$

For $m > 2$, we obtain

$$|I_1| \leqslant \frac{\pi A}{2N(m-1)^2}.$$

In a similar way, we will have

$$|I_2| \leqslant \frac{\pi A}{2N(m-1)^2}.$$

Therefore, for $m > 2$, we conclude that

$$|c_{nm}| =| \frac{1}{4\pi}\sqrt{\frac{2}{N}}[I_1 + I_2] |\leqslant \frac{1}{4\pi}\sqrt{\frac{2}{N}}\frac{\pi A}{N(m-1)^2} \leqslant \frac{A}{2\sqrt{2}}\frac{1}{n^{\frac{3}{2}}(m-1)^2}. \quad (10)$$

Note that $f'(x)$ is bounded on $[0, 1)$ due to the fact that $|f''(x)| \leqslant A$, indeed, by the Differential Mean Value Theorem and for any $t \in (0, 1)$, there exists some $\gamma_x \in (0, x)$ such that

$$f'(x) - f'(0) = f''(\gamma_x)x,$$

So

$$|f'(x)| \leqslant |f'(0)| + A,$$

for $x \in (0, 1)$. Thus $f'(x)$ is bounded on $[0, 1)$, say $|f'(x)| \leqslant \tilde{A}$ for some constant $\tilde{A}$. Hence, by (9), we have

$$\begin{aligned}|c_{n,1}| &\leqslant \sqrt{\frac{2}{N}}\frac{1}{4\pi}[\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta + \frac{1}{2}\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta] \\ &= \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{3}{2}\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta \leqslant \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{3\pi\tilde{A}}{2} = \frac{3\tilde{A}}{4\sqrt{2}n^{\frac{1}{2}}} \quad (11)\end{aligned}$$

and

$$\begin{aligned}|c_{n,2}| &\leqslant \sqrt{\frac{2}{N}}\frac{1}{4\pi}[\frac{1}{2}\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta + \frac{1}{3}\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta] \\ &= \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{5}{6}\int_0^\pi |f'(\frac{cos\theta + 2n-1}{2N})|d\theta \leqslant \sqrt{\frac{2}{N}}\frac{1}{4\pi}\frac{5\pi\tilde{A}}{6} = \frac{5\tilde{A}}{12\sqrt{2}n^{\frac{1}{2}}} \quad (12)\end{aligned}$$

Relations $(21) - (23)$ show that the series$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}$ is absolutely convergent. For $m = 0$ and according to the definition of $\varphi_{n,0}(x)$, the series $\sum_{n=1}^{\infty} c_{n,0}\varphi_{n,0}(x)$ is convergent. Therefore, the series$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\varphi_{nm}(x)$ converges to $f(x)$ uniformly.

$$\begin{aligned}(ii)\ \beta^2_{f,n,M} &= \int_0^1 |f(x) - \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx \\ &= \int_0^1 |\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} y_{nm}\varphi_{nm}(x)|^2\omega_n(x)dx \\ &= \sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty} |y_{nm}|^2(\sqrt{\frac{2}{N}})^2\int_{\frac{n-1}{N}}^{\frac{n}{N}} V_m(2Nx-2n+1)^2\sqrt{\frac{1+(2Nx-2n+1)}{1-(2Nx-2n+1)}}dx.\end{aligned}$$

Let $t = 2Nx - 2n + 1$ then $dt = 2ndx$. Therefore

$$\beta^2_{f,n,M} = \sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} |c_{nm}|^2 \frac{1}{N^2} \int_{-1}^{1} V_m^2(t) \sqrt{\frac{1+t}{1-t}} dt,$$

we have

$$\int_{-1}^{1} V_m^2(t) \sqrt{\frac{1+t}{1-t}} dt = \pi,$$

where the last equality follows due to the orthogonality of $\varphi_{nm}(x)$. Together with (10) we get

$$\beta^2_{f,n,M} \leqslant \frac{\pi A^2}{8} \sum_{n=N+1}^{\infty} \sum_{m=M}^{\infty} \frac{1}{n^5(m-1)^4}.$$

6. Solution of Lane-Emden Type Equation

In this section, consider Lane-Emden equation given in (1). In order to use (HTKCPBPF), we first approximate $y(x), f(x, y)$ and $g(x)$ as:

$$\begin{aligned} y(x) &= C^T \varphi(x), \\ f(x, y) &\approx f(x, C^T \varphi(x)), \\ g(x) &= G^T \varphi(x). \end{aligned}$$

By using (8), we have

$$\begin{aligned} y'(x) &= C^T D \varphi(x), \\ y''(x) &= C^T D^2 \varphi(x). \end{aligned}$$

With substituting in (1) we have

$$C^T D^2 \varphi(x) + \frac{2}{x} C^T D \varphi(x) + f(x, C^T \varphi(t) = G^T \varphi(x).$$

Also, the initial and boundary conditions from (2) yields

$$C^T \varphi(0) = a, \quad C^T D \varphi(0) = b. \tag{13}$$

The residual $R_N(x)$ for (1) can be written as

$$R_N(x) = C^T D^2 \varphi(x) + \frac{2}{x} C^T D \varphi(x) + f(x, C^T \varphi(t) - G^T \varphi(x).$$

As in a typical tau method, we generate $NM - 1$ equation by applying

$$(R_N(x), \varphi_i(x)) = \int_0^1 \omega(x) R_N(x) \varphi_i(x) dx = 0, \quad \imath = 0, 1, ..., NM - 1. \tag{14}$$

Equation (30) and (32) generate a set of $NM + 1$ linear or nonlinear equations.

7. Numerical Examples

In this section, linear and nonlinear Lane-Emden type equations have been solved using the proposed method.

Example 7.1. At first we consider the equation

$$y''(x) + \frac{2}{x}y'(x) + y^n(x) = 0, \quad 0 < x \leqslant 1, \quad y(0) = 1, y'(0) = 0, \tag{15}$$

where n is a constant. Substituting $n = 0, 1$ and 5 into (33) leads to the exact solution

$$y(x) = 1 - \frac{x^2}{6}, \quad y(x) = \frac{sin(x)}{x}, \quad y(x) = (1 + \frac{x^2}{3})^{-1/2},$$

respectively.

1) For $n = 0$, $f(x, y) = y^0(t) = 1$ and $g(x) = 0$, we apply the method that was explained in Section 6 for $N = 1, M = 3$, thus assume that

$$\begin{aligned} y(x) = C\varphi^T(x) &= \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \\ &= c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x). \end{aligned}$$

Our aim is to determine the unknown coefficients c_0, c_1 and c_2 by using Tau method. Also the operational matrix of third kind Chebyshev polynomials and Block-pulse functions and its square is as follows:

$$D^1 = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 8 & 0 \end{pmatrix}, \qquad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}.$$

By using (32) we have

$$\langle R_2(x), \varphi_0(x) \rangle = \int_0^1 \omega(x) R_2(x) \varphi_0(x) dx = 0,$$

where

$$
\begin{aligned}
R_2(x) &= xy_2''(x) + 2y_2'(x) + x, \\
&= x \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} D^2 \varphi^T(x) + 2 \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} D\varphi^T(x) + x, \\
&= x \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} + \\
&\quad 2 \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 8 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0(x) \\ \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} + x \\
&= 8\sqrt{2}c_1 - 40\sqrt{2}c_2 + 96\sqrt{2}c_2 x + x.
\end{aligned}
$$

Therefore

$$
\int_0^1 (8\sqrt{2}c_1 - 40\sqrt{2}c_2 + 96\sqrt{2}c_2 x + x)\sqrt{2}\sqrt{\frac{1+(2x-1)}{1-(2x-1)}}dx = \frac{3}{8\sqrt{2}} + 4c_1 + 16c_2 = 0. \tag{16}
$$

By applying the initial conditions we have

$$
\begin{aligned}
&4\sqrt{2}c_1 - 20\sqrt{2}c_2 = 0 \\
&\sqrt{2}c_0 - 3\sqrt{2}c_1 + 5\sqrt{2}c_2 = 1.
\end{aligned} \tag{17}
$$

Solving$(16) - (17)$ yields

$$
c_0 = \frac{43}{48\sqrt{2}}, \qquad c_1 = -\frac{5}{96\sqrt{2}}, \qquad c_2 = -\frac{1}{96\sqrt{2}}.
$$

Thus

$$
y(x) = c_0\varphi_0(x) + c_1\varphi_1(x) + c_2\varphi_2(x), \tag{18}
$$

$$
= \begin{pmatrix} \frac{43}{48\sqrt{2}} & -\frac{5}{96\sqrt{2}} & -\frac{1}{96\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \sqrt{2}(4x-3) \\ \sqrt{2}(16x^2 - 20x + 5) \end{pmatrix} = 1 - \frac{x^2}{6}, \tag{19}
$$

which is the exact solution.

2) For $n = 1$, we solve (15) by the method discussed in this paper with $N = 1$ and $M = 6$, we have

$$
\begin{aligned}
&c_0 = 0.636292, \quad c_1 = -0.0349386, \quad c_2 = -0.00656114, \\
&c_3 = 0.00019763, \quad c_4 = 0.0000203165, \quad c_5 = -4.44649 \times 10^{-7},
\end{aligned}
$$

and consequently

$$y(x) = C^T\varphi(x) = 1-0.166578x^2-0.000432929x^3+0.00912614x^4-0.000643921t^5.$$

Table (1) shows some values of the solutions and absolute errors at some x, and plot of the exact and approximate solutions are shown in Figure (1).
3) For $n = 5$, we solve (15) by the method discussed in this paper with $N = 1$ and $M = 4$, we have

$$c_0 = 0.6448062055, \quad c_1 = -0.0290922, \quad c_2 = -0.00417201608, \quad c_3 = 0.00058801185,$$

and consequently

$$y(x) = C^T\varphi(x) = 1 - 2.77556 \times 10^{-17}x - 0.187538x^2 + 0.0532208x^3,$$

Table (2) shows some values of the solutions and absolute errors at some x, and plot of the exact and approximate solutions are shown in Figure (2).

Table 1: Estimated and exact values of Example 7.1 for $n = 1$

x	Exact solution	Hybrid function	Absolute error
0.0	1.00000	1.00000	0.00000
0.1	0.998334	0.998335	0.000001
0.2	0.993347	0.993348	0.000001
0.3	0.985067	0.985069	0.000002
0.4	0.973546	0.973547	0.000001
0.5	0.958851	0.958852	0.000001
0.6	0.941071	0.941071	0.000000
0.7	0.920311	0.920311	0.000000
0.8	0.896695	0.896696	0.000001
0.9	0.870363	0.870364	0.000001

Table 2: Estimated and exact values of Example 7.1 for $n = 5$

x	Exact solution	Hybrid function	Absolute error
0.0	1.0000000000	1.0000000000	0.000000000
0.1	0.9983374884	0.9988937246	0.000159650
0.2	0.9933992679	0.9955756673	0.000475033
0.3	0.9853292781	0.9900455719	0.000770762
0.4	0.9743547036	0.9823034383	0.000954699
0.5	0.9607689228	0.9723492664	0.001000900
0.6	0.9449111826	0.9601830564	0.000929277
0.7	0.9271455411	0.9458048082	0.000784575
0.8	0.9078412992	0.9292145218	0.000616766
0.9	0.8873556093	0.9104121973	0.000464579
1	0.8660254038	0.8893978346	0.000342920

Table 3: Estimated and exact values of Example 7.4

t	Hybrid function	Exact solution	Absolute error
0.0	1.0000258671	1.0000000000	0.0000258672
0.1	0.9950313704	0.9950041652	0.0000272052
0.2	0.9800965668	0.9800665778	0.0000299890
0.3	0.9553665715	0.9553364891	0.0000300824
0.4	0.9210887371	0.9210609940	0.0000277432
0.5	0.8776073749	0.8775825618	0.0000248131
0.6	0.8253584756	0.8253356149	0.0000228607
0.7	0.7648644306	0.7648421872	0.0000222433
0.8	0.6967287530	0.6967067093	0.0000220437
0.9	0.6216307990	0.6216099682	0.0000208308
1	0.5403204885	0.5403023058	0.0000181827

Example 7.2. Consider the Lane-Emden equation give in [5] by

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 6 + 12x + x^2 + x^3, \quad 0 < x \leqslant 1, \qquad y(0) = 0, y'(0) = 0,$$

with the exact solution $y = x^2 + x^3$. We apply the method that was explained in Section 6 for $N = 1, M = 4$. After performing some manipulations, the components of the vector C are given by

$$c_0 = \frac{75}{64\sqrt{2}}, \qquad c_1 = \frac{41}{64\sqrt{2}}, \qquad c_2 = \frac{11}{64\sqrt{2}}, \qquad c_3 = \frac{1}{64\sqrt{2}},$$

and consequently

$$y(x) = C^T\varphi(x) = x^2 + x^3,$$

which is the exact solution.

Example 7.3. Consider the following Lane-Emden equation:

$$y''(x)+\frac{8}{x}y'(x)+xy(x) = x^5-x^4+44x^2-30x, \quad 0 < x \leqslant 1, \qquad y(0) = 0, y'(0) = 0,$$

with the exact solution $y = x^4 - x^3$. We apply the method that was explained in Section 6 for $N = 1, M = 6$. After performing some manipulations, the components of the vector C are given by

$$c_0 = -\frac{7}{128\sqrt{2}}, \quad c_1 = 0, \quad c_2 = \frac{1}{32\sqrt{2}}, \quad c_3 = \frac{5}{256\sqrt{2}}, \quad c_4 = \frac{1}{256\sqrt{2}}, \quad c_5 = 0,$$

and consequently

$$y(x) = C^T\varphi(x) = x^4 - x^3,$$

which is the exact solution.

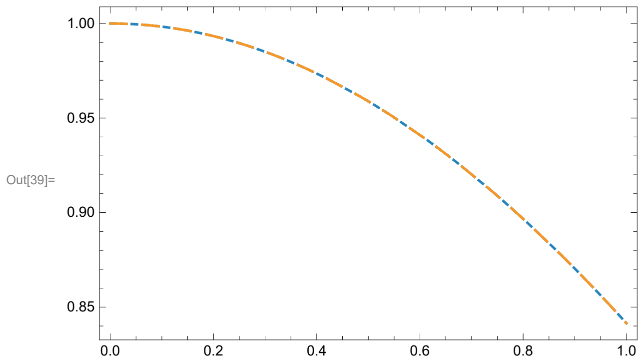

Figure 1. The exact and Presented method solution of Example1 for n=1.

Example 7.4. Consider the following nonlinear Lane-Emden equation:

$$y''(x) + \frac{2}{x}y'(x) - 2(2x^2+3)y(x) = 0, \quad 0 < x \leqslant 1, \qquad y(0) = 0, y'(0) = 0,$$

which has the following exact solution:

$$y(x) = e^{x^2}.$$

We apply the method that was explained in Section 6 for $N = 1, M = 6$. After performing some manipulations, the components of the vector C are given by:

$$c_0 = 1.380375498, \qquad c_1 = 0.3834019355, \qquad c_2 = 0.1207976086,$$
$$c_3 = 0.02368446796, \qquad c_4 = 0.005513677977, \qquad c_5 = 0.0009893435493,$$

Table (3) shows some values of the solutions and absolute errors at some x.

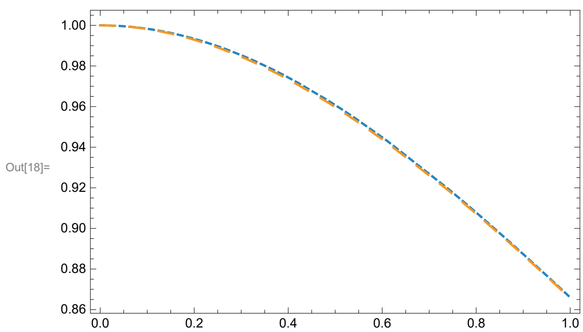

Figure 2. The exact and Presented method solution of Example1 for n=5.

8. Conclusion

In this paper, we constructed operational matrix of derivative of hybrid the third kind Chebyshev polynomials and Block-pulse functions. Also, we applied these matrices to convert solving differential equations to solving linear algebraic equations. As to validity and efficiency of the proposed method, we presented some numerical examples.

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