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A New Perspective via Fractional Calculus for the Radial Schrödinger Equation

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Abstract. Differintegral theorems are applied to solve some ordinary differential equations and fractional differential equations. By using these theorems, we obtain different results in the fractional differintegral forms. In this paper, we aim to solve the radial Schrödinger equation under the potential $V(r) = H/r^2 - K/r + Lr^\kappa$ in $\kappa = 0, -1, -2$ cases. We also obtain the solutions in the hypergeometric forms.

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1. Introduction

As is known, the order of derivative consists of an integer in ordinary calculus; differentiation with integer order is always provided for a favorable function. What's interesting is that a function can be held differentiation with any arbitrary order in fractional calculus that is an area of mathematics that grows out of the traditional definitions of the calculus integral and derivative operators. And, same situation is valid for the integration. So, differentiation and integration are generalized such as differintegral in fractional calculus theory. This theory has an important position in areas of science and engineering such as robot technology, PID control systems, Schrödinger equation, heat transfer, relativity theory, economy, filtration, controller design, mechanics, optics, modelling and so on [12, 14, 16, 17, 21]. There has been an important development in fractional calculus in recent years; see the monographs of Kilbas et al. [7] and the fractional differintegral equations of Tu et al. [19], Lin et

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al. [10], Ortigueira [15] and etc.
Riemann-Liouville differintegral definitions are, respectively,

$$ {}_aD_t^{-\mu}\varphi(t) = [\varphi(t)]_{-\mu} = \frac{1}{\Gamma(\mu)}\int_a^t \frac{\varphi(\omega)}{(t-\omega)^{1-\mu}}d\omega \quad (t>a, \mu>0), \tag{1} $$

and,

$$ {}_aD_t^{\mu}\varphi(t) = [\varphi(t)]_{\mu} = \frac{1}{\Gamma(k-\mu)}\frac{d^k}{dt^k}\int_a^t \frac{\varphi(\omega)}{(t-\omega)^{\mu+1-k}}d\omega, \tag{2} $$
$$ (k-1 \leqslant \mu < k, k \in \mathbf{N}). $$

The Schrödinger equation has an important place in fractional calculus. In this contex, many scientific works was suggested. For instance, in [4], based on the Riesz definition of the fractional derivative the fractional Schrödinger equation with an infinite well potential is studied by Herrmann. And, Laskin [9] presented some properties of the fractional Schrödinger equation. In [8], the path integrals over the Levy paths are defined and fractional quantum and statistical mechanics have been developed via new fractional path integrals approach. A fractional generalization of the Schrödinger equation has been found by Laskin. Wang [20] expressed fractional Schrödinger equations with potential and optimal controls. The fractional Schrödinger equation that contains the quantum Riesz fractional derivative instead of the Laplace operator is revisited for the case of a particle moving in the infinite potential well by Luchko in [11]. Bayin [2] showed effective potential approach and presented a free particle solution for the space and time fractional Schrödinger equation in general coordinates in terms of Fox's H-functions. In [18], some applications of a fractional approach to the Schrödinger equation are discussed by Rozmej. Jeng [5] studied on the one-dimensional infinite square well and presented that the purported ground state, which is based on a piecewise approach, is definitely not a solution of the fractional Schrödinger equation for the general fractional parameter. In [6], Khan formed approximate solutions of the time-fractional Schrödinger equations, with zero and nonzero trapping potential, by homotopy analysis method HAM.
In present paper, we deal with radial part of the Schrödinger equation given by the potential $V(r) = H/r^2 - K/r + Lr^{\kappa}$. The radial equation is a second-order homogeneous ordinary differential equation with variable coefficients. And, fractional calculus theorems can be apply to such equations.

2. Preliminaries

Definition 2.1. *If $\varphi(z)$ is analytic and has no branch point inside and on C, where $C := \{C^-, C^+\}$, C^- is a contour along the cut joining the points z and*

$-\infty + iIm(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point $-\infty$, and C^{+} is a contour along the cut joining the points z and $\infty + iIm(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$\varphi_{\mu}(z) := \frac{\Gamma(\mu+1)}{2\pi i}\int_{C}\frac{\varphi(t)dt}{(t-z)^{\mu+1}} \quad (\mu \notin \mathbf{Z}^{-}),$$

and,

$$\varphi_{-k}(z) := \lim_{\mu\to -k}\varphi_{\mu}(z) \quad (k \in \mathbf{Z}^{+}),$$

where $t \neq z$, $-\pi \leqslant \arg(t-z) \leqslant \pi$ for C^{-} and, $0 \leqslant \arg(t-z) \leqslant 2\pi$ for C^{+}, then $\varphi_{\mu}(z)$ $(\mu > 0)$ is said to be the fractional derivative of $\varphi(z)$ of order μ and, $\varphi_{\mu}(z)$ $(\mu < 0)$ is said to be fractional integral of $\varphi(z)$ of order $-\mu$, provided that $\mid \varphi_{\mu}(z) \mid < \infty$ $(\mu \in \mathbf{R})$ [3, 13, 15].

Lemma 2.2. [Linearity] *Let $\varphi(z)$ and $\psi(z)$ be analytic and single-valued functions. If $\varphi_{\mu}(z)$ and $\psi_{\mu}(z)$ exist, then*

$$[\alpha\varphi(z) + \beta\psi(z)]_{\mu} = \alpha\varphi_{\mu}(z) + \beta\psi_{\mu}(z), \tag{3}$$

where α and β are constants and, $\mu \in \mathbf{R}, z \in \mathbf{C}$ [21].

Lemma 2.3. [Index law] *Let $\varphi(z)$ be an analytic and single-valued function. If $(\varphi_{\eta})_{\mu}(z)$ and $(\varphi_{\mu})_{\eta}(z)$ exist, then*

$$(\varphi_{\eta})_{\mu}(z) = (\varphi_{\eta+\mu})(z) = (\varphi_{\mu})_{\eta}(z), \tag{4}$$

where $\mu, \eta \in \mathbf{R}, z \in \mathbf{C}$ and $\left|\frac{\Gamma(\mu+\eta+1)}{\Gamma(\mu+1)\Gamma(\eta+1)}\right| < \infty$ [21].

Lemma 2.4. [Generalized Leibniz rule] *Let $\varphi(z)$ and $\psi(z)$ be analytic and single-valued functions. If $\varphi_{\mu}(z)$ and $\psi_{\mu}(z)$ exist, then*

$$[\varphi(z)\psi(z)]_{\mu} = \sum_{k=0}^{\infty}\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}\varphi_{\mu-k}(z)\psi_{k}(z), \tag{5}$$

where $\mu \in \mathbf{R}, z \in \mathbf{C}$ and $\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)}\right| < \infty$ [21].

Remark 2.5. *Let ϑ be a constant as $\vartheta \neq 0$. Then [21],*

$$(e^{\vartheta z})_{\mu} = \vartheta^{\mu}e^{\vartheta z} \quad (\mu \in \mathbf{R}, z \in \mathbf{C}), \tag{6}$$

$$(e^{-\vartheta z})_{\mu} = e^{-i\pi\mu}\vartheta^{\mu}e^{-\vartheta z} \quad (\mu \in \mathbf{R}, z \in \mathbf{C}), \tag{7}$$

$$(z^{\vartheta})_{\mu} = e^{-i\pi\mu}\frac{\Gamma(\mu-\vartheta)}{\Gamma(-\vartheta)}z^{\vartheta-\mu} \quad \left(\mu \in \mathbf{R}, z \in \mathbf{C}, \left|\frac{\Gamma(\mu-\vartheta)}{\Gamma(-\vartheta)}\right| < \infty\right). \tag{8}$$

Remark 2.6.

$$\Gamma(\mu-k) = (-1)^k\frac{\Gamma(\mu)\Gamma(1-\mu)}{\Gamma(k+1-\mu)} \quad (k \in \mathbf{Z}^+ \cup \{0\}, \mu \in \mathbf{R}). \tag{9}$$

The following theorem given by Lin et al. [10] is the main theory of this paper:

Theorem 2.7. *Let $\varphi_{-\mu} \neq 0$ where φ is a given function and, $M(z;m)$ and $N(z;n)$ be polynomials in z of degrees m and n, respectively, given by*

$$M(z;m) = \sum_{k=0}^{m} a_k z^{m-k} = a_0\prod_{j=1}^{m}(z-z_j) \quad (a_0 \neq 0, m \in \mathbf{N}), \tag{10}$$

and,

$$N(z;n) = \sum_{k=0}^{n} b_k z^{n-k} \quad (b_0 \neq 0, n \in \mathbf{N}). \tag{11}$$

Thus, the nonhomogeneous linear ordinary fractional differintegral equation

$$M(z;m)\chi_{\eta}(z) + \left[\sum_{k=1}^{m}\binom{\mu}{k}M_k(z;m) + \sum_{k=1}^{n}\binom{\mu}{k-1}N_{k-1}(z;n)\right]\chi_{\eta-k}(z)$$

$$+\binom{\mu}{k}n!b_0\chi_{\eta-n-1}(z) = \varphi(z) \quad (m,n \in \mathbf{N}, \mu,\eta \in \mathbf{R}), \tag{12}$$

has a particular solution as follows

$$\chi(z) = \left[\left(\frac{\varphi_{-\mu}(z)}{M(z;m)}e^{\sigma(z;m,n)}\right)_{-1} e^{-\sigma(z;m,n)}\right]_{\mu-\eta+1}, \tag{13}$$

$$(z \in \mathbf{C}\backslash\{z_1,...,z_m\}),$$

where for suitable condition,

$$\sigma(z;m,n) = \int^{z}\frac{N(\zeta;n)}{M(\zeta;m)}d\zeta \quad (z \in \mathbf{C}\backslash\{z_1,...,z_m\}), \tag{14}$$

confirmed that the second component of Equ. (13) exists. Moreover, the homogeneous linear ordinary fractional differintegral equation

$$M(z;m)\chi_{\eta}(z) + \left[\sum_{k=1}^{m}\binom{\mu}{k}M_k(z;m) + \sum_{k=1}^{n}\binom{\mu}{k-1}N_{k-1}(z;n)\right]\chi_{\eta-k}(z)$$

$$+\binom{\mu}{k} n! b_0 \chi_{\eta-n-1}(z) = 0 \quad (m, n \in \mathbf{N}, \mu, \eta \in \mathbf{R}), \tag{15}$$

has solutions as follows

$$\chi(z) = \alpha[e^{-\sigma(z;m,n)}]_{\mu-\eta+1}, \tag{16}$$

where $\sigma(z;m,n)$ is given by Equ. (14) and, α is an arbitrary constant [10].

3. Main Results

The radial Schrödinger equation under the potential $V(r) = H/r^2 - K/r + Lr^\kappa$ is defined as [1]

$$R_2(r) + \frac{2m}{\hbar^2}\Big[E - \frac{H}{r^2} + \frac{K}{r} - Lr^\kappa - \frac{l(l+1)\hbar^2}{2mr^2}\Big]R(r) = 0, \tag{17}$$

where H, K and L are positive constants. Now, we get

$$\begin{gathered} -\lambda^2 = \frac{2mE}{\hbar^2}, \quad \mathbf{H} = \frac{2mH}{\hbar^2}, \quad \mathbf{K} = \frac{2mK}{\hbar^2}, \\ \mathbf{L} = \frac{2mL}{\hbar^2}, \quad \rho = l(l+1), \quad \tau = \mathbf{H} + \rho. \end{gathered} \tag{18}$$

And, by substituting (18) into (17), we have

$$r^2 R_2(r) - (\lambda^2 r^2 - \mathbf{K}r + \mathbf{L}r^{\kappa+2} + \tau)R(r) = 0. \tag{19}$$

According to the values of $\kappa = 0, -1, -2$, Equ. (19) is given by

$$r^2 R_2(r) - [(\lambda^2 + \mathbf{L})r^2 - \mathbf{K}r + \tau]R(r) = 0 \quad (\kappa = 0), \tag{20}$$

$$r^2 R_2(r) - [\lambda^2 r^2 + (\mathbf{L} - \mathbf{K})r + \tau]R(r) = 0 \quad (\kappa = -1), \tag{21}$$

$$r^2 R_2(r) - [\lambda^2 r^2 - \mathbf{K}r + (\mathbf{L} + \tau)]R(r) = 0 \quad (\kappa = -2). \tag{22}$$

In this paper, we use the fractional calculus theorems for (20), (21) and (22) and so, we find particular solutions of the radial Schrödinger equation in the fractional differintegral forms.

Theorem 3.1. *If $\mid \varphi_\mu(z) \mid < \infty \quad (\mu \in \mathbf{R})$ and $\varphi_{-\mu} \neq 0$, then*

$$Az^2\chi_2 + Bz\chi_1 + (Dz^2 + Ez + F)\chi = \varphi, \tag{23}$$

$$(A, D \neq 0, z \in \mathbf{C}\backslash\{0\}, \chi = \chi(z)),$$

has a particular solution such as:

$$\chi = z^{\nu} e^{\vartheta z} \left\{ \left[A^{-1} z^{-(\mu+1)+\frac{2A\nu+B}{A}} e^{2\vartheta z} \right.\right.$$

$$\left.\left. .\left(z^{-(\nu+1)} e^{-\vartheta z} \varphi \right)_{-\mu} \right]_{-1} z^{\mu-\frac{2A\nu+B}{A}} e^{-2\vartheta z} \right\}_{\mu-1}, \tag{24}$$

where ν, ϑ and μ are in the form:

$$\nu = \frac{A - B \pm \sqrt{(A-B)^2 - 4AF}}{2A}, \quad \vartheta = \pm i\sqrt{\frac{D}{A}}, \tag{25}$$

and,

$$\mu = \frac{(2A\nu + B)\vartheta + E}{2A\vartheta}. \tag{26}$$

Moreover,

$$Az^2\chi_2 + Bz\chi_1 + (Dz^2 + Ez + F)\chi = 0, \tag{27}$$

$$(A, D \neq 0, z \in \mathbf{C}\backslash\{0\}, \chi = \chi(z)),$$

has the particular solution in the form:

$$\chi = \alpha z^{\nu} e^{\vartheta z} \left(z^{\mu-\frac{2A\nu+B}{A}} e^{-2\vartheta z} \right)_{\mu-1} \tag{28}$$

where ν and ϑ are given by Equ. (25), and μ is given by (26) and, α is an arbitrary constant [10].

Theorem 3.2. *According to the expression of Theorem 3.1, radial Schrödinger equation in Equ. (20) has the particular solution in the form:*

$$R(r) = \alpha r^{\nu} e^{\vartheta r} \left(r^{\mu-2\nu} e^{-2\vartheta r} \right)_{\mu-1}, \tag{29}$$

where

$$\nu = \frac{1 \pm \sqrt{1+4\tau}}{2}, \quad \vartheta = \pm\sqrt{\lambda^2 + \boldsymbol{L}},$$

and,

$$\mu = \nu + \frac{\boldsymbol{K}}{2\vartheta}.$$

Theorem 3.3. *According to the expression of Theorem 3.1, radial Schr..... dinger equation in Equ. (21) has the particular solution in the form:*

$$R(r) = \alpha r^{\nu} e^{\vartheta r} \left(r^{\mu-2\nu} e^{-2\vartheta r} \right)_{\mu-1}, \tag{30}$$

where

$$\nu = \frac{1 \pm \sqrt{1+4\tau}}{2}, \quad \vartheta = \pm\lambda,$$

and,

$$\mu = \nu + \frac{\boldsymbol{K} - \boldsymbol{L}}{2\vartheta}.$$

Theorem 3.4. *According to the expression of Theorem 3.1, radial Schrödinger equation in Equ. (22) has the particular solution in the form:*

$$R(r) = \alpha r^{\nu} e^{\vartheta r} \left(r^{\mu-2\nu} e^{-2\vartheta r} \right)_{\mu-1}, \tag{31}$$

where

$$\nu = \frac{1 \pm \sqrt{1+4(\boldsymbol{L}+\tau)}}{2}, \quad \vartheta = \pm\lambda,$$

and,

$$\mu = \nu + \frac{\boldsymbol{K}}{2\vartheta}.$$

Theorem 3.5. *Let* $\mid (r^{\mu-2\nu})_k \mid < \infty$ *and* $\mid \frac{-1}{2\vartheta r} \mid < 1$ $(r \neq 0, k \in \mathbf{Z}^+ \cup \{0\})$*. The solution in the form:*

$$R(r) = \alpha r^{\nu} e^{\vartheta r} \left(r^{\mu-2\nu} e^{-2\vartheta r} \right)_{\mu-1},$$

is written as

$$R(r) = r^{\mu-\nu} e^{-\vartheta r} {}_2F_0 \left[1-\mu, 2\nu-\mu; \frac{-1}{2\vartheta r} \right],$$

where ${}_2F_0$ *is the Gauss hypergeometric function.*

Proof. By means of Equ. (5), we have

$$R(r) = \alpha r^{\nu} e^{\vartheta r} \sum_{k=0}^{\infty} \frac{\Gamma(\mu)}{\Gamma(\mu-k)k!} (r^{\mu-2\nu})_k (e^{-2\vartheta r})_{\mu-1-k}. \tag{32}$$

By using (7), (8) and (9), we rewrite Equ. (32) as follows:

$$R(r) = \alpha r^{\mu-\nu} e^{-\vartheta r} (2\vartheta e^{-i\pi})^{\mu-1} \times \sum_{k=0}^{\infty} \frac{\Gamma(k+1-\mu)}{\Gamma(1-\mu)} \frac{\Gamma(k+2\nu-\mu)}{\Gamma(2\nu-\mu)} \frac{1}{k!} \left(\frac{-1}{2\vartheta r} \right)^k.$$

Then,

$$R(r) = r^{\mu-\nu} e^{-\vartheta r} \sum_{k=0}^{\infty} (1-\mu)_k (2\nu-\mu)_k \frac{1}{k!} \Big(\frac{-1}{2\vartheta r}\Big)^k.$$

where $1/\alpha = (2\vartheta e^{-i\pi})^{\mu-1}$.
Finally, we obtain

$$R(r) = r^{\mu-\nu} e^{-\vartheta r} {}_2F_0\Big[1-\mu, 2\nu-\mu; \frac{-1}{2\vartheta r}\Big]. \quad \square$$

Conclusion

In this study, we used fractional calculus theorems for the radial Schrödinger equation given by the potential $V(r) = H/r^2 - K/r + Lr^{\kappa}$. And, we obtained the hypergeometric forms of the fractional solutions.

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