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Mittag-Leffler-Hyers-Ulam Stability of Fractional Differential Equations of Second Order

V. Kalvandi
University of Mohaghegh Ardabili

N. Eghbali*
University of Mohaghegh Ardabili

J. Michael Rassias
University of Athens

Abstract. In this paper, we have presented and studied the Mittag-Leffler-Hyers-Ulam stability of a fractional differential equation of second order. We have proved that the differential equation $y''+\alpha y'+\beta y = 0$ is Mittag-Leffler-Hyers-Ulam stable. Then we consider the stability of Lane-Emden equation of second order.

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1. Introduction

The well-known Ulam stability of functional equations, which was formulated by Ulam on a talk given at Wisconsin University in 1940, is one of the central subjects in the Mathematical analysis area.

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*Corresponding author

For more details on the advancement of Ulam type stability, refer to the papers [5, 14, 15, 16, 17, 20].

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation [1]. In fact, they proved that if a differentiable function $y : I \longrightarrow \mathbb{R}$ satisfies $|y'(t) - y(t)| \leqslant \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \longrightarrow \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \leqslant 3\varepsilon$ for every $t \in I$. The result of Alsina and Ger has been generalized [11, 13, 18].

Furthermore, the result of Hyers-Ulam stability for a first-order linear differential equation has been generalized by Miura, Miyajima and Takahasi [12], by Takahasi, Takagi, Miura and Miyajima [19], and also by Jung [9]. They this problem for the nonhomogeneous linear differential equation of first order $y' + p(t)y + q(t) = 0$.

Jung [10] proved the generalized Hyers-Ulam stability of differential equations of the form $ty'(t) + \alpha y(t) + \beta t^r x_0 = 0$ and also applied this result to the investigation of the Hyers-Ulam stability of the differential equation $t^2y''(t) + \alpha ty'(t) + \beta y(t) = 0$.

Recently, Li and Shen [4] discussed the Hyers-Ulam stability of the following linear differential equations of second order:

$$y'' + \alpha y' + \beta y = 0$$

and

$$y'' + \alpha y' + \beta y = f(x)$$

where $y \in C^2[a, b], f \in C[a, b]$ and $-\infty < a < b < +\infty$.

Recently some authors ([6], [8], [18], [21] and [22]) extended the Ulam stability problem from an integer-order differential equation to a fractional-order differential equation.

There are different types of fractional integral equations. In [2], authors by having defined the types of Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation proved that some kind of fractional integral equations can be somehow approximated by an exact solution of the considered equation.

An important example of second order differential equations is the Lane-Emden equation which has been widely used to describe a variety of phenomena in physics and astrophysics, including aspects of stellar structure, the thermal history of a spherical cloud of gas, isothermal gas spheres, and thermionic currents. Lane-Emden type equations, first published by Jonathan Homer Lane in 1870 [3], and further explored in detail by Emden. Here, we presented similar definitions with [2] and prove stability results for this equation.

In this paper, at first we present the Mittag-Leffler-Hyers-Ulam stability for the following differential equations of second order:

$$y'' + \alpha y' + \beta y = 0 \tag{1}$$

and then as an example of an important second order fractional differential equations we investigate the Mittag-Leffler-Hyers-Ulam stability of Lane-Emden equation.

Definition 1.1. *The fractional order integral of the function f of order $\alpha > 0$ is defined by*

$$I_a^\alpha f(t) = \int_a^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau.$$

Note that when $a = 0$, we write $I_a^\alpha f(t) = f(t) * \Phi_\alpha(t)$, where $(*)$ denotes the convolution product, $\Phi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, t > 0$ and $\Phi_\alpha(t) = 0, t \leqslant 0$ and $\Phi_\alpha \to \delta(t)$ as $\alpha \to 0$ where $\delta(t)$ is the delta function.

Definition 1.2. *The fractional order derivative of the function f of order $\alpha > 0$ is defined by*

$$D_a^\alpha f(t) = \frac{d}{dt}\int_a^t \frac{(t-\tau)^{-\alpha}}{\Gamma(1-\alpha)} f(\tau)d\tau = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

Remark 1.3. *From Definitions (1.1) and (1.2), we have*

$$D^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}, \mu > -1; 0 < \alpha < 1$$

and $I^{\alpha}t^{\mu} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}, \mu > -1; \alpha > 0.$

Definition 1.4. *The Mittag-Leffler function of one parameter is denoted by* $E_{\alpha}(z)$ *and defined as,*

$$E_{\alpha}(z) = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k+1)} z^k, \tag{2}$$

where $z, \alpha \in \mathbb{C}$ *and* $Re(\alpha) > 0$.
If we put $\alpha = 1$, *then the above equation becomes*

$$E_1(z) = \sum_{k=o}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=o}^{\infty} \frac{z^k}{k} = e^z. \tag{3}$$

Definition 1.5. *The generalization of* $E_{\alpha}(z)$ *was studied by Wiman (1905) [5], Agarwal [6] and Humbert and Agarwal [7] defined the function as,*

$$E_{\alpha,\beta}(z) = \sum_{k=o}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} z^k, \tag{4}$$

where $z, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0$ *and* $Re(\beta) > 0$.

2. Mittag-Leffler-Hyers-Ulam Stability of Linear Differential Equations of Second Order

Now, the main result of this work is given in the following theorem.

Definition 2.1. *We call that equation (1) has the Mittag-Leffler-Hyers-Ulam stability if there exists a constant* $K > 0$ *with the following property:*
for every $\varepsilon > 0, y \in C^2[a,b]$, *where* $-\infty < a < b < +\infty$, *if*

$$|y'' + \alpha y' + \beta y| \leqslant \varepsilon E_q,$$

where E_q *is a Mittag-Leffler function, then there exists some* $u \in C^2[a,b]$, *satisfying*

$$u'' + \alpha u' + \beta u = 0,$$

such that $|y(x) - u(x)| < K\varepsilon E_q(x^q)$. *We call* K *a Mittag-Leffler-Hyers-Ulam stability constant for equation (1).*

Theorem 2.2. *If the characteristic equation* $\lambda'' + \alpha\lambda' + \beta = 0$ *has two differential positive roots, then equation (1) has the Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let $\varepsilon > 0$ and $y \in C^2[a, b]$, such that

$$|y'' + \alpha y' + \beta y| \leqslant \varepsilon E_q.$$

We will show that there exists a constant K independent of ε and y such that $|y - u| < K\varepsilon E_q$ for some $u \in C^2[a, b]$ satisfying $u'' + \alpha u' + \beta u = 0$. Let λ_1 and λ_2 be the roots of the following characteristic equation $\lambda'' + \alpha\lambda' + \beta = 0$.
Define $g(x) = y'(x) - \lambda_1 y(x)$. Then $g'(x) = y''(x) - \lambda_1 y'(x)$ thus

$$\begin{aligned} |g'(x) - \lambda_2 g(x)| &= |y''(x) - \lambda_1 y'(x) - \lambda_2(y'(x) - \lambda_1 y(x))| \\ &= |y''(x) + \alpha y'(x) + \beta y(x)| \leqslant \varepsilon E_q(x^q). \end{aligned}$$

So, we have

$$|g'(x) - \lambda_2 g(x)| \leqslant \varepsilon E_q(x^q).$$

Equivalently, g satisfies

$$-\varepsilon E_q(x^q) \leqslant g'(x) - \lambda_2 g(x) \leqslant \varepsilon E_q(x^q).$$

Multiplying this formula by the function $e^{-\lambda_2(x-a)}$, we obtain

$-\varepsilon E_q(x^q)e^{-\lambda_2(x-a)} \leqslant g'(x)e^{-\lambda_2(x-a)} - \lambda_2 g(x)e^{-\lambda_2(x-a)} \leqslant \varepsilon E_q(x^q)e^{-\lambda_2(x-a)}$.

For the case $0 < \lambda_2 \leqslant 1$, with application of the Archimedes Principle, there exists $M > 0$ such that $M\lambda_2 > 1$; so without loss of generality, we may assume that $\lambda_2 > 1$, thus

$$\begin{aligned} -\lambda_2\varepsilon E_q(x^q)e^{-\lambda_2(x-a)} &\leqslant g'(x)e^{-\lambda_2(x-a)} - \lambda_2 g(x)e^{-\lambda_2(x-a)} \\ &\leqslant \lambda_2\varepsilon E_q(x^q)e^{-\lambda_2(x-a)}. \end{aligned} \tag{5}$$

For any $x \in [a, b]$, integrating the inequality from x to b, we get

$$|\int_x^b g'(s)e^{-\lambda_2(s-a)} - \lambda_2 g(s)e^{-\lambda_2(s-a)}ds| \leqslant \int_x^b \lambda_2\varepsilon E_q(s^q)e^{-\lambda_2(s-a)}ds$$
$$= \varepsilon\lambda_2 \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \int_x^b e^{-\lambda_2(s-a)}s^{kq}ds$$
$$\leqslant \varepsilon\lambda_2 \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \int_x^b s^{kq}ds$$
$$\leqslant \varepsilon\lambda_2 \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \frac{b^{kq+1}}{kq+1} - \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \frac{x^{kq+1}}{kq+1}$$
$$= \varepsilon\lambda_2(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

Thus

$$|g(b)e^{-\lambda_2(b-a)} - g(x)e^{-\lambda_2(x-a)}| \leqslant \lambda_2\varepsilon(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

Multiplying this formula by the function $e^{\lambda_2(x-a)}$, we obtain

$$-\varepsilon\lambda_2 e^{\lambda_2(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)) \leqslant g(b)e^{\lambda_2(b-x)} - g(x)$$
$$\leqslant \varepsilon\lambda_2 e^{\lambda_2(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

Let $z(x) = g(b)e^{\lambda_2(x-b)}$. Thus $z'(x) - \lambda_2 z(x) = 0$ and

$$|g(x) - z(x)| \leqslant \varepsilon\lambda_2 e^{\lambda_2(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, therefore

$$-\varepsilon\lambda_2 e^{\lambda_2(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)) \leqslant y'(x) - \lambda_1 y(x) - z(x)$$
$$\leqslant \varepsilon\lambda_2 e^{\lambda_2(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

By multiplying this formula by the function $e^{-\lambda_1(x-a)}$, we obtain

$$-\varepsilon\lambda_2 e^{(\lambda_2-\lambda_1)(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q))$$
$$\leqslant y'(x)e^{-\lambda_1(x-a)} - \lambda_1 e^{-\lambda_1(x-a)}y(x) - e^{-\lambda_1(x-a)}z(x)$$

$$\leqslant \varepsilon\lambda_2 e^{(\lambda_2-\lambda_1)(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

Without loss of generality, we may assume that $\lambda_1 > 1$, thus

$$-\varepsilon\lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q))$$

$$\leqslant y'(x)e^{-\lambda_1(x-a)} - \lambda_1 e^{-\lambda_1(x-a)}y(x) - e^{-\lambda_1(x-a)}z(x)$$

$$\leqslant \varepsilon\lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)(x-a)}(bE_{q,2}(b^q) - xE_{q,2}(x^q)).$$

For any $x \in [a, b]$, integrating the inequality from x to b, we get

$$|\int_x^b y'(s)e^{-\lambda_1(s-a)} - \lambda_1 e^{-\lambda_1(s-a)}y(s) - e^{-\lambda_1(s-a)}z(s)ds|$$

$$\leqslant \int_x^b \varepsilon\lambda_1\lambda_2 e^{(\lambda_2-\lambda_1)(s-a)}(bE_{q,2}(b^q) - sE_{q,2}(s^q))ds =$$

$$\varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)} - e^{(x-a)}) - \varepsilon\lambda_1\lambda_2\int_x^b se^{(\lambda_2-\lambda_1)(s-a)}E_{q,2}(s^q)ds$$

$$= \varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)} - e^{(x-a)}) - \varepsilon\lambda_1\lambda_2\sum_{m=0}^{\infty}\frac{1}{\Gamma(mq+2)}\int_x^b s^{mq+1}e^{(\lambda_2-\lambda_1)(x-a)}dx.$$

Again, without loss of generality, we may assume that $\lambda_2 > \lambda_1$, thus $e^{(\lambda_2-\lambda_1)(x-a)} > 1$. Therefore

$$|\int_x^b y'(s)e^{-\lambda_1(s-a)} - \lambda_1 e^{-\lambda_1(s-a)}y(s) - e^{-\lambda_1(x-a)}z(s)ds| \leqslant$$

$$\varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)} - e^{(x-a)}) - \varepsilon\lambda_1\lambda_2\sum_{m=0}^{\infty}\frac{1}{\Gamma(mq+2)}\int_x^b s^{mq+1}dx$$

$$\leqslant \varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)}-e^{(x-a)})-$$
$$\varepsilon\lambda_1\lambda_2\sum_{m=0}^{\infty}\frac{1}{\Gamma(mq+2)}\frac{1}{mq+2}(b^{mq+2}-x^{mq+2})\leqslant$$
$$\varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)}-e^{(x-a)})-\varepsilon\lambda_1\lambda_2 b^2E_{q,3}(b^q)+$$
$$\varepsilon\lambda_1\lambda_2 x^2E_{q,3}(x^q).$$

Let

$$\varepsilon\lambda_1\lambda_2 bE_{q,2}(b^q)\frac{1}{\lambda_2-\lambda_1}e^{\lambda_2-\lambda_1}(e^{(b-a)}-e^{(x-a)})-\varepsilon\lambda_1\lambda_2 b^2E_{q,3}(b^q)=K.$$

We have

$$|\int_x^b y'(s)e^{-\lambda_1(s-a)}-\lambda_1e^{-\lambda_1(s-a)}y(s)-e^{-\lambda_1(s-a)}z(s)ds|\leqslant$$
$$K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q).$$

So

$$-(K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q))\leqslant$$
$$y(b)e^{-\lambda_1(b-a)}-y(x)e^{-\lambda_1(x-a)}-\int_x^b z(s)e^{-\lambda_1(s-a)}ds$$
$$\leqslant K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q).$$

Multiplying this formula by the function $e^{\lambda_1(x-a)}$, we obtain

$$-(K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q))e^{\lambda_1(x-a)}\leqslant$$
$$y(b)e^{-\lambda_1(b-a)}e^{\lambda_1(x-a)}-y(x)-e^{\lambda_1(x-a)}\int_x^b z(s)e^{-\lambda_1(s-a)}ds$$
$$\leqslant (K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q))e^{\lambda_1(x-a)}.$$

Hence we may suppose that $u(x)=y(b)e^{-\lambda_1(b-x)}-e^{\lambda_1(x-a)}\int_x^b z(s)e^{-\lambda_1(s-a)}dx$ such that

$$|y(x)-u(x)|\leqslant (K+\varepsilon\lambda_1\lambda_2x^2E_{q,3}(x^q))e^{\lambda_1(x-a)}.$$

Also $u \in C^2[a,b]$ and $u'(x) - \lambda_1 u(x) - z(x) = 0$. Hence

$$z(x) = u'(x) - \lambda_1 u(x).$$

Since $z'(x) - \lambda_2 z(x) = 0$, we obtain $u''(x) - \lambda_1 u'(x) - \lambda_2(u'(x) - \lambda_1 u(x)) = 0$. Thus

$$u''(x) + \alpha u'(x) + \beta u(x) = 0.$$

This completes the proof of our theorem. □

Theorem 2.3. *Assume that the characteristic equation* $\lambda^2 + \alpha\lambda + \beta = 0$ *has two different positive roots. Then for every* $\varepsilon > 0, f \in C[a,b]$ *and* $y \in C^2[a,b]$, *if* $|y'' + \alpha y' + \beta y - f(x)| \leqslant \varepsilon E_q(x^Q)$, *there exist some* $u \in C^2[a,b]$ *and* $K > 0$ *satisfying* $u'' + \alpha u' + \beta u = f(x)$ *such that* $|y(x) - u(x)| \leqslant K\varepsilon E_q(x^q)$ *i.e. equation* $y'' + \alpha y' + \beta y = f(x)$ *has the Mittag-Leffler-Hyers-Ulam stability.*

Proof. Similar to the proof of Theorem (2.2), let λ_1 and λ_2 be the roots of characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$. Without lose of generality, we may assume that $\lambda_1 > 1$ and $\lambda_2 > 1$. Define $g(x) = y'(x) - \lambda_1 y(x)$, we obtain $g'(x) = y''(x) - \lambda_1 y'(x)$; thus

$$|g'(x) - \lambda_2 g(x) - f(x)| = |y''(x) - \lambda_1 y'(x) - \lambda_2(y'(x) - \lambda_1 y(x)) - f(x)| = |y''(x) + \alpha y'(x) + \beta y(x) - f(x)| \leqslant \varepsilon E_q(x^q).$$

So

$$-\varepsilon E_q(x^q) \leqslant g'(x) - \lambda_2 g(x) - f(x) \leqslant \varepsilon E_q(x^q).$$

Multiplying this formula by the function $e^{-\lambda_2 x}$, we obtain

$$-\varepsilon E_q(x^q) e^{-\lambda_2 x} \leqslant g'(x) e^{-\lambda_2 x} - \lambda_2 g(x) e^{-\lambda_2 x} - e^{-\lambda_2 x} f(x) \leqslant \varepsilon E_q(x^q) e^{-\lambda_2 x}.$$

For any $x \in [a,b]$, integrating the inequality from x to b, we get

$$\left| \int_x^b g'(s) e^{-\lambda_2 s} - \lambda_2 g(s) e^{-\lambda_2 s} - e^{-\lambda_2 s} f(s) ds \right| \leqslant \int_x^b \varepsilon E_q(s^q) e^{-\lambda_2 s} ds$$

$$
\begin{aligned}
&= \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \int_x^b e^{-\lambda_2 s} s^{kq} ds \\
&\leqslant \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \int_x^b s^{kq} ds \\
&\leqslant \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \frac{b^{kq+1}}{kq+1} - \varepsilon \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} \frac{x^{kq+1}}{kq+1} \\
&= \varepsilon (b E_{q,2}(b^q) - x E_{q,2}(x^q)).
\end{aligned}
$$

Thus

$$|g(b)e^{-\lambda_2 b} - g(x)e^{-\lambda_2 x} - \int_x^b e^{-\lambda_2 s} f(s) ds| \leqslant \varepsilon (b E_{q,2}(b^q) - x E_{q,2}(x^q)).$$

Multiplying this formula by the function $e^{\lambda_2 x}$, we obtain

$$
\begin{aligned}
-\varepsilon e^{\lambda_2 x}(b E_{q,2}(b^q) - x E_{q,2}(x^q)) &\leqslant g(b) e^{\lambda_2 (x-b)} - g(x) - e^{\lambda_2 x} \int_b^x e^{-\lambda_2 s} f(s) ds \\
&\leqslant \varepsilon e^{\lambda_2 x}(b E_{q,2}(b^q) - x E_{q,2}(x^q)).
\end{aligned}
$$

Let $z(x) = g(b) e^{\lambda_2 (x-b)} - e^{\lambda_2 x} \int_x^b e^{-\lambda_2 s} f(s) ds$. So we have

$$z'(x) - \lambda_2 z(x) - f(x) = 0$$

and

$$|g(x) - z(x)| \leqslant \varepsilon e^{\lambda_2 x}(b E_{q,2}(b^q) - x E_{q,2}(x^q)).$$

Since $g(x) = y'(x) - \lambda_1 y(x)$, therefore

$$
\begin{aligned}
-\varepsilon e^{\lambda_2 x}(b E_{q,2}(b^q) - x E_{q,2}(x^q)) \leqslant y'(x) - \lambda_1 y(x) - z(x) \leqslant \\
\varepsilon e^{\lambda_2 x}(b E_{q,2}(b^q) - x E_{q,2}(x^q)).
\end{aligned}
$$

By an argument similar to the above one, we can show that there exists

$$u(x) = y(b) e^{\lambda_1 (x-b)} - e^{\lambda_1 x} \int_x^b z(s) e^{-\lambda_1 s} ds$$

such that

$$|y(x) - u(x)| \leqslant (K + \varepsilon x^2 E_{q,3}(x^q)) e^{\lambda_1 x},$$

where $u \in C^2[a, b]$ and $u''(x) + \alpha u'(x) + \beta u(x) - f(x) = 0$. Thus, the proof is complete. □

3. Mittag-Leffler-Hyers-Ulam Stability of Lane-Emden Equation

In this section, we consider the fractional Lane-Emden equations of the following form

$$D^{\beta}(D^{\alpha} + \frac{a}{t})u(t) + f(t,u) = g(t), (0 < t \leqslant 1, a \geqslant 0, 0 < \alpha, \beta \leqslant 1), \quad (6)$$

with the boundary condition

$$u(0) = \mu, \qquad u(1) = \nu,$$

where $f(t,u)$ is a continuous real valued function and $g(t) \in C[0,1]$.

Lemma 3.1. *A unique solution of the linear two-point bounded value problem for Lane-Emden equation (6) is given by*

$$u(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}(\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\alpha)} g(s)ds - \frac{a}{\tau}u(\tau))d(\tau) -$$
$$t^{\alpha}[\int_0^1 \frac{(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}(\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)} g(s)ds - \frac{a}{\tau}u(\tau))d\tau] + (\nu-\mu)t^{\alpha} + \mu,$$

where $g \in C[0,1]$.

Proof. (see [7]). □

Theorem 3.2. *Let* $f : [0,1] \times X \to X$ *be a jointly continuous function satisfying the condition*

$$|f(t,u) - f(t,v)| \leqslant L|u-v|, \qquad \forall t \in [0,1], u, v \in X.$$

Moreover, assume that $sup_{t\in[0,1]}|g(t)| = \gamma$. *Then the boundary value problem (6) has a unique solution provided that the following condition holds:*

$$l = \frac{2L}{\Gamma(\alpha+\beta+1)} + \frac{2a\Gamma(\alpha)}{\Gamma(2\alpha)} < 1, \qquad a \geqslant 0.$$

Proof. (see [7]). □

Theorem 3.3. *Let* $f : [0,1] \times X \to X$ *be a jointly continuous function and maps bounded subsets of* $[0,1] \times X$ *into relatively compact subsets of* X. *Furthermore, assume that*

$$|f(t,u) - f(t,v)| \leqslant L|u-v|, \qquad \forall t \in [0,1], u, v \in X,$$

$$|f(t,u)| \leqslant \sigma(t), \qquad \sigma \in (L^1[0,1], \mathbb{R}^+),$$

and $sup_{t\in[0,1]}|g(t)| = \gamma$. *If*

$$\frac{L}{\Gamma(\alpha+\beta+1)} + \frac{a\Gamma(\alpha)}{\Gamma(2\alpha)} < 1,$$

then the boundary value problem (6) has at least one solution on $[0,1]$.

Proof. (see [7]). □

Now we consider the Mittag-Leffler-Hyers-Ulam stability of fractional Lane-Emden equation (6).

Definition 3.4. *The equation (6) has the Mittag-Lefller-Hyers-Ulam stability if there exists a positive constant* K *with the following property :*
for every $\varepsilon > 0$ *if*

$$|D^\beta(D^\alpha + \frac{a}{t})u(t) + f(t,u) - g(t)| < \varepsilon E_q(t^q)$$

then there exists some $v \in X$ *satisfying*

$$D^\beta(D^\alpha + \frac{a}{t})v(t) + f(t,v) = g(t)$$

with $v(0) = \mu, v(1) = \nu$ *such that*

$$|u(t) - v(t)| < K\varepsilon E_q(t^q).$$

Theorem 3.5. *Let the assumptions of Theorem 3.2 hold. If*

$$sup|D^\beta(D^\alpha + \frac{a}{t})u(t)| \geqslant \frac{2(\gamma+M+Lr)}{\Gamma(\alpha+\beta+1)} + \frac{2ar\Gamma(\alpha)}{\Gamma(2\alpha)} + |\nu| + 2|\mu|$$

which $sup_{t\in[0,1]}|f(t,u)| = M$ *and* $r \geqslant \frac{1}{1-\varepsilon}(\frac{2(\gamma+M)}{\Gamma(\alpha+\beta+1)} + 2\mu + \nu)$, *then the equation (6) has the Mittag-Lefller-Hyers-Ulam stability in* X.

Proof. For every $\varepsilon > 0$ we let

$$|D^\beta(D^\alpha + \frac{a}{t})u(t) + f(t,u) - g(t)| < \varepsilon E_q(t^q).$$

Under the assumptions of Theorem (3.2), equation (6) has a unique solution in X. By using the method of proof of Theorem (3.2) [7], since

$$|u(t)| \leqslant \frac{2[\gamma + M + Lr]}{\Gamma(\alpha+\beta+1)} + \frac{2ar\Gamma(\alpha)}{\Gamma(2\alpha)} + |\nu| + 2|\mu|,$$

we have

$$sup|u| \leqslant sup|D^\beta(D^\alpha + \frac{a}{t})u(t)|.$$

Thus we have

$$sup|u(t) - v(t)| \leqslant sup|D^\beta(D^\alpha + \frac{a}{t})(u(t) - v(t))| \leqslant$$
$$sup|D^\beta(D^\alpha + \frac{a}{t})u(t) - D^\beta(D^\alpha + \frac{a}{t})v(t) - f(t,u) + f(t,v) + g(t) -$$
$$g(t)| + sup|f(t,u) - f(t,v)| \leqslant \varepsilon E_q(t^q) + Lsup|u(t) - v(t)|.$$

Hence we obtain

$$sup|u(t) - v(t)| \leqslant \frac{\varepsilon E_q(t^q)}{1-L} = K\varepsilon E_q(t^q).$$

Thus the equation (6) has the Mittag-Lefller-Hyers-Ulam stability. □

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Vida Kalvandi
Ph.D Student of Mathematics
Department of Mathematics
Faculty of Mathematical Sciences
University of Mohaghegh Ardabili
P.O. Box 56199-11367
Ardabil, Iran.
E-mail: vida.kalvandi@yahoo.com

Nasrin Eghbali
Associate Professor of Mathematics
Department of Mathematics
Faculty of Mathematical Sciences
University of Mohaghegh Ardabili
P.O. Box 56199-11367
Ardabil, Iran.
E-mail: nasrineghbali@gmail.com

John Michael Rassias
Professor of Mathematics
Pedagogical Department E. E.
Section of Mathematics and Informatics
National and Capodistrian University of Athens
E-mail: jrassias@primedu.uoa.gr
Ioannis.Rassias@primedu.uoa.gr
jrass@otenet.gr