Journal of Mathematical Extension
Vol. 12, No. 1, (2018), 13-23
ISSN: 1735-8299
URL: http://www.ijmex.com

Toroidal and Projective Cyclic Graphs

M. Naghshinehfard*
Jahrom Branch, Islamic Azad University

F. Parvaneh
Kermanshah Branch, Islamic Azad University

Abstract. All finite groups with toroidal or projective cyclic graphs are classified. Indeed, it is shown that the only finite groups with projective cyclic graphs are $S_3 \times \mathbb{Z}_2$, D_{14}, QD_{16} and $\langle x, y : x^7 = y^3 = 1, x^y = x^2 \rangle$ which all have toroidal cyclic graph too. Also, D_{16} is characterized as the only finite group whose cyclic graph is toroidal but not projective.

AMS Subject Classification: 05C25; 05C10
Keywords and Phrases: Cyclic graph, toroidal, projective

1. Introduction

The *cyclicizer* of a group G, denoted by $Cyc(G)$, is defined as the set of those elements of G which make a cyclic subgroup with any other element of G. In other words,

$$Cyc(G) = x \in G : \langle x, y \rangle \text{ is cyclic for all } y \in G.$$

It is evident that $Cyc(G) = \bigcap_{x \in G} Cyc_G(x)$, where

$$Cyc_G(x) = \{y \in G : \langle x, y \rangle \text{ is cyclic}\},$$

Received: May 2017; Accepted: December 2017
*Corresponding author

is the *cyclicizer* of the element x in G. Element cyclicizers and the cyclicizers of groups first defined and studied by Patrick and Wepsic in [19] under the condition that all element cyclicizers are subgroups. We refer the interested reader to [3, 4, 5, 11, 18, 19] for further progress and details.

The *cyclic graph* of a group G is defined as a graph whose vertex set is $G \setminus Cyc(G)$ in which two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is cyclic. The complement of a cyclic graph is already studied by Abdollahi and Hassanabadi [1, 2]. While the complement of a cyclic graph is always connected with small diameter, the cyclic graph of a group has fewer edges and better represent the relationship among elements of the group. Such a relationship will be more clear when the graph can be drawn on a surface (say a plane, torus or projective plane) in which no two edges cross. Note that embedding groups on surfaces via relation among elements dates back to 1878 where Cayley [10] defined the Cayley graphs and 1896 where Maschke [16] classified all planar finite Cayley graphs. In [17] we have initiated studying the cyclic graph of groups by determination of those finite groups with a planar cyclic graph. Remind that a graph is *planar* if it has an embedding on the plane in such a way that distinct edges intersect only at the end vertices. The aim of this paper is to consider two more popular surfaces, namely torus and projective plane, and to classify all finite groups with a toroidal or projective cyclic graph, where the terms toroidal and projective are defined as in the following. Although there are infinitely many planar cyclic graphs (see [17]), our results show that there are just a few finite groups with toroidal or projective cyclic graphs.

The orientable (resp. non-orientable) surface of genus g (resp. crosscap number g), denoted by N_g (resp. $\overline{N}_g$), is the connected sum of g torus (resp. projective planes). A graph has *genus* or *orientable genus* g if it can be embedded in N_g and that g is the least number with this property. Similarly, a graph has *crosscap number* or *non-orientable genus* g if it can be embedded in $\overline{N}_g$ and that g is minimum with respect to this property. The genus and crosscap numbers of a graph Γ are denoted by $\gamma(\Gamma)$ and $\overline{\gamma}(\Gamma)$, respectively. A *toroidal graph* is a graph of genus 1 and a *projective* graph is a graph of crosscap number 1.

Throughout this paper we use the following notations without further reference:

- $\bar{\Gamma}$: The complement of Γ obtained by complementing the edge-set of Γ;

- $\Gamma_1 \cdot \Gamma_2$: The dot product of two vertex transitive graphs Γ_1 and Γ_2 obtained by unifying a vertex of Γ_1 with a vertex of Γ_2, where by a vertex transitive graph we mean a graph whose automorphism group acts transitively on its vertex set;

- $\Gamma_1 \cup \Gamma_2$: The union of two disjoint graphs Γ_1 and Γ_2 obtained from the juxtaposition of Γ_1 and Γ_2;

- $n\Gamma$: The union of n disjoint copies of the graph Γ;

- $\Gamma_1 + \Gamma_2$: The sum of two disjoint graphs Γ_1 and Γ_2 obtained from connecting every vertex of Γ_1 to every vertex of Γ_2.

Invoking the above notations, a friendship graph is defined as a sum $K_1 + nK_2$ for some non-negative integer n.

Also, if G is a finite group, then $\omega(G) = \{|g| : g \in G\}$ illustrates the spectrum of G. An arbitrary Sylow p-subgroup of G will be denoted by $S_p(G)$ for every prime divisor p of the order of G. All groups in this paper are assume to be finite.

2. Preliminary Results

We begin with recalling two famous results on the genus and crosscap number of complete graphs, which play an important role in the proof of our main results.

Theorem 2.1. ([20]) *For any positive integer n,*

$$\gamma(K_n) = \begin{cases} 0, & n = 1, 2, 3, 4, \\ \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, & n \geqslant 5. \end{cases}$$

Theorem 2.2. ([9, 20]) *For any positive integer n,*

$$\overline{\gamma}(K_n) = \begin{cases} 0, & n = 1,2,3,4, \\ 3, & n = 7, \\ \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, & 7 \neq n \geqslant 5. \end{cases}$$

Lemma 2.3. *The following results hold.*

(1) A toroidal graph has no subgraph isomorphic to K_8, $2K_5$, $K_5 \cdot K_5$ or $3K_3 + 2\overline{K}_2$.

(2) A projective graph has no subgraph isomorphic to K_7, $2K_5$, $K_5 \cdot K_5$ or $2K_3 + 2\overline{K}_2$.

Proof. (1) It follows from Theorem 2.1 and the forbidden subgraphs $G_1 \cong 2K_5$, $G_2 \cong K_5 \cdot K_5$ and $G_4 \cong 3K_2 + 2\overline{K}_2$ as stated in [12].

(2) It follows from Theorem 2.2 and the forbidden subgraphs $A_1 \cong K_5 \cdot K_5$, $A_5 = 2K_5$ and $B_3 = 2K_2 + 2\overline{K}_2$ as stated in [13]. □

It is known that the cyclicizer of a finite group is always a cyclic central subgroup of G (see [19]). Hence, $Cyc(G) = \langle z \rangle$ for some element z. In what follows, $\overline{G}$ stands for the factor group $G/Cyc(G)$ and that $\overline{g}$ denotes the element $gCyc(G)$ of $\overline{G}$ for any $g \in G$. The following two simple results are used frequently in our proofs.

Lemma 2.4. *The Sylow p-subgroups of G are either cyclic or generalized quaternion 2-group for each prime divisor p of $|z|$.*

Proof. Let p be a prime divisor of $|z|$ and P be a Sylow p-subgroup of G. If $z_p \in P \cap \langle z \rangle$ is an element of order p and $x \in P$, then $\langle x, z_p \rangle$ is cyclic, which implies that $z_p \in \langle x \rangle$, that is, P has only one subgroup of order p. Thus, by [21, 5.3.6]djsr, P is either a cyclic group or a generalized quaternion 2-group. □

Lemma 2.5. *If $\Gamma_c(G)$ has no subgraphs isomorphic to K_8, then $|z|(|\overline{x}| - 1) \leqslant 7$ for all $x \in G \setminus \langle z \rangle$. As a result, $|z| \leqslant 7$.*

Proof. Since, for $x \in G$, the set $\langle x, z \rangle \setminus \langle z \rangle$ induces a complete subgraph of $\Gamma_c(G)$, it follows that $|\langle x, x \rangle \setminus \langle z \rangle| \leqslant 7$. Hence, $|z|(|\overline{x}| - 1) \leqslant 7$. □

3. Main Theorems

In order to classify finite groups with a toroidal or a projective cyclic graph, we first restrict ourself to those groups whose cyclic graphs have no subgraphs isomorphic to K_8, $2K_5$, $K_5 \cdot K_5$ and $3K_3 + 2\overline{K}_2$, which are forbidden subgraphs for both toroidal and projective graphs by Theorem 2.3.

Lemma 3.1. *If $\Gamma_c(G)$ is a non-planar graph with no subgraph isomorphic to K_8, $2K_5$, $K_5 \cdot K_5$ or $3K_3 + 2\overline{K}_2$, then G is isomorphic to one of the groups $S_3 \times \mathbb{Z}_2$, D_{14}, D_{16}, Q_{16}, QD_{16} or $\langle x, y : x^7 = y^3 = 1, x^y = x^2 \rangle$.*

Proof. We proceed in some steps according to the order of z. Note that, by Lemma 2.5, $|z| \leqslant 7$.

Step 1. $|z| \geqslant 4$. By Lemma 2.5, $|\overline{x}| \leqslant 2$ for all $x \in G \setminus \langle z \rangle$. Thus, $\overline{G}$ is an elementary abelian 2-group. If $|z| = 5$ or 7, then $G \cong \mathbb{Z}_{|z|} \times \mathbb{Z}_2^n$ for some $n \geqslant 2$. It is easy to see that $\Gamma_c(G)$ is a union of some complete graphs with vertex sets $x\langle z \rangle$, where x ranges over all involutions of G. But then $\Gamma_c(G)$ has a subgraph isomorphic to $2K_{|z|}$, which is a contradiction

Next assume that $|z| = 6$. Then $G \cong \langle x \rangle \times P$, where P is a Sylow 2-subgroup of G and $|x| = 3$. Since P is non-cyclic, from the Lemma 2.4, it follows that $P \cong Q_{2^n}$ is a generalized quaternion 2-group for some positive integer n. Then $\overline{G} \cong D_{2^{n-1}}$, which implies that $n = 3$ and $P \cong Q_8$. A simple verification shows that $\Gamma_c(G)$ is a union of some complete graphs with vertex sets $\langle x \rangle y^{\pm 1}$, where y ranges over all elements of $P \setminus P'$. Indeed, $\Gamma_c(G) \cong 3K_6$, which is a contradiction.

Finally assume that $|z| = 4$. Then G is a 2-group and hence $G \cong Q_{2^n}$ for some $n \geqslant 3$ by Lemma 2.4. Now, if $x \in G \setminus \langle z \rangle$ is an element of order 4, then $\langle x, z \rangle$ is cyclic from which it follows that $\langle x \rangle = \langle z \rangle$, a contradiction.

Step 2. $|z| = 3$. By Lemma 2.5, $|\overline{x}| \leqslant 3$ for all $x \in G$, that is, $\omega(\overline{G}) \subseteq \{1, 2, 3\}$. Since $S_3(G)$ is cyclic by Lemma 2.4, G is not a 3-group. If $\overline{G}$ is a 2-group, then $G \cong \mathbb{Z}_3 \times \mathbb{Z}_2^n$ for some $n \geqslant 2$ so that $\Gamma_c(G)$ is a union of some complete graphs with vertex sets $x\langle z \rangle$, where x ranges over all involutions of G. Hence, $\Gamma_c(G)$ is planar, which is a contradiction. Thus, we assume that $\overline{G}$ is neither a 2-group nor a 3-group. By [8], we have to consider two possibilities.

(i) $\overline{G} = (\langle \overline{x}_1 \rangle \times \cdots \times \langle \overline{x}_n \rangle) \rtimes \langle \overline{y} \rangle$, where $|\overline{x}_i| = 3$ for $i = 1, \ldots, n$, $|\overline{y}| = 2$ and $\overline{x}_i^{\overline{y}} = \overline{x}_i^{-1}$. Since $S_3(G)$ is cyclic, we must have $n = 1$ and consequently $G = \langle x_1 \rangle \rtimes \langle y \rangle$ where $|x_1| = 9$. It is easy to see that $x_1^y = x_1^{-1}$ so that $G \cong D_{18}$, which is impossible for $Cyc(D_{18}) = 1$.

(ii) $\overline{G} = (\overline{V}_1 \times \cdots \times \overline{V}_n) \rtimes \langle \overline{y} \rangle$, where $\overline{V}_i = \langle \overline{x}_i \rangle \times \langle \overline{x}'_i \rangle$ is a Klein 4-group for $i = 1, \ldots, n$, $|\overline{y}| = 3$ and $\overline{y}$ acts on $\overline{V}_i \setminus \{\overline{1}\}$ as the cyclic permutation $(\overline{x}_i, \overline{x}'_i, \overline{x}_i\overline{x}'_i)$ for all $i = 1, \ldots, n$. Then $G = (V_1 \times \cdots \times V_n) \rtimes \langle y \rangle$ such that $V_i = \langle x_i \rangle \times \langle x'_i \rangle$ is a Klein 4-group for $i = 1, \ldots, n$, $|y| = 9$ and y acts on $V_i \setminus \{1\}$ as the cyclic permutation $(x_i, x'_i, x_i x'_i)$ for all $i = 1, \ldots, n$. A simple verification shows that $\Gamma_c(G)$ is a union of complete subgraphs with vertex sets $\langle x, z \rangle \setminus \langle z \rangle$ and $\langle xy \rangle \setminus \langle x \rangle$ of order 3 and 6, respectively, where x ranges over all nontrivial elements of $V_1 \times \cdots \times V_n$. But then $\Gamma_c(G)$ has a subgraph isomorphic to $2K_6$, which is a contradiction.

Step 3. $|z| = 2$. By Lemma 2.5, $|\overline{x}| \leqslant 4$ for each $x \in G$ so that $\omega(\overline{G}) \subseteq \{1, 2, 3, 4\}$. Hence, $\omega(G) \subseteq \{1, 2, 3, 4, 6, 8\}$. One observes that for every $x \in G$ with $|x| = 3$ or 6, $\langle x, z \rangle \setminus \langle z \rangle$ is a connected component of $\Gamma_c(G)$ isomorphic to K_4. First assume that $8 \notin \omega(G)$. Then, by Lemma 2.4, $4 \notin \omega(\overline{G})$. Let $X = \{x \in G : |x| \in \{3, 6\}\}$. Since $\langle z \rangle$ is the only subgroup of order two of any Sylow 2-subgroup of G, it follows that $|x| = 4$ for all $x \in G \setminus \langle z \rangle \cup X$. Thus, $\Gamma_c(G)$ is a union of edges $\langle x \rangle \setminus \langle z \rangle$ with $|x| = 4$ and complete subgraphs with four vertices $\langle x, z \rangle \setminus \langle z \rangle$ for $x \in X$, hence $\Gamma_c(G)$ is planar, which is a contradiction. Therefore, $8 \in \omega(G)$. Then $|\langle x \rangle \cap \langle y \rangle| \geqslant 4$ for all $x, y \in G$ with $|x| = |y| = 8$, for otherwise $\langle x \rangle \cap \langle y \rangle \subseteq \langle z \rangle$ and consequently the subgraph induced by $\langle x \rangle \cup \langle y \rangle \setminus \langle z \rangle$ is isomorphic to $2K_6$, which is impossible. Now, we consider the following two cases:

(i) G has a unique cyclic subgroup $\langle x \rangle$ of order 8. Clearly, $\langle x \rangle \trianglelefteq G$ and subsequently $G/C_G(x)$ is isomorphic to a subgroup of $Aut(\langle x \rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that $C_G(x) = \langle x \rangle$. Thus $G/\langle x \rangle \cong \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, we have $|G| = 16$ or 32. A simple computation with GAP shows that the only group with these properties is Q_{16} whose cyclic graph is isomorphic to $K_6 \cup 4K_2$.

(ii) There are distinct cyclic subgroups $\langle x\rangle$ and $\langle y\rangle$ of order 8. Then $|\langle x\rangle \cap \langle y\rangle| = 4$. Now, for every element $g \in \langle x\rangle \cap \langle y\rangle \setminus \langle z\rangle$, the subgraph induced by $\{x^{\pm1}, x^{\pm3}, y^{\pm1}, y^{\pm3}, g\}$ is isomorphic to $K_5 \cdot K_5$, which is impossible.

Step 4. $|z| = 1$. By Lemma 2.5, $|x| \leqslant 8$ for all $x \in G$, that is, $\omega(G) \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$. We proceed in some cases:

(i) $7 \in \omega(G)$. Let P be the Sylow 7-subgroup of G. Then $P \cong \mathbb{Z}_7$ and $P \trianglelefteq G$ for otherwise $P \neq P^g$ for some $g \in G$ and hence the subgraph induced by $P \cup P^g \setminus \{1\}$ is isomorphic to $2K_6$, which is a contradiction. Clearly, $C_G(P) = P$. Thus $G/P = N_G(P)/C_G(P)$ is isomorphic to a subgroup of $Aut(P) \cong \mathbb{Z}_6$. If $G/P \cong \mathbb{Z}_6$, then G has an element g of order 6 and consequently the subgraph induced by $P \cup \langle g\rangle \setminus \{1\}$ is isomorphic to $K_5 \cup K_6$, which is impossible. Thus $|G/P| = 2$ or 3, from which it follows that $G \cong D_{14}$ or $\langle x, y : x^7 = y^3 = 1, x^y = x^2\rangle$. One can easily see that, $\Gamma_c(G) \cong K_6 \cup 7K_1$ in the former case and $\Gamma_c(G) \cong K_6 \cup 7K_2$ in the latter case.

(ii) $7 \notin \omega(G)$ and $8 \in \omega(G)$. Let $x, y \in G$ with $|x| = |y| = 8$. If $\langle x\rangle \cap \langle y\rangle = 1$, then the subgraph induced by $\langle x\rangle \cup \langle y\rangle \setminus \{1\}$ is isomorphic to $2K_7$, which is impossible. Also, if $\langle x\rangle \cap \langle y\rangle \neq 1$ and $|\langle x\rangle \cap \langle y\rangle| \leqslant 4$, then the subgraph induced by $\{x^{\pm1}, x^{\pm3}, y^{\pm1}, y^{\pm3}, g\}$ where $g \in \langle x\rangle \cap \langle y\rangle \setminus \{1\}$ is isomorphic to $K_5 \cdot K_5$, which is a contradiction. Therefore, $\langle x\rangle = \langle y\rangle$, which implies that $\langle x\rangle$ is the unique cyclic subgroup of G of order 8. Clearly, $\langle x\rangle \trianglelefteq G$ and $C_G(x) = \langle x\rangle$. Thus $G/\langle x\rangle = N_G(\langle x\rangle)/C_G(x)$ is isomorphic to a subgroup of $Aut(\langle x\rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and consequently $|G| = 16$ or 32. Now, a simple computation with GAP shows that the only groups with these properties are D_{16} and QD_{16}. It is easy to see that, $\Gamma_c(G) \cong K_7 \cup 8K_1$ in the former case and $\Gamma_c(G) \cong ((K_6 \cup 2K_2) + K_1) \cup 4K_1$ in the latter case.

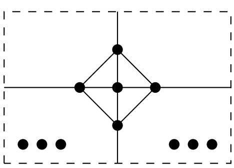
(iii) $7, 8 \notin \omega(G)$ and $6 \in \omega(G)$. Let $x, y \in G$ with $|x| = |y| = 6$. If $\langle x\rangle \cap \langle y\rangle = 1$, then $\Gamma_c(G)$ contains a subgraph isomorphic to $2K_5$, which is a contradiction. Also, if $|\langle x\rangle \cap \langle y\rangle| = 2$, then $\Gamma_c(G)$ contains a subgraph isomorphic to $K_5 \cdot K_5$, which is another contradiction. Hence, either $|\langle x\rangle \cap \langle y\rangle| = 3$ or $\langle x\rangle = \langle y\rangle$.

If G has a unique cyclic subgroup $\langle x\rangle$ of order 6, then $\langle x\rangle \trianglelefteq G$. Also, $C_G(x) = \langle x\rangle$, which implies that $G/\langle x\rangle = N_G(\langle x\rangle)/C_G(x)$ is isomorphic to a subgroup of $Aut(\langle x\rangle) \cong \mathbb{Z}_2$. As G is non-cyclic we must have $|G| = 12$. Thus $G \cong S_3 \times \mathbb{Z}_2$ or $\langle x, y : x^3 = y^4 = 1, x^y = x^{-1}\rangle$. Since the latter group has a planar cyclic graph, we have $G \cong S_3 \times \mathbb{Z}_2$ whose cyclic graph is isomorphic to $K_5 \cup 6K_1$.
Now, assume that G has $n \geqslant 2$ distinct cyclic subgroups $\langle x_1\rangle, \dots, \langle x_n\rangle$ of order 6. Then $\langle x_1\rangle \cap \cdots \cap \langle x_n\rangle = \langle y\rangle$ is a cyclic group of order 3. Clearly, the subgraph induced by $\langle x_1\rangle \cup \cdots \cup \langle x_n\rangle$ is isomorphic to $nK_3 + 2\overline{K}_2$. Hence, by hypothesis, we must have $n = 2$. Put $N = N_G(\langle x_1\rangle)$. Since the only possible conjugates of $\langle x_1\rangle$ are $\langle x_1\rangle$ and $\langle x_2\rangle$, we have $[G : N] \leqslant 2$. On the other hand, $C_G(x_1) = \langle x_1\rangle$, which implies that $N/\langle x_1\rangle = N_G(\langle x_1\rangle)/C_G(x_1)$ is isomorphic to a subgroup of $Aut(\langle x_1\rangle) \cong \mathbb{Z}_2$. Thus $|G|$ divides 24 and a simple computation with GAP shows that there is no group with these properties.

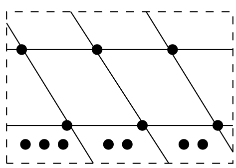
(iv) $6, 7, 8 \notin \omega(G)$. Then $\omega(G) \subseteq \{1, 2, 3, 4, 5\}$ and a simple verification shows that $\Gamma_c(G)$ is a union of some complete graphs with at most 4 vertices and some friendship graphs. Thus, $\Gamma_c(G)$ is planar, which contradicts the hypothesis. The proof is complete. $\square$

Utilizing the above lemma we are now able to state our main theorems.

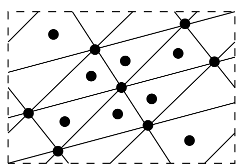
Theorem 3.2. *Let G be a finite group with a toroidal cyclic graph. Then G is isomorphic to one of the groups $S_3 \times \mathbb{Z}_2$, D_{14}, D_{16}, Q_{16}, QD_{16} or $\langle x, y : x^7 = y^3 = 1, x^y = x^2\rangle$.*



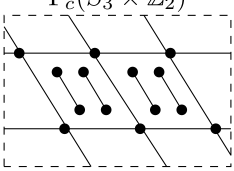
$\Gamma_c(S_3 \times \mathbb{Z}_2)$



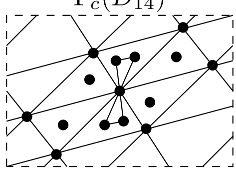
$\Gamma_c(D_{14})$



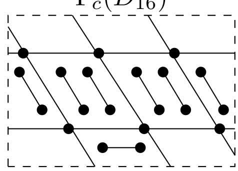
$\Gamma_c(D_{16})$



$\Gamma_c(Q_{16})$

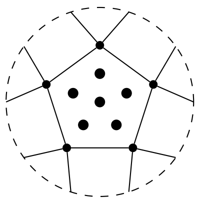


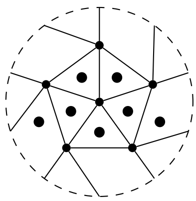
$\Gamma_c(QD_{16})$

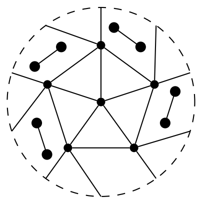


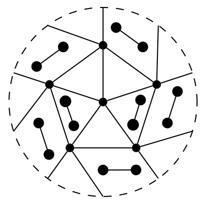
$\Gamma_c(\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$

Theorem 3.3. *Let G be a finite group with a projective cyclic graph. Then G is isomorphic to one of the groups*
$S_3 \times \mathbb{Z}_2$, D_{14}, Q_{16} *or* $\langle x, y : x^7 = y^3 = 1, x^y = x^2 \rangle$.


$\Gamma_c(S_3 \times \mathbb{Z}_2)$


$\Gamma_c(D_{14})$


$\Gamma_c(Q_{16})$


$\Gamma_c(\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$

References

[1] A. Abdollahi and A. M. Hassanabadi, Noncyclic graph of a group, *Comm. Algebra*, 35 (7) (2007), 2057-2081.

[2] A. Abdollahi and A. M. Hassanabadi, Non-cyclic graph associated with a group, *J. ALgebra Appl.*, 8 (2) (2009), 243-257.

[3] S. J. Baishya, A note on finite C-tidy groups, *Int. J. Group Theory*, 2 (3) (2013), 9-17.

[4] S. J. Baishya, On finite C-tidy groups, *Int. J. Group Theory*, 2 (4) (2013), 39-41.

[5] S. J. Baishya, Some new results on tidy groups, *Algebras Groups Geom.*, 31 (3) (2014), 251-262.

[6] W. Bannuscher and G. Tiedt, On a theorem of Deaconescu, *Rostock. Math. Kolloq.*, 47 (1994), 23-26.

[7] J. Battle, F. Harary, Y. Kodama, and J. W. T. Youngs, Additivity of the genus of a graph, *Bull. Amer. Math. Soc.*, 68 (1962), 656-568.

[8] E. D. Bolker, Groups whose elements are of order two or three, *Amer. Math. Monthly*, 79 (9) (1972), 1007-1010.

[9] A. Bouchet, Orientable and nonorientable genus of the complete bipartite graph, *J. Combin. Theory Ser. B.*, 24 (1978), 24-33.

[10] A. Cayley, Desiderata and suggestions: No. 2. The Theory of groups: graphical representation, *Amer. J. Math.*, 1 (2) (1878), 174-176.

[11] A. Erfanian and M. Farrokhi, On some classes of tidy groups, *Algebras Groups Geom.*, 25 (1) (2008), 109-114.

[12] A. Gagarina, W. Myrvold, and J. Chambers, The obstructions for toroidal graphs with no $K_{3,3}$'s, *Discrete Math.*, 309 (2009), 3625-3631.

[13] H. H. Glover, J. P. Huneke, and C. S. Wang, 103 Graphs that are irreducible for the projective plane, *J. Combin. Theory Ser. B.*, 27 (1979), 332-370.

[14] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, 1967.

[15] X. L. Ma, H. Q. Wei, and Zhong, *The Cyclic graph of a finite group. Algebar, Article ID 107265*, 2013.

[16] H. Maschke, The representation of finite groups, *Amer. J. Math.*, 18 (1896), 156-194.

[17] M. Naghshineh, On the planarity of cyclic graphs, *Rocky Mountain Journal of Mathematics,* to appear.

[18] K. O'Bryant, D. Patrick, L. Smithline, and E. Wepsic, Some facts about cycels and tidy groups, Rose-Hulman Institute of Technology, Indiana, USA, Technical report, (1992), MS-TR, 92-04.

[19] D. Patrick and E. Wepsic, cyclicizer s, centralizers, and normalizers, Rose-Hulman Institute of Technology, Indiana, USA, Technical report, (1991), MS-TR, 91-05.

[20] G. Ringel, *Map Color Theorem*, Springer-Verlag, New York, Heidelberg, 1974.

[21] D. J. S. Robinson, *A Course in the Theory of Groups*, Second Edition, Springer, New York, Berlin, Heidelberg, 1996.

Mohammad Naghshinehfard
Assistant Professor of Mathematics
Department of Mathematics
Jahrom Branch, Islamic Azad University
P.B. Box 74135/355
Jahrom, Iran
E-mail: naghshine@jia.ac.ir

Foroud Parvaneh
Assistant Professor of Mathematics
Department of Mathematics
Kermanshah Branch, Islamic Azad University
Kermanshah, Iran
E-mail: foroudparvaneh@gmail.com