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Fixed Point Results for Increasing Mapping and the Relationship Between (Relative) Algebraic Interior and Topological Interior

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Abstract. In this paper, we show that the relative algebraic interior is a suitable replacement for both of the topological interior and the algebraic interior for the cases where these are empty. Also, we present some properties of (relative) algebraic interior and some fixed point theorems for increasing mapping. The obtained results can be viewed as an extension and improvement of the known corresponding results. Also, some examples are to support our conclusions considered.

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1. Introduction and Preliminaries

In this section we recall some definitions and facts to set up our results in the next section.

Definition 1.1. ([6]) *Let X be a real vector space with the zero vector θ_X. A nonempty, nontrivial (i.e., $P \neq \{\theta_X\}$), closed subset P of X is called cone if the following conditions are satisfied:*

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(i) $2P \subseteq P$;

(ii) $tx + (1-t)y \in P, \ \forall (x, y, t) \in P \times P \times [0, 1]$;

(iii) $P \cap (-P) = \{\theta_X\}$.

A cone $P \subseteq X$ defines an ordering $\preceq_P$ on X with respect to P by letting $x \preceq_P y$ whenever $y - x \in P$. We use the notation $x \prec_p y$ for $y - x \in P \setminus \{\theta_X\}$. Also, if X is a topological vector space and P is a solid cone of X, i.e., $intP \neq \emptyset$, then we can define a pre-order (it is not reflexive) on X by

$$x \prec\prec_p y \Longleftrightarrow y - x \in intP.$$

If the cone P is known, for simplicity, we replace $\preceq_P$, $\prec_p$ and $\prec\prec_{intP}$ by $\preceq$, $\prec$ and $\prec\prec$; respectively. Note that throughout the paper we reserve the symbol $\leqslant$ and its obvious variants, for the usual order on $\mathbb{R}$.
The pair (X, P) consisting of a real normed space X and a cone P of X is called a partially ordered normed space.

Definition 1.2. ([1]) *Let S be a nonempty subset of a real linear space X. The algebraic interior of S, denoted by $cor(S)$, and the relative algebraic interior of P, denoted by $icr(P)$, are defined as follows:*
$cor(S) := \{x \in S : \forall x' \in X, \exists \lambda' > 0; \forall \lambda \in [0, \lambda'], x + \lambda x' \in S\}$,
$icr(S) := \{x \in S : \forall x' \in L(S), \exists \lambda' > 0; \forall \lambda \in [0, \lambda'], x + \lambda x' \in S\}$,
where $L(S) = span(S - S)$, and $S - S = \{s_1 - s_2 : s_i \in S, i = 1, 2\}$. If S is a subset of a topological vector space, then $intS \subseteq cor(S) \subseteq icr(S) \subseteq S$.

Definition 1.3. ([15]) *Let (X, P) be a partially ordered normed space. A cone P is said to be normal, if there exists a constant $k > 0$ such that $\theta_X \preceq x \preceq y$ implies $\| x \| \leqslant k \| y \|$ for all $x, y \in X$. The least positive constant k satisfying the above inequality is called the normal constant of P.*

Definition 1.4. ([15]) *Let (X, P) be a partially ordered normed space. A cone P is said to be (sequentially) regular, if every sequence in X which is increasing and ordered bounded above must be convergent in X.*
This means that, if $\{a_n\}$ is a sequence in X such that $a_1 \preceq a_2 \preceq \ldots \preceq$

$a_n \preceq ... \preceq m$ for some $m \in X$, then there exists $a \in X$ such that $\|a_n - a\| \to 0$ as $n \to \infty$. Equivalently, if every sequence in X which is decreasing and bounded below is convergent in X.
It is well known that any regular cone is normal and the converse is true if the space X is reflexive (see [6]).

Definition 1.5. *The mapping $T : X \to Y$ acting in partially ordered real vector linear spaces X and Y is called increasing if $x \preceq y$ implies $T(x) \preceq T(y)$. Note that if we take $X = Y = \mathbb{R}$, then Definition 1.5 collapses to the usual definition of an increasing mapping.*

The next definition was firstly introduced in [5] (for more details, see [2] in order to apply in optimization theory and then was used for equilibrium problems in [3]

Definition 1.6. ([7]) *Let (X, P) be a partially ordered topological vector space (t.v.s.) with a solid cone P and $e \in intP$. The nonlinear scalarization function $\xi_e : X \to \mathbb{R}$ is defined as follows:*

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\} = \min\{r \in \mathbb{R} : y \preceq re\}.$$

The following lemma characterizes some of the important properties of the nonlinear scalarization function, which are used in the sequel.

Lemma 1.7. ([14]) *Let (X, P) be a partially ordered normed space with a solid cone P. For each $r \in \mathbb{R}$ and $y \in X$, the following statements are satisfied:*

(i) $\xi_e(re) = r$, *particulary* $\xi_e(\theta_X) = 0$;

(ii) *if* $y_2 \preceq y_1$, *then* $\xi_e(y_2) \leqslant \xi_e(y_1)$ *for any* $y_1, y_2 \in X$;

(iii) *if* $y_2 \prec\prec y_1$, *then* $\xi_e(y_2) < \xi_e(y_1)$ *for any* $y_1, y_2 \in X$;

(iv) $\xi_e(y) \leqslant r \iff y \in re - P$;

(v) $\xi_e(y) > r \iff y \notin re - P$;

(vi) $\xi_e(y) < r \iff y \in re - intP$;

(vii) $\xi_e(y) \geqslant r \Longleftrightarrow y \notin re - intP$;

(viii) ξ_e *is subadditive on* X*, i.e.,*

$$\xi_e(x+y) \leqslant \xi_e(x) + \xi_e(y), \forall x, y \in X;$$

(ix) ξ_e *is positively homogeneous on* X*, i.e.,*

$$\xi_e(\beta x) = \beta \xi_e(x), \forall x \in X$$

and a positive real number β*;*

(x) ξ_e *is continuous on* X*.*

The following lemma guarantees the existence of two points u_0 and $v_0 \in (u_-, u_+)_c \subseteq [u_-, u_+]_o$ such that $u_0 \preceq Tu_0, Tv_0 \preceq v_0$ which plays a crucial role reaching to one of the main goal in this paper (i.e. Theorem 2.14), where

$$[u_-, u_+]_o = \{x \in X : u_- \preceq x \preceq u_+\},$$

$$(u_-, u_+)_c = \{x \in X : x = tu_+ + (1-t)u_-; 0 < t < 1\}.$$

Lemma 1.8. [6] *Let* (X, P) *be a partially ordered normed space with a solid cone* P*. Assume that there are two points* u_- *and* u_+ *in* X *and an increasing continuous mapping* $T : [u_-, u_+]_c \to X$ *such that* $u_- \preceq u_+$*,* $u_+ \prec\prec Tu_+$*, and* $Tu_- \prec\prec u_-$*. Then there exist* u_0 *and* $v_0 \in (u_-, u_+)_c$ *such that* $u_0 \preceq Tu_0, Tv_0 \preceq v_0$*, where*

$$[u_-, u_+]_c = \{x \in X : x = tu_+ + (1-t)u_-; 0 \leqslant t \leqslant 1\}.$$

2. Main Results

Now, we are in a position to give our main results. In 2012, Kostrykin and Oleynik presented Theorem 1 in [8] which is an extension of a key lemma (Lemma 2.1) of [9] that plays a key role in [9]. Moreover, it can be considered as an important existence result of the unstable bumps in neural, Integral equations and operator theory (see, for instance, [9, 10,

12, 13]).
By reviewing the proof of Theorem 1 in [8], one can check that the authors claimed that for operator $\hat{T} : [u_-, u_+]_o \to X$ defined by

$$\hat{T}u := sup\{inf\{Tu, u_+\}, u_-\},$$

every fixed point u_* of operator $\hat{T}$ satisfying $u_- \prec u_* \prec u_+$ is a fixed point of the operator T. This assertion is not true in general because $inf\{Tu_*, u_+\}$ does not exists in general. It is easy to see that Tu_* and u_+ are not comparable in general. Thus the results in [9, 12, 13], which are based on Theorem 1, can not be true. In the sequel we give two corrected versions of this theorem.
The following counter example shows that the aforementioned theorem is not correct and the results can not be true.

Example 2.1. Let $X = \mathbb{R}^2$ and $P = \{(x, y) \in \mathbb{R}^2; x, y \geqslant 0\}$. Take $u_- = (0, 0), u_+ = (1, 1)$.
Define a mapping $T : [u_-, u_+]_o \longrightarrow \mathbb{R}^2$ by

$$T(x, y) = \begin{cases} 7(x, x) + (x, y) + (-1, -3) & y \leqslant x, \\ 7(y, y) + (x, y) + (-1, -3) & x \leqslant y. \end{cases}$$

It is clear that $[u_-, u_+]_o$ equals the (full) square with the vertices $(0, 0)$, $(0, 1)$, $(1, 1)$ and $(1, 0)$ and T satisfies all the assumptions of Theorem 1 in [8], but T does not have a fixed point.

Remark 2.2. *It is worth noting that there exist some other papers whose authors have made the same mistake as mentioned above. For example, Theorem 2.3 and Theorem 2.4 (the following two theorems) together with Corollaries 3.1, 3.2, 3.3 and 3.4 in [1] are not correct. It is not hard to see that in the proof of Theorem 2.3 (that is Theorem 2.1 in [11]) the relation (2.7), i.e.,* $u_0 \preceq Av_0 \preceq Au_0 \preceq v_0$ *does not hold. Similarly, in the proof of Theorem 2.4 (that is Theorem 2.2 in [11]) the relation (2.14), i.e.,* $u_0 \preceq Av_0 \preceq Au_0 \preceq v_0$ *dose not hold. Note that for any* $x \in D$, $inf\{Ax, x\} = u_0$ *and* $sup\{Ax, x\} = v_0$ *depend on a variable* x*. Thus* u_0 *and* $Av_0 = Ax$ *(resp.* v_0 *and* $Au_0 = Ax$*) are not comparable in general.*

Theorem 2.3. [11] *Let E be an ordered Banach space with lattice structure, $D \subseteq E$ be bounded, and $A : D \longrightarrow D$ be a decreasing and condensing operator. Then the operator A has a fixed point in D.*

Theorem 2.4. [11] *Let E be an ordered Banach space with lattice structure, $P \subseteq E$ be a normal cone, and $A : E \longrightarrow E$ be a decreasing and condensing operator. Then the operator A has a fixed point in E.*

The following counter example shows that Theorem 2.3 does not hold in general, even in the special case $E = \mathbb{R}$. In the same way, one can show that Theorem 2.4 is not correct in general.

Example 2.5. Let $E = \mathbb{R}$ be endowed by the norm $\|a\| = |a|$, $P = \{x \in \mathbb{R} : x \geqslant 0\}$, and $D = [-1, 0] \setminus \{\frac{-1}{2}\}$. Define $A : D \longrightarrow D$ by $A(x) = -x - 1$. One can check that A satisfies all the assumptions of Theorem 2.4, while A does not have a fixed point.

In the following we show that the relative algebraic interior is a suitable replacement of the topological interior for the case where it is empty. Moreover, the authors establishe Theorem 2.12 in real Banach space which is an improvement version of Theorem 1 in [8] by relaxing minihedrality on cone and replacing the toplological interior of the cone by the relative algebric interior.
Now, we present the following lemma which shows that algebraic interior is not a suitable replacement of the topological interior.

Lemma 2.6. *If S is a nonempty, closed, convex subset of a Banach space X, then $cor(S) = intS$.*

Proof. It is clear that the core of S contains in its interior, i.e., int$S \subseteq$cor(S). Conversely, Let $\overline{x} \in corS$, from Definition 1.2, we have $S \setminus \{\overline{x}\}$ is absorbing. Since S is a convex closed subset of a Banach space X, then $\theta_X \in int(S \setminus \{\overline{x}\})$, which implies that $\overline{x} \in intS$. This completes the proof. $\square$

To illustrate Lemma 2.6 we provide some examples.

Example 2.7. Assume that $X = C[0, 1]$ denotes the Banach space of all real-valued continuous mapping defined on $[0, 1]$ endowed by the

$\|f\| = \max\limits_{0\leqslant t\leqslant 1} |f(t)|$. Let $S = \{f \in X : f(x) \geqslant 0, \forall x \in [0,1]\}$. It is easy to check that S is closed, convex and int$S = \{f \in X : f(x) > 0, \forall x \in [0,1]\}$. We show that int$S$=cor$(S)$. To see this, let $f \in S$ and $f > 0$. Set $m = \inf\limits_{x\in[0,1]} f(x)$. If $g \in C[0,1]$ is an orbitrary mapping, then there exists $M_g > 0$ such that $-M_g \leqslant g(x) \leqslant M_g, \forall x \in [0,1]$. Therefore there exists $\lambda' = \dfrac{m}{M_g} > 0$ such that $f + \lambda g > 0, \forall \lambda \in [0, \lambda']$. Thus cor$(S)$ =int$S$.
We know that l_∞ (that is the space consisting of all bounded sequences) with the sup norm, i.e., $\|x\|_\infty = \sup\limits_{n\in\mathbb{N}} |x_n|$, is a Banach space. Now, we give the following lemma which shows that there exists a Banach space which has no any cone, namely P, satisfying $intP \neq cor(P)$.
In other words, for each cone in l_∞ the algebraic interior of the cone coincide with interior of the cone.

Lemma 2.8. *If S is a nonempty, closed, convex subset of the Banach space l_∞, then $cor(S) = intS$.*

Proof. If $cor(S) = \emptyset$, then the proof is clear. Suppose that $cor(S) \neq \emptyset$. Let $\overline{x} \in cor(S), x \in l_\infty$ and $B = \{x \in l_\infty : \|x\|_\infty \leqslant 1\}$. Define the real valued mapping $G : B \longrightarrow \mathbb{R}^+$ by

$$G(x) = sup\{\lambda > 0 : \overline{x} + \lambda x \in S\}.$$

Put $\delta^* := \inf\limits_{x\in B} G(x) > 0$. So there exists $x_* \in B$ such that $\delta^* = G(x_*)$. Thus, $\overline{x} + \lambda x \in S$ for all $x \in B$, $0 < \lambda \leqslant \delta^*$, and so $\overline{x} \in intS$. The proof is complete. $\square$

In the following we introduce a cone of a real vector space endowed with a topology which is not a topological vector space (t.v.s.). Moreover, the relative algebraic interior of this cone coincide with interior of it, while its algebraic interior is empty. This example shows that in Lemma 2.6 it is essential that the space to be topological vector space.

Example 2.9. Let $R_1 = \mathbb{R}$ and $R_2 = \mathbb{R}$ be two topological spaces equipped with standard topology on the real line (a topology generated by the collection of all open intervals in the real line) and discrete topology(the collection of all subsets of $\mathbb{R}$), respectively. We consider

the topological space $X = R_1 \times R_2$ and the cone $P = [0, \infty) \times \{0\} \subseteq R_1 \times R_2$. It is easy to show that $intP = icr(P) = (0, \infty) \times \{0\}$ and $cor(P) = \emptyset$. Here, the scaler multiplication is not continuous, i.e, the mapping $\centerdot : \mathbb{R} \times X \to X$ definded by $\centerdot(\alpha, x) \to \alpha x$ is not continuous. Because the set $\{1\}$ is a closed set in X, but $\centerdot^{-1}\{1\}$ is not closed in $\mathbb{R} \times X$. Thus X is not a topological vector space.
In the following lemma we establish several characterizations which are used in Theorem 2.12. For the sake of the reader we give the proof.

Lemma 2.10. *Let P be a cone in a vector space X with a nonempty relative algebraic interior. Then*

(i) $\theta_X \notin icr(P)$;

(ii) if $x \in icr(P)$ and $y \in P$, then $[x, y)_c \in icr(P)$. In particular, the set icr(P) is convex;

(iii) $icr(P) \cup \{\theta_X\}$ *is a cone;*

(iv) $icr(icr(P)) = icr(P)$.

Proof. (i) On the contrary, suppose that $\theta_X \in icr(P)$. Thus

$$\forall x' \in L(P), \exists \lambda' > 0; \forall \lambda \in [0, \lambda'], \lambda x' \in P.$$

Therefore $L(P) = P$ which is contradicted by $P \cap (-P) = \{\theta_X\}$.
(ii) Let $0 \leqslant t < 1$, $z = (1 - t)x + ty, \nu \in L(P)$. Take $\lambda > 0$ such that $x + \lambda\nu \in P$. Since

$$z + (1 - t)\lambda\nu = (1 - t)(x + \lambda\nu) + ty,$$

we have $z + (1 - t)\lambda\nu \in P$. Now, the assertion easily follows.
(iii) It is clear that

$$\emptyset = icr(P) \cap -icr(P) \subseteq P \cap (-P) = \{\theta_X\}.$$

Now, applying (i) and (ii) we have (iii).
(iv) It is clear that $icr(icr(P)) \subseteq icr(P)$. Conversely, let $x \in \mathrm{icr}(P)$, $x' \in L(P)$. There exists $\lambda' > 0$ such that

$$x + \lambda x' \in P, \forall \lambda \in [0, \lambda'].$$

Applying (ii), we can write

$$t(x+\lambda x')+(1-t)x \in icr(P), \forall\ 0 \leqslant t < 1.$$

Thus $x+t\lambda x' \in icr(P),\ \forall\ t\lambda \in [0, t\lambda']$ and hence $x \in icr(icr(P))$. $\square$

The following example shows that it is possible that int$P$ =cor$(P)=\emptyset$, and $icr(P)\neq\emptyset$. Moreover, it shows that the relative algebraic interior is a suitable replacement of the topological interior and algebraic interior for the cases where these are empty.

Example 2.11. Let l_∞ be as given in Lemma 2.7 and

$$P=\{(x_1,x_2,...,x_n,0,0,0,...) : x_i \geqslant 0, 1 \leqslant i \leqslant n\}.$$

One can check that P is a cone, $intP = cor(P)=\emptyset$, and

$$icr(P)=\{(x_1,x_2,...,x_n,0,0,0,...) : x_i > 0, 1 \leqslant i \leqslant n\}.$$

It is straightforward to see that the following theorem is an improvement of Theorem 1 in [8] by using relative algebraic interior of cone.

Theorem 2.12. *Suppose that X is a real Banach space and let P be a normal cone with nonempty relative algebraic interior (i.e., $icr(P)\neq \phi$). Assume that $K = icr(P)\cup\{\theta_X\}$, there are two points u_- and u_+ in X, where $u_- \prec_P u_+$, and an increasing mapping $T : [u_-, u_+]_o \longrightarrow X$. Let $h_0 = u_+ - u_-$. If one of the following assumptions holds:*

(i) T is convex , $Tu_+ \prec_K u_+$,$u_- \preceq_P Tu_-$;

(ii) T is concave, $u_- \preceq_K Tu_-$, $Tu_+ \prec_P u_+$,

then T has a unique fixed point $x_* \in [u_-, u_+]_o$. Moreover, each iteration $Tx_n = x_{n-1}$ for all $n = 1, 2, 3, ...$ with $x_0 \in [u_-, u_+]_o$ converges to x_* and there exist $M > 0$ and $r \in (0, 1)$ such that

$$\|x_n - x_\star\| \leqslant Mr^n.$$

Proof. Assume that (i) holds(the proof is similar if condition (ii) holds). Applying Lemma 2.10 (iii), the set $K = icr(P) \cup \{\theta_X\}$ is a cone and by Lemma 2.10 (iv), we have

$$u_+ - Tu_+ \in K \setminus \{\theta_X\} = icr(P) = icr(icr(P)).$$

By the assumption $u_- \prec_P u_+$, we get $h_0 = u_+ - u_- \in P \subseteq L(P)$, where $L(P)$ is the smallest subspace containing P. So $(u_- - u_+) \in L(P)$ and there exists $\lambda' > 0$ such that

$$(u_+ - Tu_+) + \lambda(u_- - u_+) \in K = icr(P) \cup \{\theta_X\} \subseteq P, \forall \lambda \in [0, \lambda'].$$

Since $\lambda' > 0$, we can choose $\varepsilon \in (0,1)$ such that

$$Tu_+ \preceq_P u_+ - \varepsilon(u_+ - u_-).$$

This means that $Tu_+ \preceq_P u_+ - \varepsilon h_0$. Now, applying the next theorem and the inequality

$$u_- \preceq_P Tu_-,$$

complete the proof. $\square$

Theorem 2.13. [4] *Suppose that X is a real Banach space, P is a normal cone, and $u_-, u_+ \in X$ with $u_0 \prec_P v_0$. Moreover, $T : [u_-, u_+]_o \longrightarrow X$ is an increasing mapping. Let $h_0 = u_+ - u_-$. If one of the following assumptions holds:*

(i) T is convex mapping, $Tu_+ \preceq_P u_+ - \varepsilon h_0$, $u_- \preceq_P Tu_-$, where $\varepsilon \in (0,1)$ is a constant;

(ii) T is concave mapping, $u_- + \varepsilon h_0 \preceq_P Tu_-$, $Tu_+ \prec_P u_+$, where $\varepsilon \in (0,1)$ is a constant,

then T has a unique fixed point $x_* \in [u_-, u_+]_o$. Moreover, for any $x_0 \in [u_-, u_+]_o$, the iterative sequnce $\{x_n\}$ given by $x_n = Tx_{n-1}(n = 1, 2, ...)$ satisfying that

$$\|x_n - x_*\| \to 0(n \to \infty),$$

$$\|x_n - x_*\| \leqslant M(1 - r)^n(n = 1, 2, ...),$$

with M a positive constant independent of x_0.

Now, we are ready to give another corrected version of Theorem 1 in [8] by relaxing some assumptions of it (such as minihedrality on cone and the compactness on the mapping T) and an extension of it in general spaces. The following theorem can be considered as a repairment and an improvement of Theorem 1 in [8].

Theorem 2.14. *Let X be a normed spac, P be a regular cone with nonempty interior. Assume that there are two points u_- and u_+ in X such that $u_- \prec\prec u_+$ and an increasing and continuous mapping $T : [u_-, u_+]_o \longrightarrow X$. If $Tu_- \prec\prec u_-$ and $u_+ \prec\prec Tu_+$, then we have at least one of the following:*
(i) There exist u_0 and $v_0 \in (u_-, u_+)_c$ such that T has a maximal fixed point v^* and minimal fixed point u_* in $[u_0, v_0]_o$ and $u_* = \lim_{n\to\infty} u_n$, $v^* = \lim_{n\to\infty} v_n$, where $u_n = T^n u_0$, $v_n = T^n v_0 (n = 1, 2, 3, ...)$, and

$$u_0 \preceq u_1 \preceq ... \preceq u_n \preceq ... \preceq u_* \preceq v^* \preceq ... \preceq v_n \preceq ... \preceq v_1 \preceq v_0. \qquad (1)$$

(ii) For each $n \in \mathbb{N}$, there exist u_n and $v_n \in [v_{n-1}, u_{n-1}]_c$ such that $Tv_n \preceq v_n$ and $u_n \preceq Tu_n$. Moreovere, $Tv_* \preceq v_* = \lim_{n\to\infty} v_n$, $u^* = \lim_{n\to\infty} u_n \preceq Tu^*$ and $u_- \preceq v_0 \preceq v_1 \preceq ... \preceq v_n \preceq ... \preceq v_* \preceq u^* \preceq ... \preceq u_n \preceq ... \preceq u_1 \preceq u_0 \preceq u_+$.

Proof. Applying Lemma 1.8, there exist $u_0, v_0 \in (u_-, u_+)_c$ such that

$$u_0 \preceq Tu_0 \quad \textit{and} \quad Tv_0 \preceq v_0. \qquad (2)$$

Now, we have one of the following cases:

$$u_0 = v_0, \quad u_0 \prec v_0, \quad \textit{and} \quad v_0 \prec u_0.$$

Case I: $u_0 = v_0$.
In this case, $u_* = u_0 = v_0 = v^*$ is a fixed point of T.

Case II: $u_0 \prec v_0$.
In this case, set $u_n = T^n u_0 = Tu_{n-1}$ and $v_n = T^n v_0 = Tv_{n-1}$.
Since T is increasing, applying (2.2) and $u_0 \prec v_0$, we have

$$u_0 \preceq u_1 \preceq ... \preceq u_n \preceq ... \preceq v_n \preceq ... \preceq v_1 \preceq v_0. \qquad (3)$$

By the regularity of P and (3), $u_n \to u_*$ as $n \to \infty$ and $u_n \preceq u_* \preceq v_n (n = 0, 1, 2, ...)$. Since T is continuous, we have $u_{n+1} = Tu_n \to Tu_*$. So $Tu_* = u_*$. Similarly, we can show that $v_n \to v^*$ as $n \to \infty$, $Tv^* = v^*$ and

$$u_n \preceq u_* \preceq v^* \preceq v_n \qquad (n = 1, 2, ...). \tag{4}$$

Hence, applying (2.4) and (2.3), (2.1) holds. Finally, one can check that u_* and v^* are the minimal fixed point and the maximal fixed point of T in $[u_0, v_0]_c$, respectively.

Case III: $v_0 \prec u_0$.
In this case, we have $T : [v_0, u_0]_o \longrightarrow [Tv_0, Tu_0]_o (\supseteq [v_0, u_0]_o)$. Define $T^* : [v_0, u_0]_c \longrightarrow \mathbb{R}$ by $T^*t = \xi_e(t - Tt)$. Since $u_0 - Tu_0 \preceq \theta_X$ and $\theta_X \preceq v_0 - Tv_0$, applying Lemma 1.7 (parts (ii) and (ii)), we have $T^*u_0 \leqslant \xi_e(\theta_X) = 0$ and $\xi_e(\theta_X) = 0 \leqslant T^*v_0$. Applying Intermediate Value Theorem (for more details see Lemma 1.8), there exists $u_1 \in [v_0, u_0]_c$ such that $T^*u_1 = 0$. So applying Lemma 1.7(parts (iv) and (vii)) we get $u_1 - Tu_1 \in -P \ \setminus - intP$, and so $u_1 \preceq Tu_1$. Similarly, if we put $T^*t = \xi_e(Tt - t)$, there exists $v_1 \in [v_0, u_0]_c$ such that $Tv_1 \preceq v_1$.
Now, we have one of the following cases:

$$u_1 = v_1 \quad or \quad u_1 \prec v_1 \quad or \quad v_1 \prec u_1.$$

If $u_1 = v_1$ or $u_1 \prec v_1$, then T has a maximal fixed point and a minimal fixed point by repeating the process in case I or case II. If $v_1 \prec u_1$, applying case III, there exist v_2 and u_2 such that $u_2, v_2 \in [v_1, u_1]_c$, and $u_2 \preceq Tu_2, Tv_2 \preceq v_2$. By repeating this process and the regularity of cone, one can completes the proof. $\square$

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