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Derivations on the Tensor Product of Banach Algebras

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Abstract. In this paper, we study derivations on the (projective) tensor product of Banach algebras. Among other things, we show that under some mild conditions when the first cohomology group of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ with coefficients in $(\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ is zero, then $\mathcal{B}$ is $\mathcal{J}$-weakly amenable, where $\mathcal{J}$ is a closed two-sided ideal in $\mathcal{B}$. Also, we provide some concrete examples in which $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is ideally amenable.

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1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule. A *derivation* from a Banach algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-bimodule X is a bounded linear mapping $D : \mathcal{A} \longrightarrow X$ such that $D(ab) = D(a) \cdot b + a \cdot D(b)$ for every $a, b \in \mathcal{A}$. A derivation $D : \mathcal{A} \longrightarrow X$ is called *inner* if there exists $x \in X$ such that $D(a) = a \cdot x - x \cdot a = \delta_x(a) \quad (a \in \mathcal{A})$. The spaces of derivations and inner derivations from $\mathcal{A}$ into X are denoted by $Z^1(\mathcal{A}, X)$ and $N^1(\mathcal{A}, X)$, respectively. Consider the quotient space

$$H^1(\mathcal{A}, X) = \frac{Z^1(\mathcal{A}, X)}{N^1(\mathcal{A}, X)}$$

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which is called *the first cohomology group* of $\mathcal{A}$ with coefficients in X. A Banach algebra $\mathcal{A}$ is called *amenable* if every bounded derivation $D : \mathcal{A} \longrightarrow X^*$ is inner for every Banach $\mathcal{A}$-bimodule X; i.e., $H^1(\mathcal{A}, X^*) = \{0\}$, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in X^*. This definition was introduced by B. E. Johnson in [8]. Also, a Banach algebra $\mathcal{A}$ is *weakly amenable* if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Bade, Curtis and Dales introduced the notion of weak amenability for Banach algebras in [1]. They considered this concept only for commutative Banach algebras. After that Johnson defined the weak amenability for arbitrary Banach algebras and showed that for a locally compact group G, $L^1(G)$ is always weakly amenable [9]. In [4], Gordji and Yazdanpanah introduced and studied the concept of ideal amenability for a Banach algebra. Indeed, for a closed two-sided ideal $\mathcal{I}$ of a Banach algebra $\mathcal{A}$, $\mathcal{A}$ is $\mathcal{I}$-weakly amenable if $H^1(\mathcal{A}, \mathcal{I}^*) = \{0\}$. Also, $\mathcal{A}$ is called ideally amenable if $H^1(\mathcal{A}, \mathcal{I}^*) = \{0\}$ for every closed two-sided ideal $\mathcal{I}$ in $\mathcal{A}$. Ideal amenability of Banach algebras on locally compact groups and module extensions of Banach algebras are studied in [7] and [5], respectively; for more details of the hereditary properties see [6]. Also, ideal Connes-amenability of dual Banach algebras is studied by authors in [10] recently.

In [4], the authors asked the following question: if $\mathcal{A}$ and $\mathcal{B}$ are ideally amenable Banach algebras, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is ideally amenable? Mewomo in [11] answered this question by the concept of multiplier algebra. Also, the mentioned question was answered by Mewomo and Olukorede in [12] when $\mathcal{A}$ and $\mathcal{B}$ are commutative.

In this paper, we find some relationships between the ideal amenability of Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with the first cohomology group from $\mathcal{A}\widehat{\otimes}\mathcal{B}$ with coefficients in $(\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ and $(\mathcal{I}\widehat{\otimes}\mathcal{B})^*$, where $\mathcal{I}$ and $\mathcal{J}$ are closed two-sided ideals in $\mathcal{A}$ and $\mathcal{B}$, respectively. In other words, we answer the inverse of Gordji and Yazdanpanah's question.

2. Main Results

In this section, we investigate some derivations on the (projective) tensor product of Banach algebras. From now on, for a Banach algebra $\mathcal{A}$ we set $\mathcal{A}^2 = \{ab : a, b \in \mathcal{A}\}$.

Let E and F be Banach spaces. We denote the space of bounded linear operators from E to F by $\mathcal{L}(E, F)$. Also, we say $E\widehat{\otimes}$" *respects subspace isomorphically* if for every subspae G of F, then $E\widehat{\otimes}G$ is subspace of $E\widehat{\otimes}F$.

Definition 2.1. *A Banach space F is called injective if for every Banach space E, every subspace $G \subset E$ and every $T \in \mathcal{L}(G, F)$ there is an extension $\widehat{T} \in$*

$\mathcal{L}(E, F)$ *of* T.
The following result was proved in Pages 36 and 37 of [3].

Proposition 2.2. *Let* E *and* F *be Banach space.*

(i) $E\widehat{\otimes}$" *respects subspace isomorphically if and only if* E^* *is an injective Banach space;*

(ii) *If* G *is complemented subspace of* E, *then* $G\widehat{\otimes}F$ *is a subspace of* $E\widehat{\otimes}F$.

Summing up:

Lemma 2.3. *Let* $\mathcal{I}$ *and* $\mathcal{J}$ *be closed two-sided ideals in Banach algebras* $\mathcal{A}$ *and* $\mathcal{B}$, *respectively.*

(i) *If* $\mathcal{A}^*$ *is injective, then* $\mathcal{A}\widehat{\otimes}\mathcal{J}$ *is a closed two-sided ideal in* $\mathcal{A}\widehat{\otimes}\mathcal{B}$ *and is an* $\mathcal{A}\widehat{\otimes}\mathcal{B}$*-bimodule;*

(ii) *If* $\mathcal{I}$ *is complemented in* $\mathcal{A}$, *then* $\mathcal{I}\widehat{\otimes}\mathcal{B}$ *is a closed two-sided ideal in* $\mathcal{A}\widehat{\otimes}\mathcal{B}$ *and is an* $\mathcal{A}\widehat{\otimes}\mathcal{B}$*-bimodule.*

Theorem 2.4. *Let* $\mathcal{A}, \mathcal{B}$ *be Banach algebras such that* $\mathcal{A}^*$ *is injective Banach space and* $\mathcal{A}^2 \neq \{0\}$. *Also,* $\mathcal{J}$ *is a closed two-sided ideal in* $\mathcal{B}$ *and* $\mathcal{A}$ *is commutative. If* $H^1(\mathcal{A}\widehat{\otimes}\mathcal{B}, (\mathcal{A}\widehat{\otimes}\mathcal{J})^*) = \{0\}$, *then* $\mathcal{B}$ *is* $\mathcal{J}$*-weakly amenable.*

Proof. We firstly note that $\mathcal{A}\widehat{\otimes}\mathcal{J}$ is a $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule by Lemma 2.3. Let a^* be a non-zero element of $\mathcal{A}^*$. We may assume that there are $t, h \in \mathcal{A}$ such that $\langle a^*, th\rangle = 1$. Let $d : \mathcal{B} \longrightarrow \mathcal{J}^*$ be a bounded derivation. Define the map $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ via

$$\langle D(a \otimes b), c \otimes j\rangle := \langle d(b), j\rangle\langle aa^*, c\rangle \quad (a, c \in \mathcal{A}, b \in \mathcal{B}, j \in \mathcal{J}).$$

It is easily verified that D is a bounded linear map. Also, for each $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$ we have

$$\begin{aligned}\langle D((a_1 \otimes b_1) \cdot (a_2 \otimes b_2)), c \otimes j\rangle &= \langle D(a_1a_2 \otimes b_1b_2), c \otimes j\rangle \\ &= \langle d(b_1b_2), j\rangle\langle a_1a_2 \cdot a^*, c\rangle \\ &= \langle d(b_1), b_2j\rangle\langle a_1 \cdot a^*, a_2c\rangle \\ &+ \langle d(b_2), jb_1\rangle\langle a_2 \cdot a^*, ca_1\rangle.\end{aligned}$$

The above relations show that D is a derivation. Due to the $\mathcal{A}\widehat{\otimes}\mathcal{J}$-weak amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$, there is $\varphi \in (\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ such that $D = ad_\varphi$. Define $\mathfrak{I}^*$ on $\mathcal{J}$ by $\mathfrak{I}^*(j) = \varphi(th\otimes j)$ for all $j \in \mathcal{J}$. The map $\mathfrak{I}^*$ is a bounded linear functional. Now, for each $b \in \mathcal{B}$ and $j \in \mathcal{J}$, we get

$$\begin{aligned}\langle d(b), j\rangle &= \langle d(b), j\rangle\langle a^*, th\rangle = \langle d(b), j\rangle\langle ha^*, t\rangle \\ &= \langle D(h\otimes b), t\otimes j\rangle = \langle (h\otimes b\cdot\varphi - \varphi\cdot h\otimes b), t\otimes j\rangle \\ &= \langle \varphi, bj\otimes th\rangle - \langle\varphi, jb\otimes ht\rangle = \langle ad_{\mathfrak{I}^*}(b), j\rangle\end{aligned}$$

This means that d is an inner derivation. □

The proof of the next result is similar to the proof of Theorem 2.4, so is omitted.

Theorem 2.5. *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras so that $\mathcal{A}$ is commutative and $\mathcal{B}^2 \neq \{0\}$. If $\mathcal{I}$ is a closed two-sided ideal in $\mathcal{A}$ and one of the following conditions holds, then $H^1(\mathcal{A}\widehat{\otimes}\mathcal{B}, (\mathcal{I}\widehat{\otimes}\mathcal{B})^*) = \{0\}$ implies that $\mathcal{A}$ is $\mathcal{I}$-weakly amenable.*

(i) *$\mathcal{B}$ is commutative and $\mathcal{B}^*$ is injective;*

(ii) *$\mathcal{I}$ is complemented in $\mathcal{A}$.*

A special case of the condition (ii) of Theoem 2.5 is that $\mathcal{I} = \mathcal{A}$. In this case, the weak amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ necessities that $\mathcal{A}$ is weakly amenable. Here and subsequently, we denote the character space of a Banach algebra $\mathcal{A}$ by $\Phi_{\mathcal{A}}$.

Theorem 2.6. *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras such that $\mathcal{A}^*$ be injective Banach space, and $\mathcal{J}$ be a closed two-sided ideal in $\mathcal{B}$. If $\Phi_{\mathcal{A}}$ is non-empty and $H^1(\mathcal{A}\widehat{\otimes}\mathcal{B}, (\mathcal{A}\widehat{\otimes}\mathcal{J})^*) = \{0\}$, then $\mathcal{B}$ is $\mathcal{J}$-weakly amenable.*

Proof. Let $\phi \in \Phi_{\mathcal{A}}$. Choose $a_0 \in \mathcal{A}$ with $\phi(a_0) = 1$. Assume that $d : \mathcal{B} \longrightarrow \mathcal{J}^*$ is a derivation. Consider the bounded linear map $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ defined through

$$\langle D(a\otimes b), c\otimes j\rangle := \langle d(b), j\rangle\langle\phi, ca\rangle \quad (c \in \mathcal{A}, j \in \mathcal{J}).$$

We wish to show that D is a derivation. For each $a_1, a_2 \in \mathcal{A}$ and $b_1, b_2 \in \mathcal{B}$, we have

$$\begin{aligned}&\langle (a_1\otimes b_1)\cdot D(a_2\otimes b_2), c\otimes j\rangle + \langle D(a_1\otimes b_1)\cdot(a_2\otimes b_2), c\otimes j\rangle \\ &\quad= \langle D(a_2\otimes b_2), ca_1\otimes jb_1\rangle + \langle D(a_1\otimes b_1), a_2c\otimes b_2j\rangle \\ &\quad= \langle d(b_2), jb_1\rangle\langle\phi, ca_1a_2\rangle + \langle d(b_1), b_2j\rangle\langle\phi, a_2ca_1\rangle \\ &\quad= \langle D((a_1\otimes b_1)\cdot(a_2\otimes b_2)), c\otimes j\rangle.\end{aligned}$$

So, D is a derivation. Since $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is $\mathcal{A}\widehat{\otimes}\mathcal{J}$-weakly amenable, there is $\psi \in (\mathcal{A}\widehat{\otimes}\mathcal{J})^*$ such that $D = ad_\psi$. Define $\mathfrak{I}^* \in \mathcal{J}^*$ by $\mathfrak{I}^*(j) = \varphi(a_0^2\otimes j)$ for all

$j \in \mathcal{J}$. For each $b \in \mathcal{B}$ and $j \in \mathcal{J}$, we obtain

$$\begin{aligned}\langle d(b), j\rangle &= \langle d(b), j\rangle\langle \phi, a_0^2\rangle \\ &= \langle D(a_0 \otimes b), a_0 \otimes j\rangle = \langle (a_0 \otimes b \cdot \psi - \psi \cdot a_0 \otimes b), a_0 \otimes j\rangle \\ &= \langle \psi, a_0^2 \otimes jb - bj\rangle = \langle ad_{\mathfrak{J}^*}(b), j\rangle\end{aligned}$$

Therefore d is an inner derivation. □

The proof of the upcoming result is similar to the proof of Theorem 2.6. We include it without proof.

Theorem 2.7. *Let $\mathcal{A}, \mathcal{B}$ be Banach algebras, and $\mathcal{I}$ be a complemented closed two-sided ideal in $\mathcal{A}$. If $\Phi_{\mathcal{B}}$ is non-empty and $H^1(\mathcal{A}\widehat{\otimes}\mathcal{B}, (\mathcal{I}\widehat{\otimes}\mathcal{B})^*) = \{0\}$, then $\mathcal{A}$ is $\mathcal{I}$-weakly amenable.*

Let $\mathcal{A}$ be a non-unital Banach algebra. Then $\mathcal{A}^\# = \mathcal{A} \oplus \mathbb{C}$, the unitization of $\mathcal{A}$, is a unital Banach algebra with unit element $e_{\mathcal{A}}$ which contains $\mathcal{A}$ as a closed ideal.

We bring the following theorem from [4, Theorem 1.13] which plays a fundamental role to arrive our purpose in this paper.

Theorem 2.8. *Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{J}$ be a closed two-sided ideal in $\mathcal{A}$ with a bounded approximate identity. Then, for every closed two-sided ideal $\mathfrak{I}$ in $\mathcal{J}$, $\mathcal{J}$ is $\mathfrak{I}$-weakly amenable if and only if $\mathcal{A}$ is $\mathfrak{I}$-weakly amenable.*

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. If $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is ideally amenable, then it is always weakly amenable. Hence, $\mathcal{A}$ and $\mathcal{B}$ are weakly amenable by [13, Theorem 2.3]. Since $\mathcal{A}$ and $\mathcal{B}$ are commutative, they are ideally amenable by [4, Theorem 1.3]. Now, suppose that $\mathcal{A}$ and $\mathcal{B}$ are commutative ideally amenable Banach algebras. Then, they are weakly amenable, and thus $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is weakly amenable by [2, Propostion 2.8.71]. Since $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is commutative and weakly amenable, it is ideally amenable by [4, Theorem 1.3]. In other words, this fact is taken from the proof of [12, Theorem 4.3]. However, the direct affects of ideals of $\mathcal{A}$ and $\mathcal{B}$ are not seen in that proof. We bring a different proof in details as follows.

Theorem 2.9. *Let $\mathcal{A}, \mathcal{B}$ be commutative Banach algebras such that $(A^\#)^*$ is injective Banach space. Let $\mathcal{J}$ be closed two-sided ideal in $\mathcal{B}$ and $\mathcal{B}$ has bounded approximate identity in $\mathcal{J}$. If $\mathcal{A}^\#$ is weakly amenable and $\mathcal{B}$ is $\mathcal{J}$-weakly amenable, then $H^1(\mathcal{A}^\#\widehat{\otimes}\mathcal{B}^\#, (\mathcal{A}^\#\widehat{\otimes}\mathcal{J})^*) = \{0\}$.*

Proof. Let $\mathcal{A}^\#$ and $\mathcal{B}^\#$ be unitizations of $\mathcal{A}$ and $\mathcal{B}$ respectively. Assume that $D : \mathcal{A}^\#\widehat{\otimes}\ \mathcal{B}^\# \longrightarrow (\mathcal{A}^\#\widehat{\otimes}\ \mathcal{J})^*$ be a bounded derivation. Then $\mathcal{A}^\#\widehat{\otimes}\mathcal{J}$ is a Banach $\mathcal{B}^\#$-bimodule with respect to the map

$$\mathcal{B}^\# \times \mathcal{A}^\#\widehat{\otimes}\mathcal{J} \longrightarrow \mathcal{A}^\#\widehat{\otimes}\mathcal{J} : (b, x) \mapsto (e_{\mathcal{A}}\widehat{\otimes}b) \cdot x \qquad (b \in \mathcal{B}^\#, x \in \mathcal{A}^\#\widehat{\otimes}\mathcal{J}).$$

Clearly, $D|_{e_{\mathcal{A}}\widehat{\otimes}\mathcal{B}^{\#}}$ belongs to $Z^1(\mathcal{B}^\sharp, (\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{J})^*)$. Now, by Theorem 2.8 $Z^1(\mathcal{B}^\sharp, \mathcal{J}^*)$={0} and $Z^1(\mathcal{J}, \mathcal{J}^*)$={0}. We claim that

$$Z^1(\mathcal{B}^\sharp, (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{J})^*) = \{0\}.$$

Suppose contrary to our claim, that there is a non-zero derivation D in $Z^1(\mathcal{B}^{\#}, (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{J})^*)$. Since the closure of $\mathcal{J}^2$ is $\mathcal{J}$, there exists a_0 in $\mathcal{J}$ such that $D(a_0^2) \neq 0$. We choose λ in $(\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{J})^{**}$ such that $\langle \lambda, a_0 \cdot D(a_0)\rangle = 1$. Define $R_\lambda : (\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{J})^* \longrightarrow \mathcal{J}^*$ via $\langle R_\lambda(f), j\rangle := \langle \lambda, j \cdot f\rangle$ for $f \in (\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{J})^*$ and $j \in \mathcal{J}$. It is obvious that R_λ is a bounded linear $\mathcal{B}^{\#}$-bimodule homomorphisem. Thus, $R_\lambda \circ D|_{e_{\mathcal{A}}\widehat{\otimes}\mathcal{B}^{\#}}$ is a bounded linear derivation. Set $d := D|_{e_{\mathcal{A}}\widehat{\otimes}\mathcal{B}^{\#}}$. Then, $R_\lambda \circ d \in Z^1(\mathcal{B}^{\#}, \mathcal{J}^*)$. On the other hand $\langle R_\lambda \circ D(a_0), a_0\rangle = 1$. This leads to a contradiction with $Z^1(\mathcal{B}^{\#}, \mathcal{J}^*)$={0}. One can show that in a similar way $Z^1(\mathcal{A}^{\#}, (\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{J})^*) = \{0\}$. Hence, $Z^1(\mathcal{A}^{\#}\widehat{\otimes}\, \mathcal{B}^{\#}, (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{J})^*) = \{0\}$. □

We note that in the above theorem we can remove injectivity of $(A^{\#})^*$ and replace the condition that $\mathcal{J}$ is a complemented closed two-sided ideal in $\mathcal{B}$.

A (continuous) function ω from a locally compact group G to $(0, \infty)$ is called a weight function if $\omega(st) \leqslant \omega(s)\omega(t)$ for all $s, t \in G$. Let us consider the space

$$L^1(G, \omega) = \left\{f : G \longrightarrow \mathbb{G} : f\omega \in L^1(G)\right\}.$$

Proposition 2.10. *Let G_1 and G_2 be two locally compact Abelian groups and let w_1 and w_2 be weights on them, respectively. Then the (projective) tensor product algebra $L^1(G_1, w_1)\widehat{\otimes}L^1(G_2, w_2)$ is weakly amenable if and only if both $L^1(G_1, w_1)$ and $L^1(G_2, w_2)$ are weakly amenable.*

Remark 2.11. *The last result was proved in [14, Corollary 3.10]. So, if G_1 and G_2 are as in the above proposition, by the paragraph preceeding Theorem 2.9 and Proposition 2.10, we conclude that the (projective) tensor product algebra $L^1(G_1, w_1)\widehat{\otimes}L^1(G_2, w_2)$ is ideally amenable if and only if $L^1(G_1, w_1)$ and $L^1(G_2, w_2)$ are ideally amenable.*

We finish the paper by some examples.

Example 2.12. Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{\#}$ is injective Banach space and $0 \neq \phi$ in $Ball(\mathcal{A}^*)$. Then, $\mathcal{A}$ with the product $a.b = \phi(a)b$ for all a, b in $\mathcal{A}$, becomes a Banach algebra. This algebra is denoted by $\mathcal{A}_\phi$. It is easily to verified that $\Phi(\mathcal{A}_\phi) = \{\phi\}$. If $\mathcal{B}$ is a Banach algebra such that $\mathcal{A}_\phi\widehat{\otimes}\mathcal{B}$ is ideally amenable then so is $\mathcal{B}$ by Theorem 2.6.

To present next examples we need the following result which is proved in [14, Corollary 3.6].

Proposition 2.13. *Let G be a locally compact Abelian group and ω be a weight* *on G. If for each $t \in G$*

$$\lim inf_{n\to\infty} \frac{\omega(t^n)\omega(t^{-n})}{n} = 0,$$

then $L^1(G,\omega)$ is weakly amenable.

Remark 2.14. *Let G be a locally compact Abelian group and ω be a weight on G. By the hypothesis of Proposition 2.13 and [4, Theorem 1.3] we canclude that $L^1(G,\omega)$ is ideally amenable.*

Example 2.15. Fix $k \in \mathbb{N}$ and consider the group $(\mathbb{Z}^k,+)$ such that $\mathbb{Z}^k$ is the cartesian product of the set of integers k times. Then, $(\mathbb{Z}^k,+)$ is a locally compact Abelian group. Define $\omega_\alpha(t) = (1+\|t\|)^\alpha$ for $t \in \mathbb{Z}^k$ in which $0 < \alpha < \frac{1}{2}$, where $\|t\|$ is the Euclidean norm. Obviously, $\|t^n\| = n\|t\|$ for all $n \in \mathbb{N}$. We get

$$\begin{aligned}\lim inf_{n\to\infty} \frac{\omega(t^n)\omega(t^{-n})}{n} &= \lim inf_{n\to\infty} \frac{(1+n\|t\|)^\alpha(1+n\|-t\|)^\alpha}{n} \\ &= \lim inf_{n\to\infty} \frac{(1+n\|t\|)^{2\alpha}}{n} = 0\end{aligned}$$

Thus, by Remark 2.14 we conclude that $l^1(\mathbb{Z}^k,\omega_\alpha)$ for $\alpha < \frac{1}{2}$ is ideally amenable. For any $n,m \in \mathbb{N}$ and $0 < \alpha,\beta < \frac{1}{2}$, by Remark 2.11 we see that $l^1(\mathbb{Z}^n,\omega_\alpha)\widehat{\otimes}l^1(\mathbb{Z}^m,\omega_\beta)$ is ideally amenable.

Example 2.16. Consider the locally compact Abelian group $(\mathbb{R},+)$. Define the weight $\omega : \mathbb{R} \longrightarrow \mathbb{R}^+$ via $\omega(t) = e^{-t^2}$. Then, $\omega(0) = 1$ and $\omega(x+y) = e^{-(x+y)^2} \leqslant e^{-x^2}e^{-y^2} = \omega(x)\omega(y)$. So ω is a weight on $\mathbb{R}$. For all $t \in \mathbb{R}$, we find

$$\lim inf_{n\to\infty} \frac{\omega(nt)\omega(-nt)}{n} = \lim inf_{n\to\infty} \frac{e^{-n^2t^2}e^{-n^2t^2}}{n} = 0.$$

Hence, $L^1(\mathbb{R},\omega)$ is ideally amenable. Therefore, $L^1(R,\omega)\widehat{\otimes}L^1(R,\omega)$ is ideally amenable.

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