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A Generalization of Clique Polynomials and Graph Homomorphism

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Abstract. The clique polynomial of a graph G is the ordinary generating function of the number of complete subgraphs (cliques) of G. In this paper, we introduce a new vertex-weighted version of these polynomials. We also show that these weighted clique polynomials have always a real root provided that the weights are non-negative real numbers. As an application, we obtain a no-homomorphism criteria based on the largest real root of our vertex-weighted clique polynomial.

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1. Introduction

The dependence polynomial was first introduced by Fisher [1], while working on the problem of counting the number of words of length n from the alphabet of m letters so that some pairs of them can commute. Fisher and Solow [2] introduced the dependence polynomial, as follows:

$$f_G(x) = 1 - c_1x + c_2x^2 - c_3x^3 + \cdots + (-1)^{\omega}c_{\omega}x^{\omega}$$

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where ω is the size of the largest clique in G and c_i denotes the number of complete subgraphs of size i in G. Fisher [1], showed that the generating function of the above word-counting problem is $\frac{1}{f_G(x)}$.

If we change the sign of all negative coefficients in $f_G(x)$ to positive signs, we obtain a polynomial which is called the clique polynomial and denoted by $C(G, x)$. Hajiabolhasan and Mehrabadi [3] showed that for any simple graph G, the clique polynomial of G has always a real root using basic counting techniques, induction and the intermediate value theorem. As an immediate consequence, they obtained a new generating function proof of Mantel's theorem $[4, p.41]$ for *triangle-free* graphs. In this paper, we will continue the same line of research by introducing a new weighted version of the clique polynomial. Our main goal here is to show that how one can use the largest real root of this new graph polynomial to obtain a no-homomorphism criteria.

2. Weighted Clique Polynomials

Throughout the paper we will assume that G is a simple undirected graph. The graph terminology that we use is standard and generally follows from [4]. For a given graph G, we denoted by $V(G)$ its set of vertices and by $E(G)$ its set of edges. When $S \subseteq V(G)$, the *induced* subgraph $G[S]$ consists of S and all edges whose endpoints are connected in S. The *neighborhood* of a vertex u, written $N(u)$, is the set of vertices adjacent to u. We write $G-u$ for the subgraph of G obtained by deleting a vertex u. We also write $G - uv$ for the subgraph obtained by deleting an edge $uv \in E(G)$. Here by an *i-clique*, $i \geqslant 1$, we mean a complete subgraph of G with i vertices. The *clique number* of a graph G denoted by ω is the size of the largest clique in G. For simplicity of our arguments, we will assume that $V(G) = [n] = \{1, 2, \ldots, n\}$.
For a given set A, it's *power set* denoted by $\mathcal{P}(A)$, is the set of all subsets of A. We will associate a real *weight* function $\boldsymbol{w} : V(G) \mapsto \mathbb{R}$ with the given graph G, by associating an indeterminate w_i to each vertex i of G which can be viewed as the weight of the vertex i. More precisely, we

have

$$\begin{aligned} \boldsymbol{w} &: V(G) \mapsto \mathbb{R} \\ \boldsymbol{w}(i) &= w_i \quad \forall i \in V(G). \end{aligned}$$

The above mapping can be simply extended to the following multiplicative weight function (we will use the same symbol for the extended version)

$$\begin{aligned} \boldsymbol{w} &: \mathcal{P}(V(G)) \mapsto \mathbb{R} \\ \boldsymbol{w}(S) &= \prod_{i \in S} \boldsymbol{w}_i, \ \forall \ S \subseteq V(G). \end{aligned}$$

That is, the weight of any subset of vertices will be obtained as the product of the weights of those vertices. By convention, the empty product (corresponding to the set $S = \emptyset$) is defined to be 1.

Remark 2.1. *It is worthy to note that the above weight function on vertices of a graph G can be considered as the multiplicative version of the following additive weight function:*

$$\begin{aligned} \boldsymbol{w} &: \mathcal{P}(V(G)) \mapsto \mathbb{R} \\ \boldsymbol{w}(S) &= \sum_{i \in S} \boldsymbol{w}_i, \ \forall \ S \subseteq V(G). \end{aligned}$$

Now, we are ready to give the definition of the weighted clique polynomial.

Definition 2.2. *Let G be a graph on n vertices associated with the multiplicative weight function $\boldsymbol{w} : V(G) \mapsto \mathbb{R}$. We define the weighted clique polynomial of G denoted by $C(G, x; \boldsymbol{w})$, as follows*

$$C(G, x; \boldsymbol{w}) = \sum_{i=0}^{\omega} c_i(G, \boldsymbol{w}) x^i$$

where $\boldsymbol{w} = (w_1, \ldots, w_n)$ is the weight vector of vertices of G and $c_i(G, \boldsymbol{w}), i \geqslant 1$, the weighted-sum of all i-cliques in G, is defined by

$$c_i(G, \boldsymbol{w}) = \sum_{S:\ S \subseteq V(G),\ G[S] \ \textit{is an } i\textit{-clique}} \boldsymbol{w}(S).$$

By convention, we assume $c_0(G, \boldsymbol{w}) = 1$ for any weight vector $\boldsymbol{w}$ and any graph G. In particular, if all weights are equal to one then we obtain the clique polynomial of G [3].

Example 2.3. Consider the graph G_1 with $\boldsymbol{w} = (1,1,1,1,1)$ as depicted in Fig.1, we obtain

$$C(G_1, x; \boldsymbol{w}) = 1 + 5x + 3x^2 + x^3.$$

The above polynomial has at least one real root, because of having odd degree. Moreover, since the quadratic polynomial $\frac{d}{dx}C(G_1, x; \boldsymbol{w}) = 5 + 6x + 3x^2$ has the discriminate $\Delta = 9 - 15 = -6 < 0$, then by the first derivative criteria $C(G_1, x, \boldsymbol{w})$ is an increasing function on its domain and hence the clique polynomial of G_1 has only one real root.

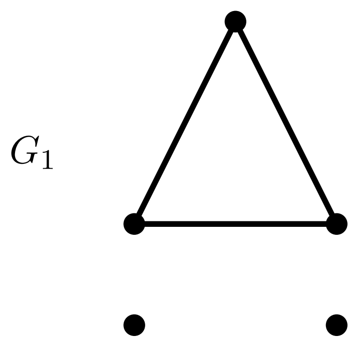

Figure 1. The clique polynomial of the graph G_1 has only one real root.

It seems that as in the case of the clique polynomials, we have always at least a real root for the weighted clique polynomials for any arbitrary choices of non-negative real weights. Next, we present the necessary tools for proving this interesting result.
The following counting lemma is key for proving the existence of a real root for the weighted clique polynomials with non-negative real weights. Here, for the given edge $e = \{u, v\}$, the notation $N(e)$ stands

for $N(e) = N(u) \cap N(v)$. We also define the multiplicative weight function associated with the subgraph H of G simply by restricting the weight function associated with G to H. More precisely, we only need to choose the weight of a vertex which is not in $V(H)$ to be zero.

Lemma 2.4. *Let G be a graph and $u, v \in V(G)$ with non-negative real weights w_u and w_v. Then, we have*

$$\begin{array}{rrcll} i) & C(G,x;\boldsymbol{w}) & = & C(G-u,x;\boldsymbol{w}_1) + w_u x C(G[N(u)],x;\boldsymbol{w}_2), & (1) \\ ii) & C(G,x;\boldsymbol{w}) & = & C(G-e,x;\boldsymbol{w}_3) & \\ & & + & w_u w_v x^2 C(G[N(e)],x;\boldsymbol{w}_4) & (2) \end{array}$$

where $e = uv \in E(G)$ and $\boldsymbol{w}_1$, $\boldsymbol{w}_2$, $\boldsymbol{w}_3$ and $\boldsymbol{w}_4$ are the weight vectors for the subgraphs $G-u$, $G[N(u)]$, $G-e$ and $G[N(e)]$, respectively.

Proof. We first note that the following recurrence relations

$$\begin{array}{rcl} c_i(G,\boldsymbol{w}) & = & c_i(G-u,\boldsymbol{w}_1) + w_u c_{i-1}(G[N(u)],\boldsymbol{w}_2), \quad (i \geqslant 1) \\ c_i(G,\boldsymbol{w}) & = & c_i(G-e,\boldsymbol{w}_3) + w_u w_v c_{i-2}(G[N(e)],\boldsymbol{w}_4), \quad (i \geqslant 2), \end{array}$$

can be easily proved using a simple counting and case analysis arguments. Now, by multiplying both sides of the above relations to x^i and summing over all i's we obtain the desired results. □

The *join* of two simple graphs G and H, written $G \vee H$, is defined as a graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{xy | x \in V(G) \wedge y \in V(H)\}$. Considering the definition of the weighted clique polynomials and using Lemma 2.4, we get the following multiplicative property for the weighted clique polynomials.

Proposition 2.5. *Let G and H be arbitrary graphs with their weight vectors $\boldsymbol{w}_1 = (w_1^{(g)}, \ldots, w_m^{(g)})$ and $\boldsymbol{w}_2 = (w_1^{(h)}, \ldots, w_n^{(h)})$. Then, we have*

$$C(G \vee H, x; \boldsymbol{w}) = C(G,x;\boldsymbol{w}_1) C(H,x;\boldsymbol{w}_2)$$

where

$$\boldsymbol{w} = (w_1^{(g)}, \ldots, w_m^{(g)}, w_1^{(h)}, \ldots, w_n^{(h)}).$$

Definition 2.6. *Let G be a graph and $\mathcal{Z}(G)$ be the set of all* negative *real roots of $C(G, x; \boldsymbol{w})$. We define ζ_G by*

$$\zeta_G = \begin{cases} \max \mathcal{Z}(G) & \text{if } \mathcal{Z}(G) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

Theorem 2.7. *Let G be a graph and H be the induced subgraph of G. Then $\zeta_H \leqslant \zeta_G$.*

Proof. We proceed by strong induction on $|V(G)| = n$. If $n = 1, 2$, then the assertion is easily followed. Moreover, if $H = G$ then we are also done. Therefore, we will assume that H is a proper induced subgraph of G. Since any proper induced subgraph H of G can be obtained from it by a sequence of vertex-deletion operations and the binary relation $\leqslant$ is transitive, it is sufficient to prove the assertion for $H = G - u$. If $\mathcal{Z}(G-u) = \emptyset$, by the definition of ζ_{G-u}, we are done. Otherwise, plugging $\alpha = \zeta_{G-u}$ (< 0) into both sides of (1), we get

$$C(G, \alpha; \boldsymbol{w}) = w_u \alpha C(G[N(u)], \alpha; \boldsymbol{w}_2).$$

Now, we distinguish between two cases.

Case 1. If $C(G[N(u)], \alpha; \boldsymbol{w}_2) \geqslant 0$, then by positivity of the weights we conclude that $C(G, \alpha; \boldsymbol{w}) \leqslant 0$. This implies that, using intermediate value theorem and $C(G, 0, \boldsymbol{w}) = 1$, the polynomial $C(G, x; \boldsymbol{w})$ has a real root in the interval $[\alpha, 0)$. Hence, we immediately obtain $\zeta_{G-u} = \alpha \leqslant \zeta_G$.

Case 2. Otherwise, $C(G[N(u)], \alpha; \boldsymbol{w}_2) < 0$. Now, we claim that this case is impossible. To do this, note that the last inequality implies that $C(G[N(u)], x; \boldsymbol{w}_2)$ has a real root in the interval $[\alpha, 0)$ (let say β), once again by applying intermediate value theorem. Hence we conclude that

$$\zeta_{G-u} = \alpha \leqslant \beta \leqslant \zeta_{G[N(u)]}.$$

But this last inequality is impossible by the induction hypothesis, since $G[N(u)]$ is an induced subgraph of $G - u$ and we obtain the inequality

$$\zeta_{G[N(u)]} \leqslant \zeta_{G-u}.$$

Thus, the only possibility is Case 1 which implies the desired inequality $\zeta_{G-u} \leqslant \zeta_G$. This completes the proof by mathematical induction. □

Corollary 2.8. *For any graph G, let w_u be the weight of the vertex u which has the maximum weights among all vertices. Then, $-\frac{1}{w_u} \leqslant \zeta_G < 0$.*

Proof. First of all, by definition of ζ_G it is clear that $\zeta_G < 0$. Next, let u be the vertex of G with the maximum weight w_u and let H be the induced subgraph $G[u]$. Then, clearly $C(H, x; w_u) = 1 + w_u x$. Hence, $\zeta_H = -\frac{1}{w_u}$. Now applying Theorem 2.7, we get $-\frac{1}{w_u} \leqslant \zeta_G$. □

Remark 2.9. *It is worth to note that the above corollary shows that the weighted clique polynomial has always a real root, provided that the weights are non-negative real numbers and the weight vector is not identically zero.*
As we already saw, when H is an induced subgraph of G we obtain $\zeta_H \leqslant \zeta_G$. Next, we show that for a spanning subgraph H of G we have the reverse inequality; that is, $\zeta_H \geqslant \zeta_G$. Recall that a *spanning* subgraph H of a given graph G is the one with the same vertex-set as G; that is, $V(H) = V(G)$.

Theorem 2.10. *Let G be a graph and H be the spanning subgraph of G. Then $\zeta_H \geqslant \zeta_G$.*

Proof. We proceed by strong induction on the number of edges. It is sufficient to prove the assertion for the case $H = G - e$, where $e = uv$ is an edge of G. Now by substituting ζ_G in both sides of (2), we get

$$C(G - uv, \zeta_G; \boldsymbol{w}_3) = -w_u w_v \zeta_G^2 C(G[N(e)], \zeta_G; \boldsymbol{w}_4). \qquad (3)$$

Since $G[N(e)]$ is an induced subgraph of G, then by Theorem 2.7 the right-hand side of (3) is negative which implies that $C(G - uv, \zeta_G; \boldsymbol{w}_3)$ is also negative. Considering the fact that $C(G - uv, 0; \boldsymbol{w}_3) = 1$ and applying the intermediate value theorem, we get the desired result. □

Definition 2.11. *An independent set in a graph is a set of pairwise nonadjacent vertices. The* independence number *of a graph G, written $\alpha(G)$, is the maximum size of an independent set of vertices.*

Proposition 2.12. *Let G be a graph with n vertices and $\alpha(G)$ its independence number. Let $\boldsymbol{w} = (w_1, \ldots, w_n)$ be the weight vector of G with $w = \min_{1 \leqslant i \leqslant n} w_i$. Then, we have $\alpha(G) \leqslant -\frac{1}{w\zeta_G}$.*

Proof. Assume that $S = \{i_1, i_2, \ldots, i_k\}$ is an independent set of size $\alpha(G) = k$ in G and H is the induced subgraph $G[S]$. Since H has no edges, we obtain

$$C(H, x; \boldsymbol{w}) = 1 + (w_{i_1} + w_{i_2} + \cdots + w_{i_k})x.$$

Now, set $w = \min_{1 \leqslant i \leqslant n} w_i$. Then $\xi_H = -\frac{1}{w_{i_1} + \cdots + w_{i_k}} \geqslant -\frac{1}{\alpha(G) w}$, and since $\zeta_H \leqslant \zeta_G$ by Theorem 2.7, we finally get

$$\alpha(G) \leqslant -\frac{1}{w\zeta_G}. \quad \square$$

3. Weighted Clique Polynomials and Homomorphisms

In this section we will discuss about one of the applications of the weighted clique polynomials for obtaining a no-homomorphism criteria. We first review some basics of graph homomorphism. The reader may consult the reference [5].

Definition 3.1. *Let G and H be two simple graphs. A* homomorphism *of G to H, written as $f : G \longrightarrow H$ is a mapping $f : V(G) \longrightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. A homomorphism of G to H is also called an H-*coloring *of G. we shall call a homomorphism $f : G \longrightarrow H$* surjective*, if the mapping $f : V(G) \longrightarrow V(H)$ is* surjective.

Definition 3.2. *Let G and H be two simple graphs and $f : G \longrightarrow H$ a homomorphism. We associate a* partition *θ_f with f consisting of the preimages of f, i.e., the set $f^{-1}(x)$, $x \in V(H)$. Clearly the set $S_x = f^{-1}(x)$ is an independent set, if there is no loop at vertex $x \in V(H)$. Thus, the mapping θ_f partitions the vertex set $V(G)$ into independent sets.*

Remark 3.3. *It is not hard to see that every weighted clique polynomial with non-negative integer weights can be viewed as the clique polynomial* *with clusters of vertices. To do this, We need the following definition of blow-up graphs.*

Definition 3.4. *For a given graph $G = (V, E)$ with the non-negative integer weight function $\boldsymbol{w} : V(G) \mapsto \mathbb{R}$ and the vertex set $V = \{1, 2, \ldots, n\}$, the* blow-up *graph G_b of G is defined as a graph $G_b = (V_b, E_b)$ such that $V_b = \{A_1, \ldots, A_n\}$ where A_i is the set of w_i vertices with no edges among them, that we will call it a cluster of vertices ($|A_i| = w_i$). There is an edge $A_iA_j \in E_b$ if there is an edge between vertices $i, j \in V$.*

Remark 3.5. *Note that to obtain a blow-up graph G_b form a graph $G = (V, E)$, we replace each vertex $i \in V$ with a cluster of vertices of size w_i (blowing-up process) and then we replace each edge $e = ij \in E$ in G with a complete bipartite graph K_{w_i,w_j} with bipartition (A_i, A_j) between two clusters of vertices of sizes w_i and w_j (see Fig 2). Now, using a simple counting argument based on inclusion - exclusion principle, one can show that*

$$C(G, x; \boldsymbol{w}) = C(G_b, x), \tag{4}$$

provided that the weights are non-negative integers.

Example 3.6. In the following picture we depicted a graph G with the weight vector $\boldsymbol{w} = (1, 2, 1, 3)$, and it's blow-up graph G_b. Note that G_b has four clusters of vertices of sizes 1, 2, 1 and 3. It is not hard to see that

$$C(G, x; \boldsymbol{w}) = 1 + 7x + 8x^2 + 2x^3 = C(G_b, x).$$

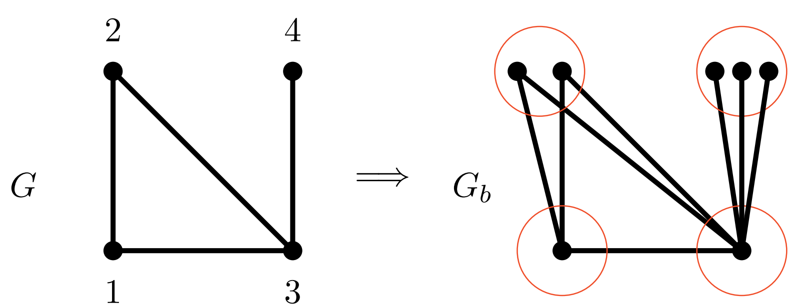

Figure 2. A graph G and it's blow-up graph G_b.

Now we are at position to state the *main* result of this section.

Theorem 3.7. *Let G and H be two simple graphs and $f : G \longrightarrow H$ be a surjective homomorphism. Then, we have*

$$\zeta_G \geqslant \zeta_H.$$

Proof. Let $V(H) = \{1, 2, \ldots, m\}$. Set

$$\boldsymbol{w} = (|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(m)|).$$

Since $f : G \longrightarrow H$ is a homomorphism, the partition function θ_f partitions the vertex set $V(G)$ into independent sets $A_i, i = 1, \ldots, m$, with $A_i = f^{-1}(i)$. Now the blow-up graph G_b of the graph G with clusters of vertices A_i's has the clique polynomial $C(G_b, x)$. By surjectivity of f, its clear that the blow-up graph H_b of the graph H is an inducted subgraph of G_b. Therefore, using Theorem 2.7, we get

$$\zeta_{H_b} \leqslant \zeta_{G_b},$$

which is equivalent to $\zeta_H \leqslant \zeta_G$, applying the identity (4). □

Corollary 2.8. *Let G and H be two simple graphs such that $\zeta_G < \zeta_H$. Then, there is no surjective homomorphism from G to H.*

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