Journal of Mathematical Extension
Vol. XX, No. XX, (2016), pp-pp (Will be inserted by layout editor)
ISSN: 1735-8299
URL: http://www.ijmex.com

A Generalization of Clique Polynomials and Graph Homomorphism

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Abstract. The clique polynomial of a graph G is the ordinary generating function of the number of complete subgraphs (cliques) of G. In this paper, we introduce a new vertex-weighted version of these polynomials. We also show that these weighted clique polynomials have always a real root provided that the weights are non-negative real numbers. As an application, we obtain a no-homomorphism criteria based on the largest real root of our vertex-weighted clique polynomial.

AMS Subject Classification: 11B83; 05A19; 11M06.
Keywords and Phrases: Weighted Clique Polynomial, Blow-Up Graph, No-Homomorphism Criteria.

1 Introduction

The *dependence* polynomial was first introduced by Fisher [1], while working on the problem of counting the number of *words* of length n

Received: XXXX; Accepted: XXXX (Will be inserted by editor)
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from the alphabet of m letters so that some pairs of them can *commute*. Fisher and Solow [2] introduced the dependence polynomial, as follows:

$$f_G(x) = 1 - c_1x + c_2x^2 - c_3x^3 + \cdots + (-1)^{\omega}c_{\omega}x^{\omega}; \qquad (1)$$

where ω is the size of the *largest* clique in G and c_i denotes the number of *complete* subgraphs of size i in G. Fisher [1], showed that the *generating* function of the above *word-counting* problem is $\frac{1}{f_G(x)}$.

If we change the sign of all negative coefficients in $f_G(x)$ to positive signs, we obtain a polynomial which is called the *clique polynomial* and denoted by $C(G,x)$. Hajiabolhasan and Mehrabadi [3] showed that for any simple graph G, the clique polynomial of G has always a *real root* using basic *counting* techniques, *induction* and the *intermediate value theorem*. As an immediate consequence, they obtained a new generating function proof of Mantel's theorem [4, p.41] for *triangle-free* graphs. In this paper, we will continue the same line of research by introducing a new *weighted* version of the clique polynomial. Our *main goal* here is to show that how one can use the *largest* real root of this new graph polynomial to obtain a *no-homomorphism* criteria.

2 Weighted Clique Polynomials

Throughout the paper we will assume that G is a simple graph. The graph terminology that we use is standard and generally follows [4]. For a given graph G, we denoted by $V(G)$ its set of vertices and by $E(G)$ its set of edges. When $S \subseteq V(G)$, the *induced* subgraph $G[S]$ consists of S and all edges whose endpoints are *connected* in S. The *neighborhood* of a vertex u, written $N(u)$, is the set of vertices *adjacent* to u. We write $G - u$ for the subgraph of G obtained by deleting a vertex u. We also write $G - uv$ for the subgraph obtained by deleting an edge $uv \in E(G)$. Here by an *i-clique*, $i \geq 1$, we mean a *complete* subgraph of G with i vertices. The clique number of a graph G denoted by ω is the size of the largest clique in G. We will associate an *indeterminate* w_i with each vertex i of G which can be viewed as the *weight* of the vertex i. For our purposes, we will assume that all weights are *non-negative* integers. We define the *weight* of an i-clique as the *product* of the weights of its vertices. Now, we are ready to give the definition of the weighted clique polynomial.

Definition 2.1. Let G be a graph with n vertices. We define the weighted clique polynomial of G denoted by $C(G, x; \overrightarrow{w})$, as follows

$$C(G, x; \overrightarrow{w}) = \sum_{i=0}^{\omega} c_i(\overrightarrow{w}) x^i, \tag{2}$$

where $\overrightarrow{w} = (w_1, \ldots, w_n)$ is the weight vector of vertices of G and $c_i(\overrightarrow{w}), i \geq 1$, denotes the sum of the weights of all i-cliques in G.

By convention, we assume $c_0(\overrightarrow{w}) = 1$ for any weight vector $\overrightarrow{w}$. In particular, if all weights are equal to one then we obtain the clique polynomial of G [3].

Example 2.2. Let $G = K_3$ be the complete graph with three vertices and the weight vector $\overrightarrow{w} = (w_1, w_2, w_3)$. Then, we have

$$C(K_3, x; \overrightarrow{w}) = 1+(w_1+w_2+w_3)x+(w_1w_2+w_1w_3+w_2w_3)x^2+(w_1w_2w_3)x^3. \tag{3}$$

The generalized Newton *binomial* identity can be read, as follows

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) = \sum_{I \subseteq \{1,2,\ldots,n\}} \left(\prod_{i \in I} x_i \right). \tag{4}$$

Hence, the equality (3) is equivalent to

$$C(K_3, x; \overrightarrow{w}) = (1 + w_1x)(1 + w_2x)(1 + w_3x). \tag{5}$$

Example 2.3. Let $G = C_4$ be the cycle of length four and the weight vector $w = (1, 2, 3, 4)$. Then, we get

$$C(C_4, x; \overrightarrow{w}) = 1 + 10x + 24x^2 = (1 + 4x)(1 + 6x). \tag{6}$$

It seems that as in the case of the clique polynomials, we have always a *real* root for the weighted clique polynomials for any arbitrary choices of *non-negative* weights. Next, we present the necessary tools for proving this interesting result.
The following *counting* lemma is *key* for proving the existence of a real root for the weighted clique polynomials with non-negative weights.

Lemma 2.4. *Let G be a graph and $u, v \in V(G)$ with non-negative weights w_u and w_v. Then, we have*

$$\begin{aligned} i) \quad C(G,x;\overrightarrow{w}) &= C(G-u,x;\overrightarrow{w}_1) + w_u x C(G[N(u)],x;\overrightarrow{w}_2), \quad (7) \\ ii) \quad C(G,x;\overrightarrow{w}) &= C(G-uv,x;\overrightarrow{w}_3) \\ &+ w_u w_v x^2 C(G[N(u)\cap N(v)],x;\overrightarrow{w}_4), \quad (8) \end{aligned}$$

where $uv \in E(G)$ and $\overrightarrow{w}_i$'s are the weight vectors for their corresponding subgraphs.

Proof. Assume that K_i is an i-clique in G.

i) If $u \in K_i$, then K_i is an i-clique in $G-u$. Otherwise, $u \in K_i$ and $K_i - u$ is an $(i-1)$-clique in $G[N(u)]$. Now by the definition of the weighted clique polynomial, we get the desired result.

ii) If K_i does not contain the edge uv, then K_i is an i-clique in $G-uv$. Otherwise, $K_i - uv$ is an $(i-2)$-clique in $G[N(u)\cap N(v)]$. Hence, we get the desired result.

□

The *join* of two simple graphs G and H, written $G \vee H$, is defined as a graph with the vertex set $V(G)\cup V(H)$ and the edge set $E(G)\cup E(H)\cup \{xy | x \in V(G) \wedge y \in V(H)\}$. Considering the definition of the weighted clique polynomials and in the same *spirit* of the above proof, we get the following *multiplicative* property for the weighted clique polynomials.

Proposition 2.5. *Let G and H be arbitrary graphs with their weight vectors $\overrightarrow{w}_1 = (w_1^g, \ldots, w_m^g)$ and $\overrightarrow{w}_2 = (w_1^h, \ldots, w_n^h)$. Then, we have*

$$C(G \vee H, x; \overrightarrow{w}) = C(G,x;\overrightarrow{w}_1)C(H,x;\overrightarrow{w}_2), \quad (9)$$

where

$$\overrightarrow{w} = (w_1^g, \ldots, w_m^g, w_1^h, \ldots, w_n^h). \quad (10)$$

Definition 2.6. Let G be a graph and $\mathcal{Z}(G)$ be the set of all *negative* real roots of $C(G,x;\overrightarrow{w})$. We define ζ_G by

$$\zeta_G = \begin{cases} \max \mathcal{Z}(G) & \text{if } \mathcal{Z}(G) \neq \emptyset, \\ -\infty & \text{otherwise.} \end{cases}$$

Theorem 2.7. *Let G be a graph and H its induced subgraph. Then $\zeta_H \leq \zeta_G$.*

Proof. We proceed by induction on $|V(G)| = n$. If $n = 1, 2$, then the assertion is easily followed. Suppose H is an induced subgraph of G. Choose a vertex u of G so that H is also an induced subgraph of $G - u$. Hence, it is sufficient to prove the assertion for $G - u$. If $\mathcal{Z}(G - u) = \emptyset$, by the definition of ζ_{G-u}, we are done. Otherwise, plugging $x = \zeta_{G-u}$ into both sides of (7), we get $C(G, \zeta_{G-u}; \overrightarrow{w}_1) = w_u \zeta_{G-u} C(G[N(u)], \zeta_{G-v}; \overrightarrow{w}_2)$. Now by mathematical induction, we have $C(G[N(u)], \zeta_{G-u}; \overrightarrow{w}_2) \geq 0$, because, otherwise we get $C(G[N(u)], \zeta_{G-u}; \overrightarrow{w}_2) < 0$, where by the intermediate value theorem implies that $C(G[N(u)], \zeta_{G-u}; \overrightarrow{w}_2)$ has a real root t so that $t > \zeta_{G-u}$. This is equivalent to $\zeta_{G[N(u)]} > \zeta_{G-u}$, which is a contradiction by the induction hypothesis. Thus, we get $C(G, \zeta_{G-u}; \overrightarrow{w}_1) \leq 0$. Applying the intermediate value theorem once again, we obtain the desired result. □

Corollary 2.8. *For any graph G, let w_u be the weight of the vertex u which has the maximum weights among all vertices. Then, $\frac{-1}{w_u} \leq \zeta_G < 0$.*

Proof. Let u be the vertex of G with the maximum weight w_u and let H be the subgraph $G[u]$. Then, clearly $C(H, x; w_u) = 1 + w_u x$. Hence, $\zeta_H = \frac{-1}{w_u}$. Now applying Theorem 2.7 , we get $\zeta_G \geq \frac{-1}{w_u}$. □

Remark 2.9. It is worth to note that the above corollary shows that the weighted clique polynomial has always a real root, provided that the weights are *non-negative real* numbers and the *weight vector* is not identically *zero*. But not all roots of a weighted clique polynomial are *necessarily* real. For example, for the graph G_1 with $\overrightarrow{w} = (1, 1, 1, 1, 1)$ as depicted in Fig.1, we obtain

$$C(G_1, x; \overrightarrow{w}) = 1 + 5x + 3x^2 + x^3. \tag{11}$$

Since the quadratic polynomial $\frac{d}{dx} C(G_1, x; \overrightarrow{w}) = 5 + 6x^2 + 3x^2$ has the discriminate $\Delta = 9 - 15 = -6 < 0$, then by the first derivative criteria $C(G_1, x, \overrightarrow{w})$ is an increasing function on its domain and hence the clique polynomial of G_1 has only one real root.

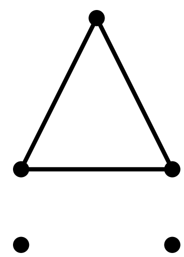

Fig 1. The clique polynomial of the graph G_1 has only one real root.

Definition 2.10. An independent set in a graph is a set of pairwise nonadjacent vertices. The independence number of a graph G, written $\alpha(G)$, is the maximum size of an independent set of vertices.

Proposition 2.11. *Let G be a graph with n vertices and $\alpha(G)$ its independence number. Let $\overrightarrow{w} = (w_1, \ldots, w_n)$ be the weight vector of G with $w = \min_{1\leq i\leq n} w_i$. Then, we have $\alpha(G) \leq \frac{-1}{w\zeta_G}$.*

Proof. Assume that $S = \{i_1, i_2, \ldots, i_k\}$ is an independent set of size $\alpha(G) = k$ in G and H is the subgraph $G[S]$. Since H has no edges, we obtain

$$C(H, x; \overrightarrow{w}) = 1 + (w_{i_1} + w_{i_2} + \cdots + w_{i_k})x. \tag{12}$$

Now, set $w = \min_{1\leq i\leq n} w_i$. Then $\xi_H = \frac{-1}{w_{i_1}+\cdots+w_{i_k}} \geq \frac{-1}{\alpha(G)w}$, and since $\zeta_H \leq \zeta_G$ by Theorem 2.7, we finally get

$$\alpha(G) \leq \frac{-1}{w\zeta_G}. \tag{13}$$

□

As we already saw, when H is an *induced* subgraph of G we obtain $\zeta_H \leq \zeta_G$. Next, we show that for a *spanning* subgraph H of G we have the *reverse* inequality; that is, $\zeta_H \geq \zeta_G$.

Theorem 2.12. *Let G be a graph and H its spanning subgraph. Then $\zeta_H \geq \zeta_G$.*

Proof. We proceed by induction on the number of edges. It is sufficient to prove the assertion for the case $H = G - e$, where $e = uv$ is an edge of G. Now by substituting ζ_G in both sides of (8), we get

$$C(G - uv, \zeta_G; \overrightarrow{w}_3) = -w_u w_v \zeta_G^2 C(G[N(u) \cap N(v)], \zeta_G; \overrightarrow{w}_4). \tag{14}$$

Since $G[N(u) \cap N(v)]$ is an induced subgraph of G, then by Theorem 2.7 the right-hand side of (14) is negative which implies that $C(G - uv, \zeta_G; \overrightarrow{w}_3)$ is also negative. Considering the fact that $C(G-uv, 0; \overrightarrow{w}_3) = 1$ and applying the intermediate value theorem, we get the desired result. □

3 Weighted Clique Polynomials and Homomorphisms

In this section we will discuss about one of the *applications* of the *weighted* clique polynomials for obtaining a *no-homomorphism* criteria. We first review some basics of graph *homomorphism*. The reader may consult the reference [5].

Definition 3.1. Let G and H be two simple graphs. A *homomorphism* of G to H, written as $f : G \longrightarrow H$ is a mapping $f : V(G) \longrightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. A homomorphism of G to H is also called an H *coloring* of G. we shall call a homomorphism $f : G \longrightarrow H$ *surjective*, if the mapping $f : V(G) \longrightarrow V(H)$ is *surjective*.

Let G and H be two simple graphs and $f : G \longrightarrow H$ a homomorphism. We associate a *partition* function θ_f with f consisting of the *preimages* of f, *i.e.*, the set $f^{-1}(x)$, $x \in V(H)$. Clearly the set $S_x = f^{-1}(x)$ must be *independent* set. Thus, the mapping θ_f *partitions* the vertex set $V(H)$ into *independent sets*.
It is not hard to see that every *weighted* clique polynomial with *non-negative integer* weights can be viewed as the *clique* polynomial with *clusters* of vertices. To see this, let G be a simple graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the corresponding weight vector $\overrightarrow{w} = (w_1, w_2, \ldots, w_n)$. We define the *blow-up* graph G_b obtained from G, as follows. The vertex set consists of the clusters of vertices $A_1, A_2, \ldots, A_n$ where $|A_i| = w_i, 1 \le i \le n$. Indeed, A_i is the blow-up of the vertex v_i with the weight w_i. To obtain G_b, we replace each *edge* e between the vertices v_i and v_j in G with the *complete bipartite* graph K_{w_i,w_j} with *bipartition* (A_i, A_j). Now using the generalized Newton *binomial* identity (4) , it is easy to see that the (*unweighed*) clique

polynomial of G is exactly the *weighted* clique polynomial G_b. That is

$$C(G,x) = C(G_b, x; \overrightarrow{w}). \tag{15}$$

Now we are at position to state the *main* result of this section.

Theorem 3.2. *Let G and H be two simple graphs and $f : G \longrightarrow H$ be a surjective homomorphism. Then, we have*

$$\zeta_G \geq \zeta_H. \tag{16}$$

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$. Set $\overrightarrow{w} = (f^{-1}(v_1), f^{-1}(v_2), \ldots, f^{-1}(v_n))$. Since $f : G \longrightarrow H$ is a homomorphism, the partition function θ_f partitions the vertex set $V(G)$ into independent sets $A_i, i = 1, \ldots, n$, with $|A_i| = f^{-1}(v_i)$. Now the blow-up graph G_b with clusters of vertices A_i's has the weighted clique polynomial $C(G_b, x; \overrightarrow{w})$. By surjectivity of f, its clear that the blow-up graph H_b of the graph H is an inducted subgraph of G_b. Therefore, using Theorem 2.7, we get

$$\zeta_{H_b} \leq \zeta_{G_b}, \tag{17}$$

which is equivalent to $\zeta_H \leq \zeta_G$, applying the identity (15). $\square$

Corollary 3.3. *Let G and H be two simple graphs such that $\zeta_G < \zeta_H$. Then, there is no surjective homomorphism from G to H.*

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