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A Generalization of Total Graphs of Modules over Commutative Rings under Multiplicatively Closed Subsets

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Abstract. Let R be a commutative ring and M be an R-module with a proper submodule N. A generalization of total graphs, denoted by $T(\Gamma_H^N(M))$, is introduced and investigated. It is the (undirected) graph with all elements of M as vertices and for distinct $x, y \in M$, the vertices x, y are adjacent if and only if $x+y \in M_H(N)$ where $M_H(N) = \{m \in M : rm \in N \text{ for some } r \in H\}$, where H is a multiplicatively closed subset of R. In this paper, in addition to studying some algebraic properties of $M_H(N)$, we investigate some graph theoretic properties of two essential subgraphs of $T(\Gamma_H^N(M))$.

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1. Introduction

Throughout, all rings are commutative with non-zero identity and all modules are unitary. Let R be a ring, M an R-module and N a proper submodule of M. The ordinary total graph of a commutative ring R,

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denoted by $T(\Gamma(R))$, was introduced by Anderson and Badawi in [6], as the graph with all elements of R as vertices and two distinct vertices $x, y \in R$ are adjacent if and only if $x+y \in Z(R)$ where $Z(R)$ denotes the set of all zero-divisors of R. The authors introduced in [5] the generalized total graph of R in which $Z(R)$ is extended to H, a multiplicative-prime subset of R, in such away that $ab \in H$ for every $a \in H$ and $b \in R$, and whenever $ab \in H$ for all $a, b \in R$, then either $a \in H$ or $b \in H$.

The concept of total graphs is a great concept that is usually used in commutative algebra to obtain many interesting graphs in this field. In [1] and [2], A. Abbasi and S. Habibi, gave a generalization of the total graph. They studied in [2] the total graph $T(\Gamma_N(M))$ of a module over a commutative ring with respect to a proper submodule. It is an undirected graph with the vertex set M, where two distinct vertices m and n are adjacent if and only if $m+n \in M(N)$, where $M(N) = \{m \in M : rm \in N$ for some $r \in R - (N : M)\}$. It is easy to see that $M(N)$ is closed under the multiplication by scalers. However $M(N)$ may not be an additive subgroup of M.

A proper submodule N of M is said to be a *prime* submodule if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $r \in (N :_R M)$. Clearly, if N is a prime submodule of M, then $P = (N :_R M)$ is a prime ideal of R. An element $a \in R$ is called prime to N if $am \in N (m \in M)$ implies that $m \in N$. Denote by $S_R(N)$ the set of all elements of R that are not prime to N (see [7]). Consider $S(I) = \{r \in R : sr \in I$ for some $s \in R - I\}$ (see [2]) needed in the context. We now define $M_H(N) = \{m \in M : rm \in N$ for some $r \in H\}$ where H is a multiplicatively closed subset of R, i.e. $ab \in H$ for all $a, b \in H$. Since N is a proper submodule of M and $N \subseteq M_H(N)$ hence $M_H(N)$ is not empty. We consider an undirected graph denoted by $T(\Gamma_H^N(M))$ with the vertex set consisting of all elements of M in which two distinct vertices m and m' are adjacent if and only if $m + m' \in M_H(N)$.

Let $\Gamma_H^N(M_H(N))$ be the (induced) subgraph of $T(\Gamma_H^N(M))$ with the vertex set $M_H(N)$ and let $\Gamma_H^N(M_H^C(N))$ be the (induced) subgraph of $T(\Gamma_H^N(M))$ with vertices of $M - M_H(N)$.

In the following, we investigate some properties of the graph $T(\Gamma_H^N(M))$.

Let G be a simple graph. We say that G is *connected* if there is a path between any two distinct vertices of G and it is *totally disconnected* if none of two vertices of G are adjacent. A subgraph G_1 of G is an *induced subgraph* if the vertex set of G_1 is contained in the vertex set of G and two vertices of G_1 are adjacent if and only if they are adjacent in G. We say that two subgraphs G_1 and G_2 of G are *disjoint* if G_1 and G_2 have no common vertices and no vertex of G_1 (resp., G_2) is adjacent (in G) to any vertex not in G_1 (resp., G_2). For vertices x and y of G, we define $d(x, y)$ to be the length of the shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path). The *diameter* of G is $diam(G) = sup\{d(x, y)|x$ and y are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is the length of a shortest cycle in G ($gr(G) = \infty$ if G contains no cycles). We denote the complete graph on n vertices by K^n and the complete bipartite graph with the partitioned vertex set as $A \cup B$ with $|A| = m$ and $|B| = n$ by $K^{m,n}$.

For a graph G, a complete subgraph of G is called a *clique*. The *clique number*, $\omega(G)$, is the greatest integer $n \geqslant 1$ such that $K^n \subseteq G$, and $\omega(G) = \infty$ if $K^n \subseteq G$ for all $n \geqslant 1$. We say that a graph G is a *forest* if it contains no cycles. A *matching* in a graph G is a set of edges such that no two have a vertex in common.

In Section 2, we show that $U^{-1}(N :_R M) = (U^{-1}(N):_{U^{-1}R}U^{-1}(M))$ whenever $U \subseteq H$ is a multiplicatively closed subset of R and N is a prime submodule. We proceed in Section 3 by studying the algebraic properties of graph $T(\Gamma_H^N(M))$ and some relationship between it's subgraphs and the subgraphs of graph $T(\Gamma_N(M))$ which introduced in [2].

2. Some Properties Concerning $M_H(N)$

In this section we obtain some properties concerning $M_H(N)$. Throughout the argument, N is a proper submodule of M over the commutative ring R. We define $M_H(N) = \{m \in M : rm \in N$ for some $r \in H\}$, where H is a multiplicatively closed subset of R, in such away that $ab \in H$ for every $a \in H$ and $b \in H$. It is easy to see that in the following cases one has $M_H(N) = M$.

1. $N = M$,
2. $0 \in H$,
3. $H \bigcap (M_H(N) : M) \neq \emptyset$,
4. $H \bigcap (0 : M) \neq \emptyset$,
5. $H \bigcap (N : M) \neq \emptyset$.

Throughout the context we suppose that $M_H(N) \neq M$.

Remark 2.1. *Note that under our notations, $M_H(N)$ is a submodule of M containing N and contained in $M(N)$.*

Example 2.2. Let $R = Z_4 \times Z_4$ and let $M = Z_4 \times Z_2$, $N = 0 \times Z_2$ and $H = \{(1, 2), (1, 0)\}$. It is clear that $(N : M) = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$, $M(N) = \{(0, 0), (0, 1), (2, 0), (2, 1)\}$ and $M_H(N) = N = \{(0, 1), (0, 0)\}$. This example shows that $M_H(N) \subsetneq M(N)$.

Proposition 2.3. *Let P be a prime submodule of M. Then $P = M_H(P)$ if and only if $H \bigcap (P : M) = \emptyset$.*

Proof. Suppose that $r \in H \bigcap (P : M)$ and $m \in M - M_H(P)$. Then $rm \in P$ and so $m \in M_H(P)$.

Conversely, it suffices to show that $M_H(P) \subseteq P$. Let $x \in M_H(P)$. Then $rx \in P$ for some $r \in H$. So, $x \in P$ or $r \in (P : M)$, by the primeness of P. The implication $r \in (P : M)$ is impossible, by the assumption, so $x \in P$ and we are done. $\square$

Remark 2.4. *(1) With our notations if N is a prime submodule of M, since $M_H(N) = M(N) = N$, we have $(M_H(N) : M) = S(N : M)$.*

We should note that in general, $(M_H(N) : M) \neq S(N : M)$. In Example 2.2, one has $(M_H(N) : M) = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$ and $S(N : M) = \{(2, 0), (2, 1), (2, 2), (2, 3), (0, 0), (0, 1), (0, 2), (0, 3)\}$.

(2) If N is a prime submodule of M, then $S_R(M_H(N)) \subseteq S(N : M)$, because if $r \in S_R(M_H(N))$, then there is an $m \in M - M_H(N)$ such that

$rm \in M_H(N)$. So, $rtm \in N$ for some $t \in H$ (note that $t \in R-(N:M)$). This shows that $rt \in (N:M)$. Hence $r \in S(N:M)$.

Proposition 2.5. *For every proper submodule N of M, $S_R(M_H^C(N)) = S(M_H^C(N):M) = \emptyset$, where $M_H^C(N) = M - M_H(N)$.*

Proof. Suppose that $r \in S_R(M_H^C(N))$. There exists $m \in M_H(N)$ such that $rm \in M_H^C(N)$. So, $tm \in N$ for some $t \in H$. Hence, $rtm \in N$. This shows that $t \notin H$, which is a contradiction. Thus, $S_R((M_H^C(N))) = \emptyset$. Now, let $r \in S((M_H^C(N)):M)$. There exists $t \notin ((M_H^C(N)):M)$ such that $rt \in ((M_H^C(N)):M)$. Because $t \notin ((M_H^C(N)):M)$, there exists $m' \in M$ such that $tm' \in M_H(N)$. Therefore, $rtm' \in M_H(N) \bigcap M_H^C(N)$, which is a contradiction. So $S((M_H^C(N)):M) = \emptyset$. $\square$

Theorem 2.6. *Let M be an R-module. Then for every multiplicatively closed subset U of R such that $U \subseteq H$, $(U^{-1}(M_H(N)):_{U^{-1}R}U^{-1}(M)) = U^{-1}(M_H(N):_R M)$.*

In particular, $H^{-1}(M_H(N):_R M) = (H^{-1}(M_H(N)):_{H^{-1}R}H^{-1}(M))$.

Proof. It suffices to show that $(U^{-1}(M_H(N)):_{U^{-1}R}U^{-1}(M)) \subseteq U^{-1}(M_H(N):_R M)$. Let $r/s \in (U^{-1}(M_H(N)):_{U^{-1}R}U^{-1}(M))$ such that $r \in R$ and $s \in U$ and let $m \in M$. Then $r/s \cdot m/1 \in U^{-1}(M_H(N))$. There exist $m' \in M_H(N)$ and $s' \in U$ such that $r/s \cdot m/1 = m'/s'$. Since $m' \in M_H(N)$, there exists $r' \in H$ such that $r'm' \in N$. Because $r/s \cdot m/1 = m'/s'$, there exists $t \in U$ such that $rms't = sm't$. Hence $r'rms't = r'sm't \in N$. Then $rm \in M_H(N)$, because $s', r', t \in H$, so $r \in (M_H(N):M)$, therefore $r/s \in U^{-1}(M_H(N):_R M)$. $\square$

Remark 2.7. *With the above notations, if N is a prime submodule of M, then $U^{-1}(N:_R M) = (U^{-1}(N):_{U^{-1}R}U^{-1}(M))$.*

Lemma 2.8. *Let $m/s \in H^{-1}N$ such that $m \in M$ and $s \in H$, then $m \in M_H(N)$.*

Proof. Let $m/s \in H^{-1}N$ such that $m \in M$ and $s \in H$. There exist $n \in N$ and $t \in H$ such that $m/s = n/t$. So, $rtm = rsn \in N$ for some $r \in H$. Hence $m \in M_H(N)$, since $rt \in H$. $\square$

Theorem 2.9. *Let M be an R-module. Then for every multiplicatively closed subset U of R such that*
$U \subseteq H$, $U^{-1}(M_H(N)) = U^{-1}(M)_{U^{-1}H}(U^{-1}(N))$.

Proof. If $m/s \in U^{-1}(M)_{U^{-1}H}(U^{-1}(N))$, where $m \in M$ and $s \in U$, then there is r/t for some $r \in H$ and $t \in U$ such that $(m/s).(r/t) \in U^{-1}(N)$. There are $n \in N$ and $s' \in U$ such that $mr/st = n/s'$. So, $umrs' = unst \in N$ for some $u \in U$. Hence $m \in M_H(N)$ and $m/s \in U^{-1}(M_H(N))$. Conversely, choose $m/s \in U^{-1}(M_H(N))$ such that $m \in M_H(N)$ and $s \in U$. Then $tm \in N$ for some $t \in H$. So $(m/s).(t/1) \in U^{-1}(N)$. Therefore, $m/s \in U^{-1}(M)_{U^{-1}H}(U^{-1}(N))$. □

3. The Generalized Total Graph

In the following, we introduce a generalized total graph $T(\Gamma_H^N(M))$ as a simple graph with vertex set M in which two distinct elements $x, y \in M$ are adjacent if and only if $x + y \in M_H(N)$. It follows from the definition that if $M_H(N) = M$, then $T(\Gamma_H^N(M))$ is complete, so we suppose that $M_H(N) \neq M$. We denote by $\Gamma_H^N(M_H(N))$ and $\Gamma_H^N(M_H^C(N))$ the (induced) subgraphs of $T(\Gamma_H^N(M))$ with vertices in $M_H(N)$ and $M_H^C(N)$, respectively, where $M_H^C(N) = M - M_H(N)$. Based our assumptions $M_H(N) \neq M$, hence $\Gamma_H^N(M_H^C(N))$ is always nonempty.

Recall from [2] that T$(\Gamma_N(M))$ is the graph with vertex set M such that two vertices m, m' are adjacent if and only if $m + m' \in M(N)$. As [2], we denote by $M(\Gamma_N(M))$ and $\overline{M}(\Gamma_N(M))$ the (induced) subgraph of T$(\Gamma_N(M))$ with vertices in $M(N)$ and $M - M(N)$, respectively. It is assumed in [2] that $M(N) \neq M$; $\overline{M}(\Gamma_N(M))$ is not a null graph.

Remark 3.1. *Let N be a submodule of M such that $M_H(N) \neq M$. Then*

(1) $\Gamma_H^N(N)$ is a complete graph.

(2) The total graph $\Gamma_H^N(M_H(N))$ is a complete graph.

(3) $\Gamma_H^N(M_H(N))$ and $\Gamma_H^N(M_H^C(N))$ are disjoint subgraphs of the total graph $T(\Gamma_H^N(M))$, so $T(\Gamma_H^N(M))$ is never connected.

(4) In the case where R is an integral domain and $H = R - \{0\}$ is a multiplicatively closed subset, then $M_H(0) = T(M)$ and $T(\Gamma_H^{(0)}(M))$ is the total torsion elements graph of a module, which studied in [8].

(5) If $\Gamma_H^N(M_H^C(N))$ just consists of one edge, then $\overline{M}(\Gamma_N(M))$ is a null graph, or just consists of a single vertex or one edge.

(6) If $x, y \in M - M(N)$ and x is adjacent to y in $\Gamma_H^N(M_H^C(N))$, then x is adjacent to y in $\overline{M}(\Gamma_N(M))$.

(7) If $\Gamma_H^N(M_H^C(N))$ is connected (complete), then $\overline{M}(\Gamma_N(M))$ is connected (complete).

Theorem 3.2. *Suppose that M is an R-module. If there exists an edge in $\Gamma_H^N(M_H^C(N))$ between two distinct vertices m_1 and m_2 such that $m_1 \neq -m_2, -m_1$, then $\Gamma_H^N(M_H^C(N))$ contains a 3 or a 4-cycle.*

Proof. Let m_1 and m_2 be distinct vertices of $\Gamma_H^N(M_H^C(N))$ which are adjacent and $m_1 \neq -m_2, -m_1$. So, $m_1 + m_2 \in M_H(N)$. If $m_2 = -m_2$, then $m_1 — m_2 — (-m_1) — m_1$ is a path in $\Gamma_H^N(M_H^C(N))$, so we have a 3-cycle. If $m_2 \neq -m_2$, there is a path $m_1 — m_2 — (-m_2) — (-m_1) — m_1$ in $\Gamma_H^N(M_H^C(N))$ and we have a 4-cycle. □

Theorem 3.3. *Suppose that $\Gamma_H^N(M_H^C(N))$ is a forest with no isolated vertices such that for every two adjacent vertices m_i and m_j, $m_i \neq -m_i, -m_j$. Then $\Gamma_H^N(M_H^C(N))$ is a matching.*

Proof. Assume one vertex m of $\Gamma_H^N(M_H^C(N))$ is adjacent to a vertex $m' \neq -m', -m$. By Theorem 3.2, we have a 3 or 4-cycle in the graph and this is a contradiction with our assumption. This yields each vertex m is just adjacent to $-m$ and so, $\Gamma_H^N(M_H^C(N))$ is a matching. □

The proof of the following lemma is similar to [2, Theorem 3.2].

Lemma 3.4. *Suppose that M is an R-module.*

(1)*If Γ is an induced subgraph of $\Gamma_H^N(M_H^C(N))$ and if m_1 and m_2 are distinct vertices of Γ that are connected by a path in Γ, then there exists* *a path in* Γ *of length at most* 2 *between* m_1 *and* m_2. *In particular, if* $\Gamma_H^N(M_H^C(N))$ *is connected, then diam(* $\Gamma_H^N(M_H^C(N)) \leqslant 2$.

(2) Let m_1 *and* m_2 *be distinct elements of* $\Gamma_H^N(M_H^C(N))$ *that are connected by a path. If* m_1 *and* m_2 *are not adjacent, then* m_1 — $(-m_1)$ — m_2 *and* m_1 — $(-m_2)$— m_2 *are paths of length* 2 *between* m_1 *and* m_2 *in* $\Gamma_H^N(M_H^C(N))$.

The proof of the following theorem is similar to [2, Theorem 3.3].

Theorem 3.5. *Let* M *be an* R*-module. Then the following statements are equivalent.*

1- $\Gamma_H^N(M_H^C(N))$ *is connected.*

2- Either $m + m' \in M_H(N)$ *or* $m - m' \in M_H(N)$ *(but not both) for all* $m, m' \in M_H^C(N)$.

3- Either $m + m' \in M_H(N)$ *or* $m + 2m' \in M_H(N)$ *for all* $m, m' \in M_H^C(N)$.
In particular, if (3) holds, then $2m \in M_H(N)$ *or* $3m \in M_H(N)$ *for all* $m, m' \in M_H^C(N)$.

Theorem 3.6. *Let* $\mid M_H(N) \mid= \alpha$, $\mid M/M_H(N) \mid= \beta$ *(we allow* α, β *to be infinite) and* $2 \in (M_H(N) : M)$. *Then* $T(\Gamma_H^N(M))$ *is a disjoint union of* β *copies of* K^α.

Proof. We at first note that the subgraph of $T(\Gamma_H^N(M))$ induced by vertices in $x + M_H(N)$ is a complete graph for each $x \in M$, because $(x + y) + (x + z) = 2x + (y + z) \in M_H(N)$ for all $y, z \in M_H(N)$. Hence the subgraph induced by $x + M_H(N)$ is isomorphic to K^α. Now assume that x_1 and x_2 are disjoint elements of M. It is easy to see that $x_1+x_2 \in M_H(N)$ if and only if $x_1-x_2 \in M_H(N)$ (because $x_1-x_2 = x_1+x_2-2x_2 \in M_H(N)$) if and only if $x_1+M_H(N) = x_2+M_H(N)$. So, x_1 and x_2 are adjacent in $T(\Gamma_H^N(M))$ if and only if $x_1+M_H(N) = x_2+M_H(N)$. By the fact that the vertex set of $T(\Gamma_H^N(M))$ is a disjont union of vertex sets in the form of $x + M_H(N)$ for $x \in M$, we are done. $\square$

Corollary 3.7. *With the above notations,* $\Gamma_H^N(M_H^C(N))$ *is a disjoint union of* $\beta - 1$ *copies of* K^α.

Example 3.8. Let $R = Z_4 \times Z_4$ and let $M = Z_2 \times Z_2$, $N = 0 \times Z_2$ and $H = \{(1,1),(3,3)\}$. It is clear to see that $N = M_H(N)$. Also, $(2,2) \in (M_H(N) : M)$ and $T(\Gamma_H^N(M))$ is a disjoint union of 2 copies of K^2. (see Figure 1).

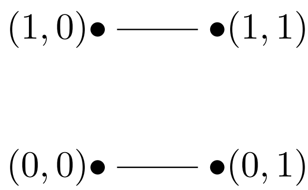

Figure 1.

Example 3.9. Let $R = M = Z_8$ and let $N = 4Z_8$ and $H = \{1,5\}$. It is clear that $N = M_H(N)$, $2 \notin H$ and $2 \notin (M_H(N) : M)$. In this case $\Gamma_H^N(M_H^C(N))$ is a disjoint union of one copy of K^2 and $K^{2,2}$. (see Figure 2).

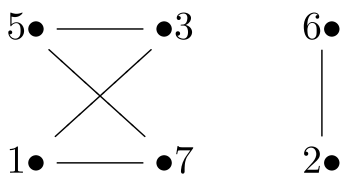

Figure 2.

Theorem 3.10. *Suppose that M is an R-module and that $\mid M_H(N) \mid= \alpha$ and $\mid M/M_H(N) \mid= \beta$ (we allow α, β to be infinite). If H contains some even elements, then $\Gamma_H^N(M_H^C(N))$ is a disjoint union of $(\beta - 1)/2$ copies of $K^{\alpha,\alpha}$.*

Proof. We assume that $2t \in H$ for some $t \in R$ and that $S = \{x_\lambda : \lambda \in \Lambda\}$ where $\mid \Lambda \mid= \beta$ is a complete representative subset of $M/M_H(N)$ such that $x_0 = 0$ and $x_i \notin M_H(N)$ for all $i \neq 0$. Let for distinct nonzero elements i, j of Λ, x_i is adjacent to x_j. Then $x_i + x_j \in M_H(N)$. Hence, $x_i = -x_j$. This means that each element of $x_i + M_H(N)$ is adjacent to every element of $x_j + M_H(N)$. On the other hand, there are no elements of the set $x_i + M_H(N)$ $(x_i \neq 0)$ adjacent to each other. Otherwise, let $x_i + y$ is adjacent to $x_i + z$ for some $y, z \in M_H(N)$. Then $2x_i \in M_H(N)$. There is $r \in H$ such that $2rx_i \in N$, so $2trx_i \in N$. This yields $x_i \in M_H(N)$ and $x_i = 0$. Therefore, for all $i \neq 0$, $x_i + M_H(N)$ and $-x_i + M_H(N)$ form a complete bipartite subgraph of $\Gamma_H^N(M_H^C(N))$ considered as a $K^{\alpha,\alpha}$. It is clear that the total number of such subgraphs is $(\beta - 1)/2$. □

Example 3.11. Let $R = Z_9 \times Z_9$ and let $M = Z_3 \times Z_3$, $N = 0 \times Z_3$ and $H = \{(2,2),(1,1),(4,4),(7,7),(5,5),(8,8)\}$. It is clear to see that $N = M_H(N)$. Also $(2,2) \in H$ and $\Gamma_H^N(M_H^C(N))$ is the complete bipartite graph $K^{3,3}(\alpha = 3$ and $\beta = 3)$.

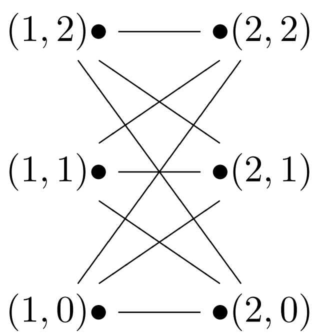

Figure 3.

Example 3.12. Let $R = M = Z_9$, $N = 3Z_9 = \{0, 3, 6\}$ and $H = \{1, 4, 7\}$. Here $2 \notin H$ but $4 \in H$. It is clear that $M_H(N) = N$ and $\Gamma_H^N(M_H^C(N))$ is isomorphic to $K^{3,3}$.

Note that $2 \notin H \bigcap (M_H(N) : M)$ by item 3 at the beginning of section 2. In the following example we see that non of the Theorems 3.6 and 3.10 are true if $2 \notin H \bigcup (M_H(N) : M)$.

Theorem 3.13. *Let x be a vertex of $T(\Gamma_H^N(M))$. Then the degree of x is either $\mid M_H(N) \mid$ or $\mid M_H(N) \mid -1$. In particular, if for every*

$m \in M_H^C(N)$, $2m \in M_H(N)$, then $T(\Gamma_H^N(M))$ is a $(\mid M_H(N) \mid -1)$-regular graph.

Proof. If $x \in M_H(N)$, then the degree of x is $|M_H(N)| - 1$, since $\Gamma_H^N(M_H(N))$ is complete.

Now, let $x \in M_H^C(N)$. If x is adjacent to y, then $x + y = a \in M_H(N)$ and hence $y = a - x$ for some $a \in M_H(N)$. We have two cases:

Case 1. Suppose $2x \in M_H(N)$. Then x is adjacent to $a - x$ for any $a \in M_H(N) - \{2x\}$. Thus the degree of x is $(\mid M_H(N) \mid -1)$. In particular, if $2m \in M_H(N)$ for every $m \in M_H^C(N)$, then $T(\Gamma_H^N(M))$ is a $(\mid M_H(N) \mid -1)$-regular graph.

Case 2. Suppose $2x \notin M_H(N)$. Then x is adjacent to $a - x$ for any $a \in M_H(N)$. Thus the degree of x is $|M_H(N)|$. $\square$

Theorem 3.14. *If $\Gamma_H^N(M_H^C(N))$ is connected, then diam $\Gamma_H^N(M_H^C(N)) \leqslant 2$ and $gr(\Gamma_H^N(M_H^C(N))) = 4$.*

Proof. Let $\mid M \mid= 2$ and $M_H(N) = \{0\}$, then diam $\Gamma_H^N(M_H^C(N) = 0$. If $\Gamma_H^N(M_H^C(N))$ is connected and there exist two elements $m_1, m_2 \in M_H^C(N)$ such that they are not adjacent (so diam $\Gamma_H^N(M_H^C(N)) \neq 1$), then by part (2) of Theorem 3.5, $m_1 - m_2 \in M_H(N)$, so we have a path m_1 — $(-m_2)$— m_2; hence diam $\Gamma_H^N(M_H^C(N)) = 2$.

Now, let $\Gamma_H^N(M_H^C(N))$ is connected and $m_1, m_2 \in M_H^C(N)$ are not adjacent. Then by Theorem 3.5, $m_1 - m_2 \in M_H(N)$. So, there exists cycle m_1 — $(-m_2)$— m_2— $(-m_1)$— m_1 and we have a 4-cycle. If $m_1, m_2 \in M_H^C(N)$ are adjacent, then there exists cycle m_1 — $(-m_1)$— $(-m_2)$— (m_2)— m_1 and we are done. $\square$

Theorem 3.15. *Let M be an R-module.*

(1) If $2 \in (M_H(N) : M)$ and $\mid (M_H(N)) \mid\geqslant 3$, then $gr(\Gamma_H^N(M_H^C(N))) = gr(T(\Gamma_H^N(M))) = 3$.

(2) If $2 \in H$ and $\mid (M_H(N)) \mid\geqslant 2$, then $gr(\Gamma_H^N(M_H^C(N))) = 4$.

Proof. (1) By Theorem 3.6, $\Gamma_H^N(M_H^C(N))$ and $T(\Gamma_H^N(M))$ are disjoint union of copies of K^α where $\mid (M_H(N)) \mid\geqslant 3$, so $gr(\Gamma_H^N(M_H^C(N))) =$

$gr(T(\Gamma_H^N(M))) = 3$.

(2) By Theorem 3.10, $\Gamma_H^N(M_H^C(N))$ is a disjoint union of copies of $K^{\alpha,\alpha}$ where $\mid (M_H(N)) \mid\geqslant 2$, so $gr(\Gamma_H^N(M_H^C(N))) = 4$. $\square$

Definition 3.16. *A vertex x of a connected graph G is called a cut-point of G if there are vertices u, w of G such that x is in every path from u to w(and $x \neq u$,$x \neq w$). Equivalently, for a connected graph G, x is called a cut-point of G if $G - \{x\}$ is not connected.*

Theorem 3.17. *If $\Gamma_H^N(M_H^C(N))$ is connected, then it has no cut-points.*

Proof. Assume that the vertex x of $\Gamma_H^N(M_H^C(N))$ is a cut-point. Then there are vertices a, b such that $a \neq b$ and x lies on every path from a to b. So, a and b are not adjacent. Hence, $a - b \in M_H(N)$, by Theorem 3.5. Similarly $b - a \in M_H(N)$. Thus there is a path a — $(-a)$— b in $\Gamma_H^N(M_H^C(N))$. Since x is a cut-point, so we must have $x = -a$. But there exists another path a — $(-b)$— b in $\Gamma_H^N(M_H^C(N))$ such that $x = -a \neq -b$ and it contradicts the fact that x is a cut-point. $\square$

Definition 3.18. *Let $m \in M$. We call the subset $m + M_H(N)$ a column of $\Gamma_H^N(M_H^C(N))$. If $2m \in M_H(N)$ for every $m \in M_H^C(N))$, then we call $m + M_H(N)$ a connected column of $\Gamma_H^N(M_H^C(N))$.*

The following proposition is clear from Definition 3.18.

Proposition 3.19. *Let M be an $R-$module, N be a submodule of M and $m \in M$. If $m + M_H(N)$ is a connected column of $\Gamma_H^N(M_H^C(N))$, then the subgraph induced by the set $m + M_H(N)$ is a complete subgraph of $\Gamma_H^N(M_H^C(N))$ and thus $\omega(\Gamma_H^N(M_H^C(N)) \geqslant\mid M_H(N) \mid$.*

Lemma 3.20. *(See [2, Theorem 3.4]) Let M be an R-module and let $\mid M(N) \mid= \alpha$ and $\mid M/M(N) \mid= \beta$ (we allow α, β to be infinite).*

(1) If $2 \in S(N : M)$, then $\overline{M}(\Gamma_N(M))$ is a disjoint union of $\beta - 1$ copies of K^{α}.

(2) If $2 \notin S(N : M)$, then $\overline{M}(\Gamma_N(M))$ is a disjoint union of $(\beta - 1)/2$ copies of $K^{\alpha,\alpha}$.

Theorem 3.21. *Let M be an $R-$ module. Then*

1. *One has $\omega(T(\Gamma_H^N(M))) \geqslant\mid M_H(N) \mid$.*
2. *If $\omega(\overline{M}(\Gamma_N(M)))$ is finite, then $\omega(\overline{M}(\Gamma_N(M))) =\mid M(N) \mid$ or 2.*
3. *If $\mid M(N) \mid\geqslant 2$ and finite, then $\omega(T(\Gamma_N(M))) =\mid M(N) \mid$.*

Proof.

1. It is clear from the fact that $M_H(N)$ induces a complete subgraph of $T(\Gamma_H^N(M))$.
2. By Lemma 3.20, If $2 \in S(N : M)$, then $\omega(\overline{M}(\Gamma_N(M))) =\mid M(N) \mid$. Otherwise, $\overline{M}(\Gamma_N(M))$ is a disjoint union of copies of $K^{\alpha,\alpha}$, so $\omega(\overline{M}(\Gamma_N(M))) = 2$.
3. It is clear from the definition of $\overline{M}(\Gamma_N(M))$ and case (2). □

Example 3.22. In the Example 2.2, we have $\omega(T(\Gamma_H^N(M))) = 2$ and $\omega(T(\Gamma_N(M))) = 4$.

Theorem 3.23. *If $\Gamma_H^N(M_H^C(N))$ is connected and has a connected column namely $m+M_H(N)$ for $m \in M_H^C(N)$ and at least one vertex $b \neq m$, then ω $(\Gamma_H^N(M_H^C(N)) \geqslant \mid M_H(N) \mid +1$.*

Proof. Because $\Gamma_H^N(M_H^C(N))$ is connected, so by Theorem 3.5, $b+m \in M_H(N)$ or $m+2b \in M_H(N)$. Then each element of the connected column $m + M_H(N)$ is adjacent to b or $2b$, and so $(m + M_H(N)) \cup \{b\}$ or $(m + M_H(N)) \cup \{2b\}$ forms a complete subgraph. □

An *independent set* is a set of vertices in a graph, no two of which are adjacent. The *vertex independence number* of a graph G, $\alpha(G)$, often called simply the "independence number", is the size of a maximum independent set.

Theorem 3.24. *Let in graph $T(\Gamma_H^N(M))$, the subgraph $\Gamma_H^N(M_H^C(N))$ be connected with $diam(\Gamma_H^N(M_H^C(N))) = 2$ and $2m \neq 0$ for every $m \in M_H^C(N)$. Then $\alpha(T(\Gamma_H^N(M))) = \frac{|M_H^C(N)|}{2} + 1$.*

Proof. Choose $x \in V(\Gamma_H^N(M_H^C(N)))$. Put $A_x = \{-y \in M_H^C(N))|y$ is adjacent to $x\}$, $A'_x = \{y \in M_H^C(N)|y \neq x$ and y is not adjacent to $x\}$ and let $P_x = A_x \cup A'_x$. For every $n(\neq x, -x) \in M_H^C(N)$, $n \in P_x$ or $-n \in P_x$.

Claim: P_x is an independent set in $\Gamma_H^N(M_H^C(N))$.

By way of contradiction, let there exist $n_1, n_2 \in P_x$ such that they are adjacent. Since $n_1, n_2 \in P_x$, so n_1, n_2 are not adjacent to x. It should be noted that for every $m \in M_H^C(N)$, either $m + x \in M_H(N)$ or $m - x \in M_H(N)$ (but not both), by Theorem 3.5, so n_1, n_2 are adjacent to $-x$. Then there is a path n_1— n_2— $(-n_2)$— x in $\Gamma_H^N(M_H^C(N))$ and $n_1 + x = (n_1 + n_2) + (-n_2 + x) \in M_H(N)$, a contradiction. Hence, P_x is an independent set in $\Gamma_H^N(M_H^C(N))$. On the other hand, for every $y(\neq x) \in V(\Gamma_H^N(M_H^C(N)))$, one has $|P_x| = |P_y|$. We show that P_x is the largest independent set in $\Gamma_H^N(M_H^C(N))$. Let there exists an independent set U in $\Gamma_H^N(M_H^C(N))$ such that $|U| > |P_x| = \frac{|M_H^C(N)|}{2}$. So, there exists $l \in V(\Gamma_H^N(M_H^C(N)))$ such that $l, -l \in U$; this implies that U is not independent. Hence, P_x is the largest independent set in $\Gamma_H^N(M_H^C(N))$.

Now, let $m \in V(\Gamma_H^N(M_H(N)))$. Then in $T(\Gamma_H^N(M)), B = P_x \cup \{m\}$ is an independent set (since $\Gamma_H^N(M_H(N))$ and $\Gamma_H^N(M_H^C(N))$ are disjoint subgraphs of $T(\Gamma_H^N(M))$ and that $\Gamma_H^N(M_H(N))$ is complete, so we can choose just one vertex $m \in M_H(N)$) that $P_x \cup \{m\}$ is a largest independent set in $T(\Gamma_H^N(M))$ (because $\Gamma_H^N(M_H(N))$ is complete and P_x is the largest independent set in $\Gamma_H^N(M_H^C(N))$), hence $\alpha(T(\Gamma_H^N(M))) = \frac{|M_H^C(N)|}{2} + 1$. □

A *vertex coloring* is an assignment of colors to the vertices of a graph G such that no two adjacent vertices have the same color. Therefore, no two vertices of an edge should be of the same color. The minimum number of colors required for vertex coloring of graph G is called as the *chromatic number* of G, denoted by $\chi(G)$.

Theorem 3.25. *Let in $T(\Gamma_H^N(M))$, the subgraph $\Gamma_H^N(M_H^C(N))$ be connected with $diam(\Gamma_H^N(M_H^C(N))) = 2$ and $2m \neq 0$ for every $m \in M_H^C(N)$. Then $\chi(T(\Gamma_H^N(M))) = |M_H(N)|$.*

Proof. Suppose that $x \in V(\Gamma_H^N(M_H^C(N)))$. Considering our hypothesis and by the proof of Theorem 3.24, for every vertex t other than x and $-x$, t is adjacent to either $-x$ or x (but not to both of them). If t is adjacent to x, then t is not adjacent to $-x$, so $t \in P_{-x}$; otherwise, $t \in P_x$. Hence $P_x \cup P_{-x} = V(\Gamma_H^N(M_H^C(N)))$. Now, we assign color a to elements of P_x and color b to elements of P_{-x} (it should be noted that, by the proof of Theorem 3.24, P_x, for every $x \in V(\Gamma_H^N(M_H^C(N)))$, is the largest independent set in $\Gamma_H^N(M_H^C(N))$). Hence $\chi(\Gamma_H^N(M_H^C(N))) = 2$. Since the subgraphs $\Gamma_H^N(M_H(N))$ and $\Gamma_H^N(M_H^C(N))$ of graph $T(\Gamma_H^N(M))$ are disjoint and $\Gamma_H^N(M_H(N))$ is complete, so we assign $|M_H(N)|-2$ colors along with colors a and b to vertices of $\Gamma_H^N(M_H(N))$. So, $\chi(\Gamma_H^N(M_H(N))) = \chi(T(\Gamma_H^N(M))) = |M_H(N)|$. $\square$

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