Journal of Mathematical Extension
Vol. 12, No. 1, (2018), 91-112
ISSN: 1735-8299
URL: http://www.ijmex.com

Absolutely Extendable Property and Stable Elements in Γ-Semihyperrings

S. Ostadhadi Dehkordi
University of Hormozgan

M. Heidari
Bu-Ali Sina University

B. Davvaz*
Yazd University

Abstract. The concept of Γ-semihyperrings is a generalization of a semiring, a generalization of a Γ-semiring, and a generalization of a semihyperering. In this paper, we define the notions of complex product, extension property and flat Γ-semihyperrings and some of their properties are obtained. In addition, we prove that every flat Γ-semihyperring is absolutely extendable. Finally, we give some characterization of stable elements.

AMS Subject Classification: 16Y99; 20N20
Keywords and Phrases: Γ-semihyperrings, absolutely extendable, stable elements, complex product, extension property

1. Introduction

The theory of hyperstructures was introduced by Marty [17] in 1934 during the 8^{th} Congress of the Scandinavian Mathematicians. Algebraic hyperstructures are a generalization of classical algebraic structures. In

Received: February 2017; Accepted: July 2017
*Corresponding author

a classical algebraic structure the composition of two elements is an element, while in an algebraic hyperstructure the composition of two elements is a non-empty set.
Let H be a non-empty set. Then, the map $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation*, where $\mathcal{P}^*(H)$ is the family of non-empty subsets of H. $(H, \circ)$ is called a *semihypergroup* if for every $x, y \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$. If for every $x \in H$, $x \circ H = H = H \circ x$, then $(H, \circ)$ is called a *hypergroup*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

Since then, hundreds of papers and several books have been written on this topic; see [2, 3, 6, 20]. A recent book on hyperstructures points out on their applications in cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [6] is devoted especially to the study of hyperring theory; several kinds of hyperrings are introduced and analyzed, and the volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. A well known type of a hyperring is called the *Krasner hyperring* [16] and then some researchers such as Davvaz et al. [1, 5, 4, 7, 8, 14, 15, 18, 22], Gontineac [13], Sen and Dasgupta [19], Vougiouklis [20, 21] and others followed him.

Definition 1.1. *A Krasner hyperring is an algebraic structure* $(R, +, \cdot)$ *which satisfies the following axioms:*

(1) $(R, +)$ is a canonical hypergroup, i.e.,

- (i) for every $x, y, z \in R$, $x + (y + z) = (x + y) + z$,
- (i) for every $x, y \in R$, $x + y = y + x$,
- (iii) there exists $0 \in R$ such that $0 + x = x$.
- (iv) for every $x \in R$ there exists a unique element $-x \in R$ such that $0 \in x + (-x)$.
- (v) $z \in x + y$ implies that $y \in -x + z$ and $x \in -y + z$.

(2) Relating to the multiplication, $(R, \cdot)$ is a semigroup having zero as a bilaterally absorbing element,

(3) The multiplication is distributive with respect to the hyperoperation $+$.

Recently, the concept of Γ-hyperstructures such as Γ-semihypergroups, Γ-hypergroups, Γ-semihyperrings and Γ-hypermodules study by many resechers. The concept of Γ-semihyperrings is a generalization of semihyperrings, generalization of a Γ-semirings and a generalization of semirings. This concept consider by Dehkordi and Davvaz [9, 10, 11, 12]. They introduced rough ideals, fundamental relations and complex product on Γ-semihyperrings. By the concept fundamental relation on Γ-semihyperrings they introduced covariant functor between the category Γ-semihyperrings and the category semirings.
We know that homological algebra is a efficient toll in the study of rings and modules. This research work deals with certain algebraic systems that is non-additive modification of classical homological structure. Motivated by the definition of flat rings in the category of rings, we define flat Γ-semihyperrings in the category of Γ-semihyperrings. We introduce the notions of *complex systems* on Γ-semihypergroups, then we prove some results in respect. Also, we introduce the notions of right(left) *flat* Γ*-semihyperring*, *extension property* and *absolutely extendable*. We prove that every flat Γ-semihyperring is absolutely extendable. Finally, we obtain a characterization of stable elements in Γ-semihyperrings.

2. Γ-Semihyperrings and Complex Product

In [10, 11], Dehkordi and Davvaz introduced the concept of Γ-semihyperrings. Now, in this section, we shall explain more about Γ-semihyperrings. We investigate the concept of left (right) Γ-funs and complex product.

Definition 2.1. *Let R and Γ be additive hypergroup and semihypergroup, respectively. Then, R is called a Γ-semihyperring if there exists a hyperoperation $R \times \Gamma \times R \longrightarrow \mathcal{P}^*(R)$ (the image of (x, α, y) is denoted by $x\alpha y$, for $x, y \in R$ and $\alpha,\ \beta \in \Gamma$) satisfies the following conditions:*

(1) $x_1\alpha(x_2 + x_3) = x_1\alpha x_2 + x_1\alpha x_3$,

(2) $(x_1 + x_2)\alpha x_3 = x_1\alpha x_3 + x_2\alpha x_3$,

(3) $x_1(\alpha + \beta)x_2 = x_1\alpha x_2 + x_1\beta x_2$,

(4) $(x_1\alpha x_2)\beta x_3 = x_1\alpha(x_2\beta x_3)$,

for all $x_1, x_2, x_3 \in R$ *and* $\alpha \in \Gamma$.
A Γ-semihyperring R is called Γ-*hyperring* if R is a canonical hypergroup. It is obvious that every Krasner hyperring is a Γ-hyperring where $x\alpha y$ denotes the product of the elements $x, y \in R$.

Example 2.2. Let $R = \{a, b\}$ and $\Gamma = \{\alpha, \beta\}$ be two sets with the following operations and hyperoperation. Then, R is a Γ-hyperring.

$+$	a	b
a	a	R
b	R	b

α	a	b
a	a	a
b	a	a

β	a	b
a	a	a
b	a	R

$+$	α	β
α	α	α
β	α	R

Example 2.3. Let $S = \{a_1, a_2, a_3, a_4\}$, $\Gamma = \{\alpha, \beta\}$. Then, S is a Γ-semihyperring with respect to the following operations and hyperoperations:

$\oplus$	a_1	a_2	a_3	a_4
a_1	a_1	a_2	$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_2	a_2	a_2	S	S
a_3	$\{a_3, a_4\}$	S	$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_4	$\{a_3, a_4\}$	S	$\{a_3, a_4\}$	$\{a_3, a_4\}$

β	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_2	$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_3	a_1	a_2	$\{a_3, a_4\}$	$\{a_3, a_4\}$
a_4	a_1	a_2	$\{a_3, a_4\}$	$\{a_3, a_4\}$

$+$	α	β
α	α	β
β	α	β

for every $x, y \in S$, $x\alpha y = a_1$.

Example 2.4. Let R be the Krasner hyperring, $R_{m\times n}$ be of all matrices over R and Γ be additive semihypergroup of all $n \times m$ matrices over R. Then, $R_{n\times m}$ is a Γ-hyperring where $a\alpha b$ denoted the usual matrix product of a, α, b where $a, b \in R_{m\times n}$ and $\alpha \in \Gamma$.

Example 2.5. Let $\mathbb{R}$ be the set of real numbers. Then, $\mathbb{R}$ is a $\widehat{\mathbb{Z}}$-semihyperring with respect to the following hyperaddition and hyperoperation:

$$\begin{aligned} x_1 \oplus x_2 &= \{z : [x_1] + [x_2] \leqslant z < [x_1] + [x_2] + 1\}, \\ x_1 \widehat{\alpha} x_2 &= \{z : \alpha[x_1][x_2] \leqslant z < \alpha[x_1][x_2] + 1\}, \end{aligned}$$

for every $x_1, x_2 \in \mathbb{R}$ and $\widehat{\alpha} \in \widehat{\mathbb{Z}}$, where $\widehat{\mathbb{Z}} = \{\widehat{\alpha} : \alpha \in \mathbb{Z}\}$.

Let A and B be non-empty subsets of Γ-semihyperring R. We define

$$A\Gamma^{\Sigma}B = \Big\{x \in R : x \in \sum_{i=1}^{n} a_i\alpha_i b_i : a_i \in A, b_i \in B, n \in \mathbb{N}\Big\}.$$

Let Γ be a semihypergroup and n be a nonzero natural number. Then, we say that

$$x\beta_n y \Longleftrightarrow \ \exists x_1, x_2, \cdots, x_n \in \Gamma : \ \{x, y\} \subseteq \prod_{i=1}^{n} x_i.$$

Let $\beta = \bigcup_{n\geqslant 1} \beta_n$. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure of β.
Let R be a Γ-semihyperring and $\mathcal{U}$ be a finite sum of elements of R. We define a relation γ on R as follows:

$$(a, b) \in \gamma \Longleftrightarrow \quad a, b \in u,$$

where $u \in \mathcal{U} = U_R \bigcup R\Gamma^{\Sigma}R \bigcup (U_R + R\Gamma^{\Sigma}R)$. We denote the transitive closure γ by γ^* and this equivalence relation is called *fundamental equivalence* relation on R. We denote the equivalence class of the element a by $\gamma^*(a)$. Hence, $\gamma^*(a_1) = \gamma^*(a_2)$ if and only if there exist $x_1, x_2, \ldots x_{n+1}$ with $x_1 = a_1$, $x_{n+1} = a_2$ and $u_1, u_2, \ldots u_n \in \mathcal{U}$ such that $\{x_i, x_{i+1}\} \subseteq u_i$, for some $i \in \{1, 2, \ldots n\}$.
Let R be a Γ-semihyperring. We define a relation θ on

$$\Big\{\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i)) : n \in \mathbb{N}, x_i \in R, \alpha_i \in \Gamma\Big\},$$

as follows:

$$\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i)),\prod_{j=1}^{m}(\gamma^*(x_j^{'}),\beta^*(\alpha_j^{'}))\Big)\in\theta$$
$$\Longleftrightarrow \sum_{i=1}^{n}\gamma^*(x_i)\widehat{\beta^*(\alpha_i)}\gamma^*(x)=\sum_{j=1}^{m}\gamma^*(x_j^{'})\widehat{\beta^*(\alpha_j^{'})}\gamma^*(x),$$

for every $\gamma^*(x)\in[R:\gamma^*]$, where γ^* is a fundamental relation on R.
Let R be a Γ-semihyperring and there exists an element

$$\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),$$

such that $\sum_{i=1}^{n}\gamma^*(e_i)\widehat{\beta^*(\delta_i)}\gamma^*(x)=\gamma^*(x)$, for all $\gamma^*(x)\in[R:\gamma^*]$. We say that this element is an *identity element(or just an identity))* of $F(R)$ and $F(R)$ is a Γ-semihyperring with identity.
Let $F(R)=\Big\{\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big):x_i\in R,\alpha_i\in\Gamma,n\in\mathbb{N}\Big\}$ and S be a non-empty set. We say that S is a left Γ-*fun* if there exists an action

$$\begin{aligned} F(R)\times S &\longrightarrow S\\ \Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big),y\Big) &\longmapsto \theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)y,\end{aligned}$$

with the following property:

$$\theta\Big(\prod_{i,j}(\gamma^*(x_i)\widehat{\beta^*(\alpha_i)}\gamma^*(y_j)),\beta^*(\gamma_j)\Big)y$$
$$=\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j)\Big)y\Big),$$
$$\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),s\Big)=s,$$

where $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)$, $\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)$ are elements of $F(R)$ and $s\in S$. In the same way, we can define right Γ-fun. Also, if R_1 and R_2 are Γ_1- and Γ_2- semihyperrings respectively, we say that S is a (Γ_1,Γ_2)-

fun if it is a left Γ_1-fun and a right Γ_2-fun, and

$$\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)y\Big)\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)$$
$$=\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big(y\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big),$$

where $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\in F(R_1)$, $\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\in F(R_2)$. It is clear that the cartesian product $X_1\times X_2$ of a left Γ_1- fun X_1 and a right Γ_2-fun X_2 becomes (Γ_1,Γ_2)-fun if we make the obvious definition:

$$\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)(x_1,x_2)=\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)x_1,x_2\Big),$$

$$(x_1,x_2)\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)=\Big(x_1,x_2\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big).$$

Suppose that A is a (Γ_1,Γ_2)-fun and B is a (Γ_2,Γ_3)-fun. Hence, $A\times B$ is a (Γ_1,Γ_3)-fun. A map $\varphi: A\times B\longrightarrow C$ is called a (Γ_1,Γ_3)-*map* if for all $a\in A$, $b\in B$ and $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\in F(R_2)$,

$$\varphi\Big(a\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big),b\Big)=\varphi\Big(a,\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)b\Big).$$

Example 2.6. Let R be a Γ-semihyperring, S be the set of all one-one and onto functions on $F(R)$. Then, S is a left Γ-fun.

Example 2.7. Let I be an ideal of Γ-semihyperring R. Then,

$$T(I)=\Big\{\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\in F(R):\ \omega\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\subseteq\gamma^*(I)\Big\},$$

is a left Γ-fun, where

$$\omega\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)$$
$$=\Big\{\bigoplus_i\gamma^*(x_i)\widehat{\beta^*(\alpha_i)}\gamma^*(x):1\leqslant i\leqslant n,\ x\in R\Big\}.$$

We say that (Γ_1, Γ_3)-fun C is a *complex product* of A and B over $F(R_2)$ if there is a (Γ_1, Γ_3)-map $\varphi : A \times B \longrightarrow C$ such that for every (Γ_1, Γ_3)-fun D and every (Γ_1, Γ_3)-map $\beta : A \times B \longrightarrow D$ there exists a unique (Γ_1, Γ_3)-map $\overline{\beta} : C \longrightarrow D$ such that $\overline{\beta} \circ \varphi = \beta$.
Suppose that ρ^* is an equivalence relation on $A \times B$ generated by the following relation:

$$\rho = \Big\{ \Big(\Big(a\theta \Big(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)), b \Big), \Big(a, \theta \Big(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) b \Big) \Big) \Big) \Big)$$

$$: a \in A, b \in B, \theta \Big(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \Big) \in F(R_2) \Big\}.$$

We define $C(A, B) = [A \times B : \rho^*]$ and denote a typical element $\rho^*(a, b)$ of $C(A, B)$ by $C(a, b)$. By definition of ρ we have that

$$C\Big(a\theta \Big(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \Big), b \Big) = C\Big(a, \theta \Big(\prod_{i=1}^{n} (\gamma^*(x_i), \beta^*(\alpha_i)) \Big) b \Big),$$

for all $a \in A$ and $b \in B$.

Proposition 2.7. *Let A be a (Γ_1, Γ_2)-fun and B be a (Γ_2, Γ_3)-fun. Then, $C(A, B)$ is a complex product of A and B over $F(R_2)$.*

Theorem 2.8. *The complex product of A and B over $F(R_2)$ is unique up to isomorphism.*

3. Flat Γ-Semihyperrings and Stable Elements

Motivated by the definition flat rings in the category of ring, we define flat Γ-semihyperrings in the category Γ-semihyperrings. This concept is a efficient tolls in the study of Γ-semihyperrings. In this section, we introduce the concept of flat Γ-semihyperrings, absolutely extendable, stable elements. Moreover, we prove that every flat Γ-semihyperring is absolutely extendable and we obtain a characterization for stable elements.

Definition 3.1. *Let R be a Γ-semihyperring and X_1, X_2 be left Γ-funs. Then by a morphism or Γ-morphism from a left Γ-fun X_1 into a left Γ-fun X_2 we mean a map $\psi : X_1 \longrightarrow X_2$ with the following property:*

$$\psi\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)x_1\Big)=\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\psi(x_1),$$

for every $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\in F(R)$ and $x_1\in X_1$.

A *congruence relation* on a left Γ-fun X is an equivalence relation on X with the following property:

$$x_1\rho x_2 \Longrightarrow \theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)x_1\rho\ \theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)x_2,$$

for every $x_1, x_2 \in X$ and $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\in F(R)$.

The quotient $[X:\rho]$ is a left Γ-fun structure by the following definition:

$$\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\cdot\rho(x)=\rho\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)x\Big).$$

We can generalize the notion of complex product for three Γ-funs. Suppose that X_1, X_2 and X_3 are (Γ_1,Γ_2)-, (Γ_2,Γ_3)-and (Γ_3,Γ_4)- funs, respectively. A map $\varphi : X_1\times X_2\times X_3 \longrightarrow X$ is called a *triple map* or (Γ_1,Γ_4)-*map*, if for $x_1\in X_1$, $x_2\in X_2$ and $x_3\in X_3$

$$\begin{aligned}&\varphi\Big(x_1\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big),x_2,x_3\Big)\\&=\varphi\Big(x_1,\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big)x_2,x_3\Big),\end{aligned}$$

where $\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big)\in F(R_2)$, and

$$\begin{aligned}&\varphi\Big(x_1,x_2\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\gamma_j))\Big),x_3\Big)\\&=\varphi\Big(x_1,x_2,\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\gamma_j))\Big)x_3\Big),\end{aligned}$$

where $\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\gamma_j))\Big)\in F(R_3)$.
We say that P is a complex product of X_1, X_2 and X_3 if there exists a unique (Γ_1,Γ_4)- map $\psi : X_1\times X_2\times X_3 \longrightarrow P$ such that for every (Γ_1,Γ_4)- fun X and (Γ_1,Γ_4)- map $\overline{\varphi} : P \longrightarrow D$, $\overline{\varphi}\circ\psi=\varphi$. One can see that $C(C(X_1,X_2),X_3)$ is a complex product of $X_1\times X_2\times X_3$ and

$$C(C(X_1,X_2),X_3)\cong C(X_1,C(X_2,X_3)).$$

Let R be a Γ-semihyperring. We say that R is *left flat* if for every left Γ-fun X and monomorphism $\psi : X_1\longrightarrow X_2$ of right Γ-funs, the induced map $\psi_C : C(X_1,X)\longrightarrow C(X_2,X)$ is injective. In the same way, we can define a *right flat* Γ-semihyperring.
Suppose that R_1 is a Γ-subsemihyperring of R. We say that R_1 has the *extension property* in R if for every right Γ-fun X_1 and left Γ- fun X_2 in R_1, the following map is injective:

$$\begin{aligned}\psi : C_{F(R_1)}(X_1,X_2) &\longrightarrow C_{F(R_1)}(C_{F(R_1)}(X_1,F(R)),X_2)\\ C(x_1,x_2) &\longmapsto C\Big(C\Big(x_1,\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),x_2\Big).\end{aligned}$$

A Γ-semihyperring R is called *absolutely extendable* if it has extension property in every Γ-semihyperring $R^{'}$ containing it as Γ-subsemihyperring.

Example 3.2. Let $(R,+,*)$ be a Krasner hyperring, $(\Gamma,+)$ be a subsemihypergroup of $(R,+)$ and $\{A_g \mid g\in R\}$ be a family of disjoint nonempty sets. Then, $S=\cup_{g\in R}A_g$ is a Γ-semihyperring with respect to the following hyperoperations:

$$x\oplus y=\bigcup_{t\in g_1+g_2}A_t,\qquad x\alpha y=\bigcup_{t=g_1*\alpha*g_2}A_t,$$

where $x\in A_{g_1}$ and $y\in A_{g_2}$. Also, R is a left Γ-fun by

$$\begin{aligned}F(S)\times R &\longrightarrow R\\ \theta\Big(\Big(\prod_{i=1}^{n}(\gamma^*(s_i),\beta^*(\alpha_i))\Big),x\Big) &\longrightarrow x,\end{aligned}$$

where $x \in R$, $\gamma^*(s_i) \in [S : \gamma^*]$ and $\beta^*(\alpha_i) \in [\Gamma : \beta^*]$. Let X_1 and X_2 be left Γ-funs and $\psi : X_1 \longrightarrow X_2$ be a monomorphism. Then, $\psi_C : C(X_1, R) \longrightarrow C(X_2, R)$ is injective. Indeed,

$$\psi_C(\rho^*(x_1, r_1)) = \psi_C(\rho^*(x_2, r)),$$

where $x_1 \in X$ and $r \in R$. By definition, ρ^*, we have $\psi(x_1) = \psi(x_2)$ and $r_1 = r_2$. Since ψ is one to one, we have $x_1 = x_2$. Therefore, $\rho^*(x_1, r_1) = \rho^*(x_2, r_2)$. Therefore, S is a flat and absolutely extendable Γ-semihyperring.

Proposition 3.3. *Every flat Γ-semihyperring is absolutely extendable.*

Proof. Suppose that R is a flat Γ-semihyperring and R_1 is a Γ-semihyperring containing R as a Γ-subsemihyperring. We show that the map

$$\psi : C_{F(R)}(X_1, X_2) \longrightarrow C_{F(R)}(C_{F(R)}(X_1, F(R_1)), X_2),$$

is injective. We note that the map

$$X_1 \cong C_{F(R)}(X_1, F(R)) \longrightarrow C_{F(R)}(X_1, F(R_1)),$$

is injective. Since R is flat, the following map is one-one. Hence,

$$\begin{aligned} &C_{F(R)}(X_1, X_2) \\ &\cong C_{F(R)}(C_{F(R)}(X_1, F(R)), X_2) \longrightarrow C_{F(R)}(C_{F(R)}(X_1, F(R_1)), X_2). \end{aligned}$$

Therefore, R has the extension property in R_1. This completes the proof. Let R_1 be a Γ-subsemihyperring of R such that $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i)), \beta^*(\alpha_i)\Big) \in F(R)$. We say that $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\Big)$ is *stable element* by R_1 if for every Γ'-semihyperring R' and homomorphism $\psi_1, \psi_2 : F(R) \longrightarrow F(R')$

$$\psi_1\Big(\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j), \beta^*(\gamma_j))\Big)\Big) = \psi_2\Big(\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j), \beta^*(\gamma_j))\Big)\Big),$$

for every $\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j), \beta^*(\gamma_j))\Big) \in F(R_1)$ which implies that

$$\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\Big)\Big) = \psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i), \beta^*(\alpha_i))\Big)\Big).$$

The set of elements of $F(R)$ stable by R_1 denoted by $St_R(R_1)$. It is easy to see that $F(R_1) \subseteq St_R(R_1)$ $\square$.

Theorem 3.4. *Let R_1 be a Γ-subsemihyperring of R and*

$$\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big) \in F(R).$$

Then,

$$C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),$$

$$= C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_j))\Big)\Big),$$

implies that $\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)$ is stable by R_1.

Proof. Suppose that

$$C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),$$

$$= C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_j))\Big)\Big).$$

Let we have $\Gamma^{'}$-semihyperring $R^{'}$ and homomorphism $\psi_1, \psi_2 : F(R) \longrightarrow F(R^{'})$ such that for every $\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big) \in F(R_1)$,

$$\psi_1\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big)\Big) = \psi_2\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big)\Big).$$

We define

$$\begin{aligned}&\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big)\Big)\cdot\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\gamma_j))\Big)\\ &= \psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\gamma_j))\Big).\end{aligned}$$

and

$$\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\gamma_j))\Big)\cdot\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\epsilon_j))\Big)$$
$$=\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\gamma_j))\Big)\psi_2\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big)\Big),$$

where $\theta\Big(\prod_{j=1}^{m}(\gamma^*(s_j),\beta^*(\varepsilon_j))\Big)\in F(R_1)$ and $\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\gamma_j))\Big)\in F(R^{'})$. Hence, $F(R^{'})$ is a (Γ_1,Γ_1)-funs in R_1. We define $\psi : F(R)\times F(R)\longrightarrow F(R^{'})$ by the rule that

$$\psi\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big)$$
$$=\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big)\Big)\psi_2\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big).$$

Then, ψ is a (Γ_1,Γ_1)-map in R_1. Indeed,

$$\psi\Big(\theta\Big(\prod_{i,j}(\gamma^*(t_i)\widehat{\beta^*(\delta_i)}\gamma^*(y_j),\beta^*(\gamma_j))\Big),\theta\Big(\prod_{r=1}^{m}(\gamma^*(z_r)),\beta^*(\omega_r))\Big)\Big)$$

$$=\psi_1\Big(\prod_{i,j}(\gamma^*(t_i)\widehat{\beta^*(\delta_i)}\gamma^*(y_j),\beta^*(\gamma_j))\Big)\psi_2\Big(\theta\Big(\prod_{r=1}^{m}(\gamma^*(z_r),\beta^*(\omega_r))\Big)\Big)$$
$$=\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big)\Big)\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big);$$
$$\psi_2\Big(\theta\Big(\prod_{r=1}^{m}(\gamma^*(z_r),\beta^*(\omega_r))\Big)\Big)$$
$$=\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big)\Big)\psi_2\Big(\prod_{r,j}\gamma^*(y_j)\widehat{\beta^*(\gamma_j)}\gamma^*(z_r),\beta^*(\omega_r))\Big).$$

Hence, there exists a map $\overline{\psi} : C_{F(R_1)}(F(R),F(R))\longrightarrow F(R^{'})$ such that

$$\overline{\psi}\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big)\Big)$$
$$=\psi\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big)$$
$$=\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(t_i),\beta^*(\delta_i))\Big)\Big)\psi_2\Big(\theta\Big(\prod_{j=1}^{n}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big).$$

Now, by assumption

$$\begin{aligned}
&\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\\
&=\psi_1\Big(\theta\Big(\prod_{i=1}^{m}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\psi_2\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big)\\
&=\overline{\psi}\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big)\Big)\\
&=\overline{\psi}\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\Big)\\
&=\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big)\psi_2\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big)\\
&=\psi_2\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(x_i),\beta^*(\alpha_i))\Big)\Big).
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let R_1 be a Γ-subsemihyperring of R. Then,*

$$\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big)\in St_R(R_1)$$

implies that

$$C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big)$$

$$=C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big)\Big).$$

Proof. We know that $C_{F(R_1)}(F(R),F(R))$ is a (Γ,Γ)-fun in R as follows:

$$\begin{aligned}
&\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\gamma_i))\Big)\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\alpha_j))\Big)\\
&\qquad\cdot\theta\Big(\prod_{j=1}^{m'}(\gamma^*(z_j'),\beta^*(\alpha_j'))\Big)\Big)\Big)\\
&=C_{F(R_1)}\Big(\theta\Big(\prod_{i,j}\Big(\gamma^*(y_i)\widehat{\beta^*(\gamma_i)}\gamma^*(z_j),\beta^*(\alpha_j)\Big),\theta\Big(\prod_{j=1}^{m'}(\gamma^*(z_j'),\beta^*(\alpha_j'))\Big)\Big)\Big);
\end{aligned}$$

$$C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\alpha_j))\Big),\theta\Big(\prod_{j=1}^{m'}(\gamma^*(z_j'),\beta^*(\alpha_j'))\Big)\Big)$$
$$\cdot\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\gamma_i))\Big)$$
$$=C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(z_j),\beta^*(\alpha_j))\Big),\theta\Big(\prod_{i,j}\Big(\gamma^*(z_j')\widehat{\beta^*(\alpha_j')}\gamma^*(y_i)),\beta^*(\gamma_i)\Big)\Big)\Big).$$

Let Ω be the set of all finite combinations

$$\sum_{i=1}^{n}n_i\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m_i}\gamma^*(x_{ij}),\beta^*(\alpha_{ij})\Big)\Big),\theta\Big(\prod_{j=1}^{m_i'}\gamma^*(x_{ij}'),\beta^*(\alpha_{ij}')\Big)\Big).$$

One can see that Ω is a (Γ,Γ)-fun in R. We define a binary relation on $F(R)\times\Omega$ as follows:

$$\Big(\theta\Big(\prod_{t=1}^{n}(\gamma^*(y_t),\beta^*(\alpha_t))\Big),$$
$$\sum_{i=1}^{n}n_i\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m_i}\gamma^*(x_{ij}),\beta^*(\alpha_{ij})\Big)\Big),\theta\Big(\prod_{j=1}^{m_i'}\gamma^*(x_{ij}'),\beta^*(\alpha_{ij}')\Big)\Big)\Big)\cdot$$
$$\Big(\theta\Big(\prod_{r=1}^{n'}(\gamma^*(y_r'),\beta^*(\alpha_r'))\Big),$$
$$\sum_{i=1}^{m}s_i\Big(C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m_i}\gamma^*(z_{ij}),\beta^*(\gamma_{ij})\Big)\Big),\theta\Big(\prod_{j=1}^{m_i'}\gamma^*(z_{ij}'),\beta^*(\gamma_{ij}')\Big)\Big)\Big)$$
$$=\Big(\theta\Big(\prod_{t,r}(\gamma^*(y_t)\widehat{\beta^*(\alpha_t)}\gamma^*(y_r'),\beta^*(\alpha_r'))\Big),$$
$$\sum_{i=1}^{m}n_iC_{F(R_1)}\Big(\theta\Big(\prod_{j,t}(\gamma^*(y_t)\widehat{\beta^*(\alpha_t)}\gamma^*(z_{ij})),\beta^*(\gamma_{ij})\Big),\theta\Big(\prod_{j=1}^{m_i'}\gamma^*(z_{ij}'),\beta^*(\gamma_{ij}')\Big)\Big)$$
$$+\sum_{i=1}^{n}s_iC_{F(R_1)}\Big(\theta\Big(\prod_{r,j}\gamma^*(x_{ij})\widehat{\beta^*(\alpha_{ij})}\gamma^*(y_j'),\beta^*(\alpha_r'))\Big)\Big),$$
$$\theta\Big(\prod_{j=1}^{m_i'}\gamma^*(x_{ij}'),\beta^*(\alpha_{ij}')\Big)\Big).$$

It is easy to see that this binary relation is associative. In fact $F(R)\times\Omega$ is a groupoid with identity $\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big),0\Big)$. Suppose that

$\theta\Big(\prod\limits_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big) \in St_R(R_1)$. We consider two homomorphisms ψ_1 and ψ_2 from $F(R)$ into $F(R)\times\Omega$ and show that they coincide on R_1. We define

$$\begin{aligned}
&\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big)\Big) = \Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big),0\Big)\\
&\psi_2\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big)\Big) = \Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(y_i),\beta^*(\alpha_i))\Big),\\
&\theta\Big(\prod_{i,j}\gamma^*(y_i)\widehat{\beta^*(\alpha_i)}\gamma^*(e_j),\beta^*(\delta_j))\Big) = \theta\Big(\prod_{i,j}\gamma^*(e_j)\widehat{\gamma^*(\delta_j)}\gamma^*(y_i),\beta^*(\alpha_i))\Big)\Big).
\end{aligned}$$

By a routine process, we see that ψ_1 and ψ_2 are homomorphisms. Let $\theta\Big(\prod\limits_{i=1}^{n}(\gamma^*(z_i),\beta^*(\alpha_i))\Big) \in F(R_1)$. This implies that

$$\begin{aligned}
&C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(z_i),\beta^*(\alpha_i))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i)\Big)\Big)\\
&= C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i)\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(z_i),\beta^*(\alpha_i))\Big)\Big)
\end{aligned}$$

in $C_{F(R_1)}(F(R),F(R))$ and so

$$\psi_1\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(z_i),\beta^*(\alpha_i))\Big)\Big) = \psi_2\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(z_i),\beta^*(\alpha_i))\Big)\Big).$$

Moreover, $\theta\Big(\prod\limits_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big) \in St_R(R_1)$ implies that

$$\psi_1\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big)\Big) = \psi_2\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big)\Big),$$

and so

$$\begin{aligned}
&C_{F(R_1)}\Big(\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),\\
&= C_{F(R_1)}\Big(\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\theta\Big(\prod_{j=1}^{m}(\gamma^*(x_j),\beta^*(\alpha_j))\Big)\Big).
\end{aligned}$$

This completes the proof. □

Proposition 3.6. *Let R be a Γ-semihyperring, such that X_1, X_2 and X_3 be Γ-funs and $\varphi_1 : X_1 \longrightarrow X_2$, $\varphi_2 : X_1 \longrightarrow X_3$ be Γ-morphisms. Then, there exists a Γ-fun X and $\psi_1 : X_2 \longrightarrow X$, $\psi_2 : X_3 \longrightarrow X$ such that ψ_1 and ψ_2 are Γ-homomorphisms. Moreover, if $\psi_1(x_2) = \psi_2(x_3)$, then $x_2 \in \varphi(X_1)$.*

Proof. Suppose that ρ is the equivalence relation generated by all pairs $(x_1, \varphi_1(x_1))$, $(x_1, \varphi_2(x_1))$, on $X = X_2 \cup X_3$, where $x_1 \in X_1$. The maps $\psi_1 : X_2 \longrightarrow X$, $\psi_2 : X_1 \longrightarrow X$ are given by $\psi_1(x_2) = \rho(x_2)$, $\psi_1(x_2) = \rho(x_2)$. This complete the proof. □

Lemma 3.7. *Let R_1 be a Γ-subsemihyperring of R and R_1 has extension property in R, $\varphi : X_1 \longrightarrow X_2$ be a Γ-morphism of right Γ-fun in R_1 and*

$$C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big) = C\Big(\varphi(x_1), \theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\alpha_j))\Big)\Big),$$

in $C(X_2, F(R))$. Then, $x_2 \in \varphi(X_1)$.

Proof. Suppose that X is a Γ-fun in Proposition 3.6. Consider the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{\psi_2} & X \end{array}$$

where $\psi_1 : X_2 \longrightarrow X$ and $\psi_2 : X_2 \longrightarrow X$. Hence, the following diagram is commutative:

$$\begin{array}{ccc} C(X_1, F(R)) & \xrightarrow{C(\varphi,1)} & C(X_2, F(R)) \\ \downarrow & & \downarrow \\ C(X_2, F(R)) & \xrightarrow{C(\psi_2,1)} & C(X, F(R)) \end{array}$$

By the extension property the map $x_2 \mapsto C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big)$

from $X_2 \cong C(X_2, F(R_1))$ to $C(X_2, F(R))$ is one to one. We have

$$\begin{aligned}
&C\Big(\psi_1(x_2), \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big)\\
&= C(\psi_1, 1)\Big(C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big)\Big)\\
&= C(\psi_1, 1)\Big(C\Big(\varphi(x_1), \theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\alpha_j))\Big)\Big)\Big)\\
&= C\Big((\psi_2 \circ \varphi)(x_1), \theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\alpha_j))\Big)\Big)\\
&= C(\psi_2, 1)\Big(C\Big(\varphi(x_1), \theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\alpha_j))\Big)\Big)\Big)\\
&= C(\psi_2, 1)\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big)\\
&= C\Big(\psi_2(x_2), \theta\Big(\prod_{j=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big).
\end{aligned}$$

Hence, $\psi_1(x_2) = \psi_2(x_2)$ and it follows by Proposition 3.6, $x_2 \in \varphi(X_1)$. □

Theorem 3.8. *Let R_1 be a Γ-subsemihyperring of R and suppose that R_1 has the extension property in R. Let X_1, X_2 be right Γ-funs in R_1 and $\varphi : X_1 \longrightarrow X_2$ be Γ-monomorphism in R_1 and Z be a left Γ-fun in R_1 such that $C(\varphi, 1) : C(X_1, Z) \longrightarrow C(X_2, Z)$ is also monomorphism. If*

$$\begin{aligned}
&C\Big(C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big), z\Big)\\
&= C\Big(C\Big(\varphi(x_1), \theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j), \beta^*(\gamma_j))\Big)\Big), z'\Big),
\end{aligned}$$

in $C(C(X_2, F(R)), Z)$. Then, there exists $x_1' \in X_1$ and $z_1 \in Z$ such that

$$\begin{aligned}
&C\Big(C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big), z\Big)\\
&= C\Big(C\Big(\varphi(x_1'), \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big), z_1\Big).
\end{aligned}$$

Proof. Suppose that

$$C\Big(C\Big(x_2,\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),z\Big)$$
$$=C\Big(C\Big(\varphi(x_1),\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big),z^{'}\Big),$$

in $C(C(X_2,F(R)),Z)$. By Proposition 3.6, we have the following commutative diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\varphi} & X_2 \\ \downarrow & & \downarrow \\ X_2 & \xrightarrow{\psi_2} & X \end{array}$$

such that $\psi_1 : X_2 \longrightarrow X$ and $\psi_2 : X_2 \longrightarrow X$. Hence, the following diagram is commutative:

$$\begin{array}{ccc} C(X_1,Z) & \xrightarrow{C(\varphi,1)} & C(X_2,Z) \\ \downarrow & & \downarrow \\ C(X_2,Z) & \xrightarrow{C(\psi_2,1)} & C(X,Z) \end{array}$$

is commutative. We note that

$$C\Big(C\Big(\psi_1(x_2),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),z\Big)$$
$$=C(C(\psi_1,1),1)\Big(C\Big(C\Big(x_2,\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\delta_i))\Big)\Big),z\Big)\Big)$$
$$=(C(\psi_1,1),1)\Big(C\Big(C\Big(\varphi(x_1),\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big),z^{'}\Big)\Big)$$
$$=C\Big(C\Big((\psi_1\circ\varphi)(x_1),\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big),z^{'}\Big)$$
$$=C\Big(C\Big((\psi_2\circ\varphi)(x_1),\theta\Big(\prod_{j=1}^{m}(\gamma^*(y_j),\beta^*(\gamma_j))\Big)\Big),z^{'}\Big)$$
$$\vdots$$
$$=C\Big(C\Big(\psi_2(x_2),\theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i),\beta^*(\gamma_i))\Big)\Big),z\Big).$$

By the extension property we deduced that $C(\psi_1(x_2),z)=C(\psi_2(x_2),z)$. Hence, by Proposition 3.6, there exists $C(x_1^{'},z_1)\in C(X_1,Z)$ such that

$$C(x_2,z)=C(\varphi,1)(C(x_1^{'},z_1))=C(\varphi(x_1^{'}),z_1).$$

Therefore,

$$C\Big(C\Big(x_2, \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big), z\Big)$$
$$= C\Big(C\Big(\varphi(x_1'), \theta\Big(\prod_{i=1}^{n}(\gamma^*(e_i), \beta^*(\delta_i))\Big)\Big), z_1\Big).$$

This completes the proof. □

References

[1] S. M. Anvariyeh, S. Mirvakili, and B. Davvaz, Fundamental relation on $(m, n)-$ary hypermodules on $(m, n)-$ary hyperring, *ARS Combinatoria*, 94 (2010), 273-288.

[2] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, (1993).

[3] P. Corsini and V. Leoreanu, *Applications of Hyperstructures Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, (2003).

[4] B. Davvaz, Approximations in hyperrings, *Journal of Multiple-Valued Logic and Soft Computing*, 15 (5-6) (2009), 471-488.

[5] B. Davvaz, Isomorphism theorems of hyperrings, *Indian J. Pure Appl. Math.*, 35 (3) (2004), 321-331.

[6] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, (2007).

[7] B. Davvaz and A. Salasi, A realization of hyperrings, *Comm. Algebra*, 34 (12) (2006), 4389-4400.

[8] B. Davvaz and T. Vougiouklis, Commutative rings obtained from hyperrings (H_v-rings) with α^*-relations, *Comm. Algebra*, 35 (2007), 3307-3320.

[9] S. O. Dehkordi and B. Davvaz, Γ-semihyperrings: Approximations and rough ideals, *Bulletin of the Malaysian Mathematical Sciences Society,* 35 (4) (2012), 1035-1047.

[10] S. O. Dehkordi and B. Davvaz, A strong regular relation on Γ-semihyperrings, *J. Sci. I.R.*, 22 (3) (2011), 257-266.

[11] S. O. Dehkordi and B. Davvaz, Complex product, *Extended Abstracts of the 42^th Anuual Iranian Mathematics Conference*, University of Rafsanjan, Iran, (2011), 72.

[12] S. O. Dehkordi and B. Davvaz, Γ-semihyperring: Fundamental rings and complex product, *Bulletin of the Allahabad Mathematical Society*, 29 (2) (2014), 111-135.

[13] V. M. Gontineac, On hypernear-rings and H-hypergroups, in *Algebraic Hyperstructures and Applications*, Hadronic Press, Palm Harbor, Fla, USA, (1994), 171-179.

[14] H. Hedayati and I. Cristea, Fundamental Γ-semigroups through H_v-Γ-semigroups, *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 75 (2) (2013), 33-46.

[15] H. Hedayati and B. Davvaz, Fundamental relation on Γ-hyperrings, *ARS Combinatoria*, 100 (2011), 381-394.

[16] M. Krasner, A class of hyperrings and hyperfields, *Int. J. Math. and Math. Sci.*, 2 (1983), 307-312.

[17] F. Marty, Sur une generalization de la notion de groupe, in *Proceedings of the 8^th Congress des Mathematiciens Scandinaves*, Stockholm, Sweden, (1934), 45-49.

[18] S. Mirvakili and B. Davvaz, Applications of the α^*-relation to Krasner hyperrings, *J. Algebra*, 362 (2012), 145-156.

[19] M. K. Sen and U. Dasgupta, Hypersemiring, *Bull. Cal. Math. Soc.*, 100 (2) (2008), 143-156.

[20] T. Vougiouklis, *Hyperstructures and Their Representations*, Hadronic Press, Florida, (1994).

[21] T. Vougiouklis, The fundamental relation in hyperrings. The general hyperfield, *Algebraic Hyperstructures and Applications* (Xnthi, 1990), 203-211, World Sci. Publ., Teaneck, NJ, 1991.

[22] J. Zhan and I. Cristea, Characterizations of fuzzy soft Γ-hemirings, *Journal of intelligent & fuzzy systems*, 26 (2) (2014), 901-911.

Sohrab Ostadhadi-Dehkordi
Assistant Professor of Mathematics
Department of Mathematics
University of Hormozgan
Bandar-Abbas, Iran
E-mail: Ostadhadi@hormozgan.ac.ir

Mehri Heidari
MS.c Student of Mathematics
Department of Mathematics
Bu-Ali Sina University
Hamedan, Iran
E-mail: mheidari@basu.ac.ir

Bijan Davvaz
Professor of Mathematics
Department of Mathematics
Yazd University
Yazd, Iran
E-mail: davvaz@yazd.ac.ir